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ON ASYMPTOTIC DENSITY IN n-DIMENSIONS

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The notion of asymptotic density for sets of nonnegative integers is generalized to sets of n-dimensional "nonnegative" lattice points. The additive properties of sets relative to this density are discussed. Some of the results are extended to the infinite dimensional case. Finally, natural density is defined and discussed.

To the author's knowledge, the only attemps at generalizing asymptotic density are to be found in Christopher [3] and, more significantly, in Buck [2]. In § 5, it is proved that our density always differs on certain sets from those given in these articles. Moreover, neither of the above mentioned papers discusses additive properties.

In § 2 the asymptotic density for subsets of the set of n-tuples of nonnegative integers is defined and various equivalent forms are considered. In § 3 some density results involving the sum of sets are obtained. In § 4 some of these results are extended to the infinite dimensional case. Finally, in § 5, upper asymptotic density and natural density are defined and a measure theoretic property of the latter is proved.

2. Definitions, etc. Let n be a positive integer and S the set of all n-tuples of nonnegative integers. The element $(0, \cdots, 0)$ will be denoted by $\mathbf{0}$ and generally the element $(x_1, \cdots, x_n)$ by $\boldsymbol{x}$. For $\boldsymbol{x} \in S$, let

$$L(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \in S, y_i \leqq x_i \ (i = 1, \cdots, n)\}$$

and

$$U(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{x} \in L(\boldsymbol{y})\} .$$

For a set $X \subset S$ and element $x \in S$ denote by $X \backslash \boldsymbol{x}$ the set of all $\boldsymbol{y} \in X$, $\boldsymbol{y} \neq \boldsymbol{x}$. In general the set theoretic difference will be denoted by$\backslash$(rather than —).

Define $\mathscr{K}$ to be the set

$$\{F \mid F \cap (S \backslash \mathbf{0}) \text{ is nonempty and finite; } \boldsymbol{x} \in F \Rightarrow L(\boldsymbol{x}) \subset F\} .$$

For $F \in \mathscr{K}$, let

$$F^* = \{\boldsymbol{x} \mid \boldsymbol{x} \in F; \boldsymbol{x} \in L(\boldsymbol{y}) \backslash \boldsymbol{y} \Rightarrow \boldsymbol{y} \notin F\} .$$

F^* is just the set of maximal points of F with respect to the partial ordering $\prec$ determined by the equivalence

$$x \prec y \Leftrightarrow x \in L(y) .$$

It is then clear that, for each $F \in \mathscr{K}$, $F = \cup \{L(x) \mid x \in F^*\}$.

For sets $A \subset S$ the "counting function" of A is defined as follows: for each $X \subset S$, $A(X)$ is the cardinality of the set $(A \cap X)\backslash \mathbf{0}$. Use of the counting function will be made only when X is finite. Note especially that $\mathbf{0}$ is never counted.

Given $A \subset S$, the K-density of A is defined to be

$$d(A) = \text{glb}\left\{\frac{A(F)}{S(F)} \middle| F \in \mathscr{K}\right\} .$$

In the case $n = 1$ this definition reduces to the ordinary Schnirelmann density of the set A. This generalization has been considered by Kvarda [6] and the author [4]. Further generalizations of Schnirelmann density have been considered by the author in [5].

A property of K-density [4, Lemma 1] can be noted here: if $d(A) < 1$(i.e., if $A\backslash\mathbf{0} \neq S\backslash\mathbf{0}$), then

$$(1) \quad d(A) = \text{glb}\left\{\frac{A(F)}{S(F)} \middle| F \in \mathscr{K}, F^* \subset S\backslash A\right\} .$$

For a nonnegative integer N, let

$$J(N) = \{x \mid x \in S, \min\{x_1, \cdots, x_n\} \leqq N\} .$$

It can be noted that $J(N) = S\backslash U((N+1, \cdots, N+1))$.

