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AUTOMORPHISMS OF GROUPS OF SIMILITUDES OVER F_3

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Let $Q(x)$ be a quadratic form defined on a vector space M of dimension $n = 2m > 4$ $(n \neq 8)$ over the field with three elements. The purpose of this paper is to show that any automorphism of the projective group of similitudes or of the projective group of proper similitudes can not take the coset of an $(n-2, 2)$ involution into the coset of an $(n-p, p)$ involution or into the coset of a similitude T of ratio ρ where $T^2 = \rho_L$ (left multiplication by ρ) and where ρ is not a square in F_3. This result, together with some results of Wonenburger, shows that any such automorphism is induced by an automorphism of the group of similitudes.

In [6] Wonenburger proved the above statements except when the field K had 3 or 5 elements. Only the corollary to Lemma 3 of [6] needs to be verified to complete this result for all fields of characteristic not 2. The case in which $K = F_5$ can be proved by a slight alteration of Wonenburger's argument, so it will not be considered in this paper.

A linear transformation T of a vector space M of dimension $n = 2m$ is called a similitude of ratio ρ $(0 \neq \rho \in K)$, of the quadratic form Q, if for every $x \in M$, $Q(xT) = \rho Q(x)$. The similitudes of ratio 1 make up the orthogonal group $O(Q)$. A similitude T of ratio ρ is called a proper similitude if the determinant of the matrix of T, Det $T = \rho^m$, and improper if Det $T = -\rho^m$. Let $S(Q)$ $(S^+(Q))$ denote the group of similitudes (proper similitudes). In the projective groups $PS(Q) = S(Q)/K^*$ and $PS^+(Q) = S^+(Q)/K^*$ we will use the symbol $\bar{T}$ to denote the coset of the similitude T.

A projective involution is a similitude T such that $\bar{T}^2 = \bar{1}$, the coset of the identity. There are three types of projective involutions (see [6, p. 608]). First there are scalar multiples of orthogonal involutions. With respect to an orthogonal involution T, we can decompose M into a "minus space" M^- of dimension p such that $xT = -x$ for all $x \in M^-$, and a "plus space" M^+ of dimension $n-p$ such that $xT = x$ for all $x \in M^+$. Such a transformation will be called an $(n-p, p)$ involution and its image in $PS(Q)$ or in $PS^+(Q)$ will be called an $(n-p, p)$ coset. A P-involution is a similitude T of ratio ρ with $T^2 = -\rho_L$, left multiplication by $-\rho$. All P-involutions are proper similitudes (see [3, p. 64]). A P'-involution is a similitude T of ratio ρ such that $T^2 = \rho_L$, ρ not a square in K. If $K = F_3$, it can be easily shown that the determinant of the matrix of a P'-involution is 1. Therefore a P'-involution is proper if and only if the dimension of M is divisible by 4; i.e., if and only if $(-1)^m = 1$.

THEOREM. *Let M be a vector space of dimension $n = 2m > 4$ over the field $K = F_3$, and let Q be a nondegenerate quadratic form on M. No automorphism of $PS(Q)$ or of $PS^+(Q)$ can take an $(n-2, 2)$ coset into an $(n-p, p)$ coset, $p \neq 2$, $n-2$, or into the coset of a P'-involution.*

The proof is by comparing the orders of the centralizers of these projective involutions. First we need two lemmas. It shall be assumed that the hypotheses of the theorem hold.

LEMMA 1. *If $\bar{T}$ is an $(n-p, p)$ coset and $p = 2r$, then the highest power of the prime 3 dividing the order of the centralizer of $\bar{T}$ in $PS(Q)$ or in $PS^+(Q)$ is*

$$(p(p-2) + (n-p)(n-p-2))/4 .$$

Proof. The centralizer of $\bar{T}$ in $PS^+(Q)$, which we shall denote by $C_{PS+}(\bar{T})$, contains all cosets of elements of $0^+(Q^+) \times 0^+(Q^-)$, where $0^+(Q^+)$ $(0^+(Q^-))$ is the group of all proper orthogonal transformations of the plus (minus) space of T. Since F_3 has only two nonzero elements, the index $[S^+(Q^+): 0^+(Q^+)] = 2$. Hence the order

