

FREE CURVES IN E^3

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A 2-sphere in Euclidean 3-dimensional space E^3 is called free if it can be pushed into either complementary domain by a map moving no point more than ε , for arbitrary ε . Such 2-spheres have been the object of much recent attention, although the basic problem of whether they must be tame or not remains unsolved. The purpose of this paper is to take a different direction in this study. We introduce a natural generalization of the term free so that it can be used to describe a k -sphere in E^n , then direct our attention to free 1-spheres and 2-spheres in E^3 .

Our main tool is Theorem 1, which, roughly speaking, should be viewed as follows: It is well known that if D and E are both polyhedral disks in E^3 intersecting only in their interiors (in general position), then E may be altered via a disk replacement process to miss D . Theorem 1 states that even if D were a singular disk in E^3 , this process would remain valid to an extent.

This fact is proven through a proof of Dehn's lemma given by Shapiro and Whitehead [12].

Theorem 1 is then used to prove a theorem of Hempel [8], which asserts that free 2-spheres in E^3 are tame under an additional assumption. This assumption is defined in § 4. Moreover, an analogous theorem is then established for free 1-spheres in E^3 . Surprisingly enough, the version for free 1-spheres then can be used to deduce another theorem about free 2-spheres; they can be pierced at every point by a tame arc. This fact was recently shown by McMillan, using other methods [10].

In § 5 we deal with free 1-spheres whose wild points form a 0-dimensional set. If such a 1-sphere bounds a disk, then it must be tame.

The paper is concluded with a section of conjectures.

2. Preliminaries. We assume throughout that all polyhedral sets and maps are placed in general position. By a loop L in the space X is meant a map $L: S^1 \rightarrow X$, where S^1 denotes the 1-dimensional sphere; we set $|L| = L(S^1)$. Similarly, a disk D in X is a map $D: \Delta \rightarrow X$, where Δ is a 2-cell. The disk D is *singular* or *nonsingular* according as D is not or is one-to-one. We set $|D| = D(\Delta)$ and $\partial D = D(\partial\Delta)$, if D is singular; a nonsingular disk will be identified with its image and this notation therefore is unnecessary.

If J is a simple closed curve in a nonsingular disk E , then by $\text{Int}_E J$ we denote the domain of $E - J$ disjoint from ∂E . Similarly, if L is a polyhedral loop in E , then $\text{Int}_E L(\text{Ext}_E L)$ denotes all points p in $E - L$ such that a polyhedral arc from p to ∂E intersects $|L|$ an odd (even) number of times.

The term ‘‘regular neighborhood’’ is to be understood in the sense of Whitehead [14].

3. Simplifying the intersections of disks. All sets in this section are polyhedral.

THEOREM 1. *Let M be a 3-manifold, D a singular disk in M , E a nonsingular disk in M , with $|D| \cdot \partial E = \partial D \cdot E = \phi$. Let J be a simple closed curve in $\text{Int}_E(|D| \cdot E)$, and R a regular neighborhood of $|D|$. Then there exists a nonsingular disk E' such that $\partial E' = \partial E$ and $E' \subset \text{Ext}_E J + R$.*

Proof. The set $D^{-1}(|D| \cdot E)$ consists of a finite number of disjoint simple closed curves K_1, \dots, K_k in Δ ; the restriction of D to K_i may be viewed as a loop L_i in E . We may assume that $J \subset \text{Int}_E L_1$, $J \subset \text{Ext}_E L_i$, for $i = 2, 3, \dots, k$, and $K_1 = \partial \Delta$, by astute selection of a subdisk of Δ . As in [12], we now cut down M to a 3-manifold-with-boundary V_0 by selection of a regular neighborhood of $|D| + E$, taking care near ∂E to ensure that $\partial E \subset \partial V_0$.