The asymptotic density of a set $A \subset S$ is defined to be

$$\delta(A) = \lim_{N\to\infty} d(A \cup J(N)) .$$

With little difficulty it can be proved directly that, in the case $n = 1$, $\delta(A)$ is the usual asymptotic density of A, i.e., $\delta(A) = \underline{\lim}_{n\to\infty} A(n)/n$. However, a slightly different proof will be given at the end of this section. For any dimension n, the asymptotic density of a set A exists since $d(A \cup J(N))$ forms a nondecreasing sequence bounded above by 1.

We proceed to investigate some equivalent forms for $\delta(A)$ as well as some other proporties. First, a number of structure lemmas are needed concerning lower bounds on the quotients of the number of elements in certain subsets of S. These seem to be interesting in themselves.

For integers M, N, n with $M > N \geqq 0$, $n > 0$, let

$$g(M, N, n) = \min\left\{\frac{(M+2)^m - 1}{(M+2)^m - (M-N+1)^m} \middle| m = 1, \cdots, n\right\} .$$

We note that, for fixed N, n, $g(M, N, n) \to \infty$ as $M \to \infty$. Also, for fixed M, N, if $n_1 > n_2$ then $g(M, N, n_1) \leqq g(M, N, n_2)$.

LEMMA 2.1. *Let $0 \leqq N < M$ and $\boldsymbol{x} \in S$ such that*

$$M + 1 \leqq x_i (i = 1, \cdots, n) .$$

If

$$f(M, N, n, \boldsymbol{x}) = \frac{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M)}{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - N)} ,$$

then $f(M, N, n, \boldsymbol{x}) \geqq g(M, N, n)$.

Proof. For $n = 1$, $f(M, N, 1, \boldsymbol{x}) = (M + 1)/(N + 1) = g(M, N, 1)$. Also, for any n, if $\boldsymbol{x} = (M + 1, \cdots, M + 1)$, then

$$f(M, N, n, \boldsymbol{x}) = \frac{(M + 2)^n - 1}{(M + 2)^n - (M - N + 1)^n} \geqq g(M, N, n) .$$

We perform a multiple induction. Let $k > 1$ and assume the lemma true for all $M, N, n, \boldsymbol{x}$ with $n < k$. Let $\boldsymbol{x} = (x_1, \cdots, x_k)$ be such that $x_i > M (i = 1, \cdots, k)$ and, for some j, $x_j > M + 1$, and assume for each $\boldsymbol{y} = (y_1, \cdots, y_k)$ with $M < y_i \leqq x_i (i = 1, \cdots, k)$ and, for some j, $y_j < x_j$, that $f(M, N, k, \boldsymbol{y}) \geqq g(M, N, k)$. Without loss of generality we may assume that $x_1 > M + 1$. Now

$$\begin{aligned} &f(M, N, k, \boldsymbol{x}) \\ &= \frac{[(x_1 - 1 + 1) \prod_{i=2}^{k} (x_i + 1) - (x_1 - 1 - M) \prod_{i=2}^{k} (x_i - M)] + [\prod_{i=2}^{k} (x_i + 1) - \prod_{i=2}^{k} (x_i - M)]}{[(x_1 - 1 + 1) \prod_{i=2}^{k} (x_i + 1) - (x_1 - 1 - N) \prod_{i=2}^{k} (x_i - N)] + [\prod_{i=2}^{k} (x_i + 1) - \prod_{i=2}^{k} (x_i - N)]} \\ &\geqq \min \{f(M, N, k, (x_1 - 1, x_2, \cdots, x_k)), f(M, N, k - 1, (x_2, \cdots, x_k))\} \\ &\geqq \min \{g(M, N, k), g(M, N, k - 1)\} = g(M, N, k) . \end{aligned}$$

This completes the proof.

Note the following simple formulae:

(2) For each $\boldsymbol{x} \in S$, $S(L(\boldsymbol{x})) = \prod_{i=1}^{n} (x_i + 1) - 1$.

(3) For $\boldsymbol{x}, \boldsymbol{y} \in S$, $\boldsymbol{y} \in L(\boldsymbol{x})$,

$$S(L(\boldsymbol{x}) \cap U(\boldsymbol{y})) = \prod_{i=1}^{n} (x_i - y_i + 1) - \eta(\boldsymbol{y})$$

where $\eta(\boldsymbol{y}) = 0$ if $\boldsymbol{y} \neq \boldsymbol{0}$ and $\eta(\boldsymbol{y}) = 1$ if $\boldsymbol{y} = \boldsymbol{0}$.