$$[PS^+(Q^+): 1] = [0^+(Q^+): 1] .$$

An element $\bar{U}$ will be contained in $C_{PS+}(\bar{T})$ if and only if U leaves invariant both the plus and minus spaces of T, except in the case in which $p = n-p$ and the discriminant of Q^+, the restriction of Q to the plus space of T, equals the discriminant of Q^-. In this latter case there will exist similitudes which will interchange the plus and minus spaces of T. The number of these elements will equal the number of similitudes which leave invariant both the plus and minus spaces. Therefore we have $[C_{PS}(\bar{T}): 1] = 2[C_{PS+}(\bar{T}): 1] = k[0^+(Q^+): 1]\cdot [0^+(Q^-): 1]$ where k is a power of 2. But by [1, p. 147] we have that

$$[0^+(Q^-): 1] = 3^{p(p-2)/4}(3^{p/2} - \varepsilon) \prod_{i=1}^{(p-2)/2} (3^{2i} - 1)$$

where ε is always either $+1$ or -1. Replacing p by $n-p$ gives us an expression for the order of $0^+(Q^+)$. By factoring out the powers of 3 we get the desired result.

LEMMA 2. *Let the dimension of $M = n = 4r > 4$. The highest* *power of the prime* 3 *dividing the order of the centralizer of the coset of a P'-involution T in $PS^+(Q)$ is $n(n-4)/8$.*

Proof. Let $F = K[i] = K[x]/(x^2+1)$ where $i^2 = -1$. Then F is the field with $3^2 = 9$ elements. Let $M_F = M \otimes_K F$. This is a vector space of dimension n over F. The extension Q_F of Q to M_F is non-degenerate since Q is (see [5]). The P'-involution T can be extended to a transformation T_F on M_F, and T_F is a scalar multiple of an (m, m) involution T_F^*. In [5] it was shown that the centralizer of T in $0^+(Q)$ is isomorphic to the orthogonal group $0(Q_F^+)$ of the plus space of T_F^*. Hence by an argument similar to the one in the last lemma we get that $[C_{PS^+}(\bar{T}): 1] = k[0^+(Q_F^+): 1]$ where k is a power of 2. Also

$$[0^+(Q_F^+): 1] = 9^{m(m-2)/4}(9^{m/2} - \varepsilon) \prod_{i=1}^{(m-2)/2} (9^{2i} - 1) ,$$

where $m = n/2$ and ε is either 1 or -1. The power of 3 dividing this expression is $n(n-4)/8$.

Proof of theorem. Since the commutator subgroup $(PS(Q))' = PO^+(Q)$ (see [3, p. 58]), and since

$$PS^+(Q) \cap PO(Q) = PO^+(Q) ,$$

no automorphism of $PS(Q)$ or of $PS^+(Q)$ can take an $(n-2, 2)$ coset into an $(n-p, p)$ coset for p an odd integer.

Suppose φ is an automorphism of $PS^+(Q)$. Then $C_{PS^+}(\varphi(\bar{T})) = \varphi(C_{PS^+}(\bar{T}))$ and the order of the centralizer of $\bar{T}$ is the same as the order of the centralizer of $\varphi(\bar{T})$. In particular if $\bar{T}$ is an $(n-2, 2)$ coset, the power of 3 dividing the order of $C_{PS^+}(\bar{T})$ must be the same as that dividing $C_{PS^+}(\varphi(\bar{T}))$. Suppose $\varphi(\bar{T})$ is an $(n-p, p)$ coset. Then $(n-2)(n-4)/4 = p(p-2)/4 + (n-p)(n-p-2)/4$ and $p = 2$ or $p = n-2$. The same holds if φ is an automorphism of $PS(Q)$.

Suppose $\varphi(\bar{T})$ is the coset of a P'-involution. Then the dimension of M must be divisible by 4, since otherwise $\varphi(\bar{T})$ would be an improper similitude. By Lemma 2 we have $(n-2)(n-4)/4 = n(n-4)/8$ and $n = 4$ which is a contradiction.

An automorphism of $PS(Q)$ can not take an $(n-2, 2)$ coset into the coset of a P'-involution since it must leave $(PS(Q))' = PO^+(Q)$ invariant.

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