A tower of 2-sheeted coverings is constructed as in [11] and [12]. The 0th story is given by the disks $D = D_0$ and $E = E_0$, the curve $J = J_0$, and the regular neighborhood V_0 . The n th story will consist of a connected 3-manifold M_n , a nonsingular disk E_n and a singular disk D_n in M_n with $\partial D_n \subset E_n$ and $\partial E_n \subset \partial M_n$, a curve $J_n \subset E_n$, and a regular neighborhood V_n of $|D| + E_n$ in M_n . Connecting the n th and $(n + 1)$ st story is a map $p_{n+1}: M_{n+1} \rightarrow V_n$ which is a double covering, with D_{n+1} covering D_n , E_{n+1} covering E_n , and J_{n+1} covering J_n . If one views D_n and E_n as a single map of the set $\Delta + \Delta$ into M_n , the singularity of this map can be used to prove that the tower is finite, as in Lemma 3 of [13]. Call the top story M_m . Then V_m has a boundary consisting of a finite number of 2-spheres.

We will establish the conclusion of the lemma first in the m th story. That is, there exists a disk E'_m such that $\partial E'_m = \partial E_m$ and

$$E'_m \subset (\text{Ext}_{E_m} J_m) + R_m$$

where R_m is a regular neighborhood of D_m . Note that E_m separates V_m into two components W and X , because the first homology group of V_m with Z_2 coefficients is trivial [9, pp. 461–462]. Let W be the

component containing $|D_m|$ near to ∂D_m . If Q is the 2-sphere in ∂V_m which contains ∂E_m , then ∂E_m divides Q into two disks A and B . Call A the disk in W ; this is the disk we seek, modulo a minor modification.

We regard V_m as composed of a regular neighborhood Y_m of E_m and Z_m of $|D_m|$, and assume that Z_m lies strictly in the interior of R_m . The set $Y_m - Z_m$ may be viewed as a collar on an open subset Y of E_m , that is, $Y_m - Z_m = Y \times [-1, 1]$ with $Y \times (0, 1] \subset W$. We now project the set $Y \times 1$ onto $Y \times 0$; this projection carries A onto a new disk E'_m in V_m , which we verify to satisfy all desired properties. It should be noted that a bit of care must be taken to modify this projection near the edges of Y so that we have continuity of this map when viewed in V_m . This can be done in Z_m , so that the goal of this modification of A , that of pulling A into $E_m + R_m$, remains valid.

It would be disastrous if this projection carried A onto J_m . The key property of V_m that prevents this from occurring is the trivial first homology with Z_2 coefficients. This implies that a polyhedral 1-cycle and 2-cycle will intersect in an even number of points. Thus, if J is a simple closed curve in V_m and $J \cdot E_m = \phi$, then $J \cdot |D_m|$ contains an even number of points. If $E'_m \cdot J_m \neq \phi$, then we can find an arc α in $\text{Int } A$ joining a point of $\partial E_m \times 1$ to a point of $J_m \times 1$. Select an arc β in $E_m \times 1$ connecting these same two points. Then β intersects D_m an odd number of times. To see this, we use induction on m . If $m = 0$, then it is certainly true. If $m = 1$, then the loop L_1 in E_0 is covered by an identical loop in E_1 , making an odd contribution. The loops $L_i, i = 2, \dots, k$ are either covered by identical loops in E_1 , or have totally disappeared; in either case, their contribution is zero, mod 2. The argument proceeds similarly. Then $\alpha + \beta$ forms a curve γ in V_m with $\gamma \cdot E_m = \phi$ and $\gamma \cdot |D_m|$ containing an odd number of points, which gives the desired contradiction.

We now assume that there exists a point

$$p \in E'_m \cdot \text{Int}_{E_m} J_m$$

and reason to a contradiction. Let P denote the 2-sphere in ∂V_m which contains $J_m \times 1$. By the previous paragraph, P is not the same 2-sphere as Q . Select an arc r in A joining $p \times 1$ to ∂A , an arc s in E_m which completes $r + (p \times [0, 1])$ to a simple closed curve. Now construct a 2-sphere by selecting a subdisk of P bounded by $J_m \times 1$, combined with

$$J_m \times [-1, 1] + \text{Int}_{E_m \times -1} J \times -1 .$$

Again we have a 2-cycle and a 1-cycle whose intersection is $s \cdot J_m$, an odd number of points, which can not happen. Thus, the disk E'_m has

all desired properties.