(4) For $M \geqq 0$, $\boldsymbol{x} \notin J(M)$,

$$S(L(\boldsymbol{x}) \cap J(M)) = \prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M) - 1 .$$

This last follows since

$$L(\boldsymbol{x}) \cap J(M) = L(\boldsymbol{x}) \backslash [L(\boldsymbol{x}) \cap U((M+1, \cdots, M+1))] .$$

We can now prove

LEMMA 2.2. *If* $0 < N < M$ *and* $\boldsymbol{x} \notin J(M)$, *then*

$$\frac{S(L(\boldsymbol{x}))}{S(L(\boldsymbol{x}) \cap J(N))} > \frac{S(L(\boldsymbol{x}) \cap J(M))}{S(L(\boldsymbol{x}) \cap J(N))} > g(M, N, n) .$$

Proof. The first inequality is obvious. The middle fraction is

$$\frac{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M) - 1}{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - N) - 1} > f(M, N, n, x) \geqq g(M, N, n) .$$

LEMMA 2.3. *Assume* $0 < N < M, x \notin J(M)$, $\varnothing \neq P \subset \{1, \cdots, n\}$ *and let* $\boldsymbol{y}$ *be defined by*

$$y_i = \begin{cases} 0 \text{ if } i \in P \\ x_i \text{ if } i \notin P . \end{cases}$$

Then

$$\frac{S(L(\boldsymbol{x}) \cap U(\boldsymbol{y}))}{S(L(\boldsymbol{x}) \cap U(\boldsymbol{y}) \cap J(N))} \geqq \frac{1}{2} \cdot g(M, N, n) .$$

Proof. Let $P = \{i_1, \cdots, i_m\}$ where $0 < m \leqq n$. Let S' be the set of m-tuples of nonnegative integers. Define $h: S \to S'$ so that $h(\boldsymbol{w}) = (w_{i_1}, w_{i_2}, \cdots, w_{i_m})$. For $z \in S'$ and $K \geqq 0$, define as before $L'(z)$ and $J'(K)$. From (2) and (3) above we have

$$\begin{aligned} S(L(\boldsymbol{x}) \cap U(\boldsymbol{y})) &= \prod_{i=1}^{n} (x_i - y_i + 1) - \eta(\boldsymbol{y}) \\ &= \prod_{j=1}^{m} (x_{i_j} + 1) - \eta(\boldsymbol{y}) \\ &= S'(L'(h(\boldsymbol{x}))) + 1 - \eta(\boldsymbol{y}) . \end{aligned}$$

Note that $h(\boldsymbol{x}) \notin J'(M)$. Also, the function h establishes a one-one correspondence between the sets $L(\boldsymbol{x}) \cap U(\boldsymbol{y}) \cap J(N)$ and $L'(h(\boldsymbol{x})) \cap J'(N)$. Nothing that $\mathbf{0}$ is in the first set if and only if $\boldsymbol{y} = \mathbf{0}$ we get

$$S(L(\boldsymbol{x}) \cap U(\boldsymbol{y}) \cap J(N)) = S'(L'(h(\boldsymbol{x})) \cap J'(N)) + 1 - n(\boldsymbol{y}).$$

Thus, using 2.2 and the fact that $S'(L'(h(\boldsymbol{x})) \cap J(N)) > 0$, we get

$$\frac{S(L(\boldsymbol{x}) \cap U(\boldsymbol{y}))}{S(L(\boldsymbol{x}) \cap U(\boldsymbol{y}) \cap J(N))} = \frac{S'(L'(h(\boldsymbol{x}))) + 1 - \eta(\boldsymbol{y})}{S'(L'(h(\boldsymbol{x})) \cap J'(N)) + 1 - \eta(\boldsymbol{y})}$$
$$\geqq \frac{S'(L'(h(\boldsymbol{x})))}{2 \cdot S'(L'(h(\boldsymbol{x}) \cap J'(N))}$$
$$\geqq \frac{1}{2} g(M, N, m) \geqq \frac{1}{2} g(M, N, n)$$

and the lemma is proved.