The proof is completed by projecting E'_m down the tower, story by story, and removing the double simple closed curves as they appear, by the standard method. By requiring that $P_{n+1}(R_{n+1}) \subset R_n$, we guarantee that

$$P_{n+1}((\text{Ext}_{E_{n+1}} J_{n+1}) + R_{n+1}) \subset (\text{Ext}_{E_n} J_n) + R_n .$$

4. Free spheres in E^3 . The key to this section is the following :

DEFINITION. A k -sphere J^k embedded in E^n is *free* if, given $\varepsilon > 0$, there exists a map $f: J^k \times S^{n-k-1} \rightarrow E^n - J^k$ such that

- (i) For $p \in J^k$, $f(p \times S^{n-k-1})$ lies in an ε -neighborhood of p .
- (ii) For $p \in J^k$, $f(p \times S^{n-k-1})$ links J^k (homologically with Z_2 coefficients).

For 2-spheres in E^3 , this definition corresponds to the standard one, which was stated in the introduction.

NOTE. In the above definition, if ε is sufficiently small, then a ray R from J^k to ∞ in general position intersects $f(J^k + S^{n-k-1})$ in an odd number of points. To see this, let D be a singular, polyhedral $(n - k)$ -disk in E^n in general position such that ∂D links J^k . Such a disk is easily built by using condition (ii) to select ∂D , then taking the cone over ∂D . If ε is small, then the $(n - k - 1)$ -cycle

$$C = D \cdot f(J^k \times S^{n-k-1})$$

is homologous to a meridian $f(p \times S^{n-k-1})$. Thus C links J^k . We now may find \tilde{J}^k , a polyhedral, (singular) close approximation to J^k , and place it in general position. Thus C links \tilde{J}^k , which implies that $\tilde{J}^k \cdot D$ is odd, and that a general positioned subarc of D from \tilde{J}^k to ∂D hits C in an odd number of points. This subarc yields the ray R .

We now establish a result of Hempel [8]. His definition of Condition (A) for 2-spheres in E^3 generalizes to: A k -sphere $J^k \subset E^n$ satisfies Condition (A) provided that whenever D is a polyhedral (2-dimensional) disk in E^n with $\partial D \subset E^n - J^k$ and V is any neighborhood of D , there is a disk D' (not necessarily tame) such that

- (i) $\partial D' = \partial D$
- (ii) $D' \subset V$, and
- (iii) if C is the component of $D' - S$ which contains $\partial D'$, then $D' - C$ has finitely many components.

THEOREM 2. (Hempel). *Let S be a 2-sphere in E^3 . If S is free and satisfies Condition (A), then S is tame.*

Proof. We wish to show that the complement of S is 1 – ULC so that we may apply [1]. Given $\varepsilon > 0$, there exists $\delta > 0$ such that a δ -set (meaning a set of diameter less than δ) in S lies in an $\varepsilon/6$ -disk in S . Let E be a polyhedral δ -disk in E^3 with ∂E contained in one complementary domain U of S . We wish to modify E on its interior, forming a singular disk in U which still has diameter less than ε . Although seemingly weaker than being 1 – ULC, this property is strong enough so that tameness of S will follow from [1].

We now invoke Condition (A); let E' denote the δ -disk replacing E , and A_1, \dots, A_k the finite set of components of $E' - C$. Select disjoint subdisks E_1, \dots, E_k of E' with $A_i \subset \text{Int } E_i$. Select a map $f: S \rightarrow E^3 - S$ such that any polyhedral arc from ∂E_i to S hits $f(S)$ in an odd number of points, for $i = 1, 2, \dots, k$. We now modify E_1 to miss S : There exists a simple closed curve $J \subset E_1$ such that

$$A_1 \subset \text{Int}_{E_1} J \quad \text{and} \quad J \subset \text{Int}_{E_1} f(S) \cdot E_1.$$

We may require that there is a singular disk D , obtained by restricting the map f , such that $f(S) \cdot E_1 = |D| \cdot E_1$ and $|D|$ is an $\varepsilon/6$ -set, by choosing f to move points a small distance. Applying Theorem 1 to singular disk D , nonsingular disk E_1 , and curve J , we find an $\varepsilon/3$ -disk F_1 which is disjoint from S . The same argument is used to modify E_2, \dots, E_k .

COROLLARY 2.1. (*Hempel*). *If S is a free 2-sphere in E^3 , and the wild points of S form a tame 0-dimensional set, then S is tame.*

In order to prove a similar theorem for a free 1-sphere K in E^3 , we need the extra hypothesis that K bounds a disk, because otherwise knotted counterexamples can be found. Although the following theorem parallels Theorem 2 in its statement, the proof turns out to be surprisingly different.

THEOREM 3. *Let K be a 1-sphere in E^3 . If K is free, satisfies Condition (A), and bounds a disk in E^3 , then J is tame.*

Proof. Note first that if the hypothesized map $f: K \times S^1 \rightarrow E^3$ were always taken to be one-to-one, then Lemma 3.5 would follow as a corollary of Theorem 1 of [4]. Our approach is, in fact, to mimic the proof of Theorem 1 of [4], so closely that we number the steps in this proof to correspond precisely with those of Theorem 1 of [4]. Thus, we omit justification of all steps where the argument is to be found in detail in [4]. We assume K lies in a 2-sphere S which is locally polyhedral, mod K [2]. Given a point $p \in K$, and $\varepsilon > 0$, we

wish to find a $\delta > 0$ such that: if B is a disk in a δ -neighborhood of p with $\partial B \cdot S = \phi$, then B may be modified on its interior to form an ε -disk in $E^3 - S$. To make notation correspond, select a map $f_i: K \times S^1 \rightarrow E^3$ moving points less than $1/i$, for $i = 1, 2, \dots$, and let $T_i = f_i(K \times S^1)$.

(1) There exists an integer N_1 and a positive number γ such that if α is any subset of T_n , $n > N_1$, and if α lies in a neighborhood of p of radius γ (which we abbreviate $o_\gamma(p)$), then either $\alpha \cap S \neq \phi$, or α lies in a singular subdisk of T_n (which means formally that

$$f_n^{-1}(\alpha) \subset \Delta \subset K \times S^1,$$

where Δ is a nonsingular disk. The singular disk is then $f_n \mid \Delta$).

(2) We assume that diameter $K > \varepsilon/3$. There exists a $\delta_1 > 0$ such that any $3\delta_1$ -subset of S lies in a $\varepsilon/3$ -disk on S . There exists a $\delta_2 > 0$ and integer N_2 such that any δ_2 -subset of T_n , $n > N_2$, which lies in a singular subdisk of T_n also must lie in a singular subdisk D of T_n with diameter $|D| < \delta_1/3$. Such a δ_2 and N_2 may be found thus: Select δ_2 so that any $2\delta_2$ -subset of K lies in a $\delta_1/6$ -subarc A of K . For N_2 sufficiently large, and $n > N_2$, any δ_2 -subset of T_n lies in $f(A \times S^1)$, where diameter $A < \delta_1/6$, so diameter $f_n(A \times S^1) < \delta_1/3$. We may select D to be a further restriction of the map $f_n/A \times S^1$, since the given δ_2 -subset does lie already in a singular subdisk of T_n , by hypothesis.

(3) Select an annulus U in S containing K on its interior such that: If W is any open set containing K , and X is an open set containing S , then there exists a homeomorphism H of E^3 onto itself such that $H(S) = S$, $H = \text{identity}$ on $(E^3 - X) + K$, $H(U) \subset W$, and H moves no point of E^3 more than the minimum of the two numbers $\delta_2/3$ and $\gamma/2$.

(4) Let δ be chosen so that $\delta < \delta_2/6$, $\delta < \gamma/2$, and $o_\delta(p) \cdot (S - U) = \phi$. We now begin modification of the given δ -disk B . By Condition (A), we may assume that $B \cdot K$ has finitely many components. We select all components X_α of $B \cdot S$ such that $B \cdot K \cdot X_\alpha \neq \phi$. There are only finitely many such components, which we number X_1, \dots, X_k . Select subdisks E_1, \dots, E_k of B with $X_i \subset \text{Int } E_i$ and $\partial E_i \cdot S = \phi$. We may assume that $\partial E_i \cdot \partial E_j = \phi$, for $i \neq j$.