For an integer $M \geqq 0$ define

$$\mathscr{K}(M) = \{F \mid F \in \mathscr{K}, F^* \subset S \backslash J(M)\} .$$

LEMMA 2.4. *If* $F \in \mathscr{K}(M)$ *and* $M - 1 > N > 0$, then

$$\frac{S(F)}{S(F \cap J(N))} > \frac{S(F \cap J(M))}{S(F \cap J(N))} \geqq \frac{1}{2} g(M-1, N, n) .$$

Proof. The first inequality is obvious so we prove the second. Let $R = \{\boldsymbol{x} \mid \boldsymbol{x} \in F, x_i \geqq M (i = 1, \cdots, n) \text{ and } x_j = M \text{ for some } j\}$. First it is shown that $F \cap J(M) = \cup \{L(\boldsymbol{x}) \mid \boldsymbol{x} \in R\}$. If $\boldsymbol{y} \in F \cap J(M)$, then $\boldsymbol{y} \in L(z)$ for some $z \in F^*$ and for some i_0, $y_{i_0} \leqq M$. Let $\boldsymbol{x}$ be defined by

$$x_i = \begin{cases} z_i & \text{if } i \neq i_0 , \\ M & \text{if } i = i_0 . \end{cases}$$

Then $\boldsymbol{y} \in L(\boldsymbol{x})$ and $\boldsymbol{x} \in R$ so that $F \cap J(M) \subset \cup \{L(\boldsymbol{x}) \mid \boldsymbol{x} \in R\}$. If $\boldsymbol{y} \in \cup \{L(\boldsymbol{x}) \mid \boldsymbol{x} \in R\}$, then $\boldsymbol{y} \in L(\boldsymbol{x}) \subset F$ for some $\boldsymbol{x} \in R \subset F$, and since $x_j = M$ for some j, $y_j \leqq M$ so that $\boldsymbol{y} \in J(M)$.

For $x \in R$ let $\boldsymbol{x}'$ be defined by

$$x'_i = \begin{cases} x_i & \text{if } x_i > M \\ 0 & \text{if } x_i = M . \end{cases}$$

Let $G(\boldsymbol{x}) = \mathrm{L}(\boldsymbol{x}) \cap U(\boldsymbol{x}')$. If $\boldsymbol{x} \neq \boldsymbol{y}$, then $G(\boldsymbol{x})$ and $G(\boldsymbol{y})$ are disjoint, for, if $z \in G(\boldsymbol{x}) \cap G(\boldsymbol{y})$, then, for $x_i > M$, we have $x_i = z_i \leqq y_i$ and, otherwise, $x_i = M \leqq y_i$ so that $\boldsymbol{x} \prec \boldsymbol{y}$. Reversing the argument gives $\boldsymbol{x} = \boldsymbol{y}$.

Next it is shown that $\cup \{G(\boldsymbol{x}) \mid \boldsymbol{x} \in R\} = F \cap J(M)$. Clearly

$$\cup \{G(\boldsymbol{x}) \mid \boldsymbol{x} \in R\} \subset \cup \{L(\boldsymbol{x}) \mid \boldsymbol{x} \in R\} = F \cap J(M) .$$

If $\boldsymbol{y} \in F \cap J(M)$, then the element $\boldsymbol{s}$ defined by

$$s_i = \begin{cases} y_i \text{ if } y_i > M \\ M \text{ if } y_i \leqq M \end{cases}$$

is in R and $\boldsymbol{y} \in G(\boldsymbol{s})$.

Finally, noting that each $\boldsymbol{x} \in R$ is not in $J(M-1)$, we have

$$\begin{aligned} \frac{S(F \cap J(M))}{S(F \cap J(N))} &= \frac{\sum_{\boldsymbol{x} \in R} S(G(\boldsymbol{x}))}{\sum_{\boldsymbol{x} \in R} S(G(\boldsymbol{x}) \cap J(N))} \\ &\geqq \min_{\boldsymbol{x} \in R} \frac{S(G(\boldsymbol{x}))}{S(G(\boldsymbol{x}) \cap J(N))} \\ &\geqq \frac{1}{2} \cdot g(M-1, N, n) \,. \end{aligned}$$

The last step follows from 2.3. This completes the proof.