(5) In this step, we show that ∂B bounds a $3\delta_1$ -disk B' such that $B' \cdot K = \phi$. We do this by modifying $\text{Int } E_i$, for $i = 1, \dots, k$. We describe this process for the disk E_1 only: Let m be an integer, $m > N_1$, $m > N_2$, so that ∂E_1 lies in the unbounded component of $E^3 - T_m$, and so that a ray R from K to ∞ hits T_m an odd number of times. Thus, an arc from K to ∂E_1 hits T_m an odd number of times.

Next we construct, by Step 3, a homeomorphism H so that $H/\partial E_1 = \text{identity}$, $H(E_1)$ is a δ_2 -disk, and $H(E_1) \cdot S \cdot T_m = \phi$. Thus, $H(E_1) \cdot T_m$ lies in a singular subdisk of T_m , by step 1; in fact, by step 2, this set lies in a $\delta_1/3$ -singular subdisk D .

Since an arc from $H(X_1)$ to ∂E_1 must intersect T_m in an odd number of points, and $\partial E_1 = \partial(H(E_1))$, it follows that

$$H(X_1) \subset \text{Int}_{H(E_1)} [D \cdot H(E_1)] .$$

This enables us to apply Theorem 1 to singular disk D , nonsingular disk $H(E_1)$, and a small curve J in $H(E_1)$ about $H(X_1)$. The resulting δ_1 -disk F_1 is disjoint from K . We use it in place of E_1 . We repeat this argument k times, thus constructing the $3\delta_1$ -disk B' . (Actually, this is a singular disk, but by requiring that $T_m \cdot \partial B = \phi$, we may deduce that B' has no singularities near the boundary, so a nonsingular disk is obtained by an application of Dehn's Lemma.)

(6) The disk B' is pulled off of $S - K$ by the usual disk replacement process to yield an ε -disk in $E^3 - S$. This completes the proof.

THEOREM 4. *If K is a 1-sphere in a free 2-sphere S in E^3 , then K is free in E^3 .*

Proof. Given $\varepsilon > 0$, select an annular neighborhood A of K in S , with a homeomorphism $h : K \times [0, 1] \rightarrow A$ such that the distance from p to $h(p \times q)$ is less than $\varepsilon/2$, for all (p, q) in $K \times [0, 1]$. We select maps $f_i : A \rightarrow E^3 - S$, for $i = 1, 2$. If S^1 is viewed as a square, then we begin the definition of

$$g : K \times S^1 \rightarrow E^3 - K$$

on the top and bottom of the square by the maps

$$f_i \circ h : K \times [0, 1] \rightarrow E^3 - S ,$$

for $i = 1, 2$. On the two remaining sides of S_1 , g is extended linearly. It is not difficult to show that if f_i were selected with care, then this linear extension does not intersect K , and that $g(p \times S^1)$ links K .

Actually, this proof could be set in a wider context, to show that if K is a flat k -sphere in S^m , and S^m is free in E^n , then K is free in E^n .

COROLLARY 4.1. *If K is a 1-sphere in a free 2-sphere S in E^3 , and the set of wild points of K form a tame 0-dimensional set, then K is tame.*

Proof. By Theorem 4, K is free. The result now follows from Theorem 3. Note that this corollary may be viewed as a strengthening of Corollary 2.1.

COROLLARY 4.2. (McMillan [10]). *A free 2-sphere in E^3 can be pierced at every point by a tame arc.*

Proof. Given a point $p \in S$, there exists a simple closed curve K with $p \in K \subset S$, such that $J - p$ is locally tame [5]. By Corollary 4.1, K is tame. Thus, S can be pierced by a tame arc at p [5].

5. **Wild Cantor sets.** If one is not interested in theorems dealing with Condition (A), but only in proving Corollary 4.1, then a more direct argument is available. In fact, the following theorem implies Corollary 4.1 with the word "tame" stricken from the hypothesis.