Define $\mathscr{S}$ to be the class of all sequences (F_i) in $\mathscr{K}$ which satisfy the property that for each integer $N > 0$

$$\lim_{i \to \infty} \frac{S(F_i)}{S(F_i \cap J(N))} = \infty \,.$$

LEMMA 2.5. *If* (F_i) *is a sequence such that* $F_i \in \mathscr{K}(i)$, *then* $(F_i) \in \mathscr{S}$.

Proof. Lemma (2.4) says that for i sufficiently large

$$\frac{S(F_i)}{S(F_i \cap J(N))} \geqq \frac{1}{2} \cdot g(i-1, N, n) \longrightarrow \infty (i \longrightarrow \infty) \,.$$

THEOREM 2.6. *If* $(F_i) \in \mathscr{S}$ *and* $A \subset S$, *then*

$$\delta(A) \leqq \underline{\lim}_{i \to \infty} \frac{A(F_i)}{S(F_i)} \,.$$

Proof. Let $N > 0$. Then

$$d(A \cup J(N)) \leqq \frac{[A \cup J(N)](F_i)}{S(F_i)} \leqq \frac{A(F_i) + S(F_i \cap J(N))}{S(F_i)} \,.$$

Hence

$$\begin{aligned} d(A \cup J(N)) &\leqq \underline{\lim}_{i \to \infty} \left[\frac{A(F_i)}{S(F_i)} + \frac{S(F_i \cap J(N))}{S(F_i)} \right] \\ &\leqq \underline{\lim}_{i \to \infty} \frac{A(F_i)}{S(F_i)} + \overline{\lim}_{i \to \infty} \frac{S(F_i \cap J(N))}{S(F_i)} \end{aligned}$$

$$= \lim_{i\to\infty} \frac{A(F_i)}{S(F_i)} .$$

Letting $N \to \infty$ we have the result.

The following theorem shows that $\delta(A)$ can always be obtained as a limit of quotients $A(F_i)/S(F_i)$ where (F_i) is a sequence in $\mathscr{S}$. Actually, anticipating a subsequent application, a little more is proved.

THEOREM 2.7. *For each $A \subset S$ there exists $(F_i) \in \mathscr{S}$ such that*

$$\delta(A) = \lim_{i\to\infty} \frac{A(F_i)}{S(F_i)} .$$

Moreover, if $\delta(A) < 1$, we may choose the F_i so that $F_i^* \subset S \backslash (A \cup J(i))$.

Proof. If $\delta(A) = 1$, then for any sequence $(F_i) \in \mathscr{S}$

$$1 = \delta(A) \leqq \varliminf_{i\to\infty} \frac{A(F_i)}{S(F_i)} \leqq \varlimsup_{i\to\infty} \frac{A(F_i)}{S(F_i)} \leqq 1$$

and the theorem is proved in this case.

Suppose that $\delta(A) < 1$. For $i \geqq 1$, let $M(i)$ be such that

$$g(M(i) - 1, i, n) \geqq 2^{i+1}$$

and choose $F_i \in \mathscr{K}$ so that $F_i^* \subset S \backslash (A \cup J(M(i)))$ and

$$\frac{[A \cup J(M(i))](F_i)}{S(F_i)} < d(A \cup J(M(i))) + \frac{1}{2^i} .$$

The existence of F_i follows from (1) above. From the inclusions

$$F_i^* \subset S \backslash (A \cup J(M(i))) \subset S \backslash (A \cup J(i)) \subset S \backslash J(i)$$

it follows that $F_i \in \mathscr{K}(i)$ so that, by 2.5, $(F_i) \in \mathscr{S}$. Since also $F_i^* \subset S \backslash (A \cup J(i))$ it remains only to show that $\delta(A)$ is the limit of the quotients $A(F_i)/S(F_i)$.