THEOREM 5. *If K is a free 1-sphere in E^3 which bounds a disk and the wild points of K form an 0-dimensional set, then K is tame.*

Proof. Given $\varepsilon > 0$, and $x \in K$, we outline an argument to show that there exists a 2-sphere S of diameter less than ε , such that $x \in \text{Int } S$, and $S \cdot K$ consists of two points. The result will then follow from [6]. We select a small subarc A of K with end points v and z , and interior points w and y at which A pierces the disks W and Y respectively, so that v, w, x, y, z lie on A in that order. A map $f: A \times S^1 \rightarrow E^3 - A$ is given via the hypothesis, with

$$w \in \text{Int}_w \{W \cdot f(A \times S^1)\}$$

and

$$y \in \text{Int}_y \{Y \cdot f(A \times S^1)\}.$$

We may enlarge f to a map from a disk into E^3 by extending rays from $f(z \times S^1)$ to z . Thus, f becomes a singular disk in E^3 , W a nonsingular disk, and letting J be a small curve in W around the point w , we may apply Theorem 1. This yields a disk D with $\partial D = \partial W$. One builds the 2-sphere S from the three nonsingular disks D , W and Y by standard techniques. See Theorem 2 of [4] for details.

COROLLARY 5.1. (Hempel [8]). *If S is a free 2-sphere in E^3 where wild points form a 0-dimensional set, then S is tame.*

Proof. Let K be a simple closed curve in S containing the wild

points of S . By Theorem 4, K is free. By Theorem 5, K is tame. Thus S is tame.

6. **Some problems.** We begin with two variations of Theorem 1, each of which would imply that free 2-spheres in E^3 are tame. This paper, in fact, began as an attempt to prove Conjecture 2.

Conjecture 1. Let M, D, E and R be as in Theorem 1. Then there exists a nonsingular disk E' such that $\partial E' = \partial E$ and

$$E' \subset (\text{Ext}_E | D | \cdot E) + R .$$

Conjecture 2. Let M, D, E, R and J be as in Theorem 1. Then there exists a nonsingular disk E' and subset X of $\text{Int } E'$, consisting of a finite number of disjoint disks, such that $\partial E' = \partial E$, $E' - X \subset \text{Ext}_E J$ and $X \subset R$.

The next conjecture would also imply that free 2-spheres in E^3 are tame: it would in fact, be a very useful tool in many problems in E^3 . It is, so to speak, a Dehn's Lemma with epsilons.

Conjecture 3. If D is a nonsingular disk in E^3 (wildly embedded), then for any ε , there exists a δ such that if f is a map of D into E^3 moving no point more than δ , with no singularities near $f(\partial D)$, then there exists a homeomorphism h of D into E^3 moving no point more than ε with $\partial h(D) = \partial f(D)$.

Hempel has shown that deformation free 2-spheres in E^3 are tame [7]. The definition of *deformation free* easily generalizes to k -spheres in E^3 as did the definition of *free*: an embedding $f: S^k \rightarrow E^n$ is *deformation free* if, considering f to be a map from $S^k \times q$ into E^n , where q is the center point of the $(n - k)$ -disk D^{n-k} , then f extends to a map $F: S^k \times D^{n-k} \rightarrow E^n$ with

$$F(S^k \times (D^{n-k} - q)) \subset E^n - f(S^k)$$

and for $p \in S^k$,

$$F(p \times \partial D^{n-k}) \text{ links } f(S^k) .$$

Then deformation free spheres are clearly free.

Conjecture 4. Deformation free k -spheres in E^n are tame, for all k and n .

A step toward establishing Conjecture 4 for 1-spheres in E^3 is the following.

Conjecture 5. If D is a singular disk in E^3 , and J is a simple closed curve piercing D at p , (i.e., $J \cdot |D| = p$ and J links ∂D , mod 2), then J pierces a nonsingular disk at p .

Conjecture 6. If $n - k \neq 2$, then a free k -sphere in E^n which satisfies Condition (A) is tame.

Perhaps the results of Bryant and Seebeck [3] offer some hope of proving Conjecture 6 for $n \geq 5$, $n - k \geq 3$. One would need to prove a generalized Theorem 1 for D a singular $(n - 1)$ -disk and E a nonsingular 2-disk in E^n .

Conjecture 7. An $(n - 2)$ -sphere in E^n which is free, satisfies Condition (A), and bounds an $(n - 1)$ -disk is tame.

Conjecture 8. If "free" were defined using linking with Z coefficients instead of Z_2 , then Theorem 5 would still be valid.

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