From the inequalities

$$0 < \frac{[A \cup J(i)](F_i)}{S(F_i)} - \frac{A(F_i)}{S(F_i)} \leqq \frac{(J(i))(F_i)}{S(F_i)} = \frac{S(F_i \cap J(i))}{S(F_i)}$$
$$\leqq \frac{2}{g(M(i) - 1, i, n)} \leqq \frac{1}{2^i}$$

it follows that

$$\lim_{i\to\infty} \left[\frac{[A \cup J(i)](F_i)}{S(F_i)} - \frac{A(F_i)}{S(F_i)} \right] = 0 .$$

But also

$$\lim_{i\to\infty} \frac{[A \cup J(i)](F_i)}{S(F_i)} = \delta(A) ,$$

for

$$d(A \cup J(i)) \leqq \frac{[A \cup J(i)](F_i)}{S(F_i)} \leqq \frac{[A \cup J(M(i))](F_i)}{S(F_i)}$$
$$\leqq d(A \cup J(M(i))) + \frac{1}{2^i}$$

where both ends approach $\delta(A)$ as $i \to \infty$. This proves the theorem.

THEOREM 2.8. *For each* $A \subset S$,

$$\delta(A) = \operatorname*{glb}_{(F_i) \in \mathscr{S}} \lim_{i\to\infty} \frac{A(F_i)}{S(F_i)} .$$

Proof. Theorems 2.6 and 2.7.

For $N \geqq 0$ and $A \subset S$ define

$$d^N(A) = \operatorname{glb}\left\{\frac{A(F) + S(F \cap J(N))}{S(F)} \,\middle|\, F \in \mathscr{K}\right\} .$$

THEOREM 2.9. $\delta(A) = \lim_{N\to\infty} d^N(A)$.

Proof. Since, for each $F \in \mathscr{K}$,

$$\frac{[A \cup J(N)](F)}{S(F)} \leqq \frac{A(F) + S(F \cap J(N))}{S(F)} ,$$

it follows that $d(A \cup J(N)) \leqq d^N(A)$. Let $(F_i) \in \mathscr{S}$ such that

$$A(F_i)/S(F_i) \to \delta(A)$$

as $i \to \infty$. Then

$$d^N(A) \leqq \frac{A(F_i) + S(F_i \cap J(N))}{S(F_i)} \to \delta(A)$$

as $i \to \infty$. Hence, for each N, $d(A \cup J(N)) \leqq d^N(A) \leqq \delta(A)$ and the theorem follows.

THEOREM 2.10. $\delta(A) = \lim_{N\to\infty} d^N(A \cup J(N))$.

Proof. As in the proof of Theorem 2.9,

$$d(A \cup J(N)) = d(A \cup J(N) \cup J(N)) \leqq d^N(A \cup J(N)) \leqq \delta(A \cup J(N)) = \delta(A) .$$

The last equality follows easily from the definition of δ.

In the next theorem the asymptotic density of certain sets is calculated. They are applied in the proof of Theorem 5.1 below.

THEOREM 2.11. (i) *If* $n \geqq 2$ *and* $A \cap J(N)$ *is finite for each* $N \geqq 0$, *then* $\delta(A) = 0$.
(ii) *If* $S \backslash A \subset J(N)$ *for some* N, *then* $\delta(A) = 1$.

Proof. (i) For $N \geqq 0$, let $x_{1,N}, x_{2,N}, \cdots, x_{n-1,N}$ be chosen so large that $x_{i,N} > N (i = 1, \cdots, n-1)$ and

$$S(L((x_{1,N}, \cdots, x_{n-1,N}, N))) > N \cdot S(A \cap J(N)) .$$

Let $F_N = L((x_{1,N}, \cdots, x_{n-1,N}, N))$ so that $F_N \in \mathscr{K}$, $(F_N) \in \mathscr{S}$ (since $F_N \in \mathscr{K}(N-1)$) and $F_N \subset J(N)$. Hence

$$0 \leqq \delta(A) \leqq \varliminf_{N \to \infty} \frac{A(F_N)}{S(F_N)} \leqq \varliminf_{N \to \infty} \frac{[A \cap J(N)](F_N)}{N \cdot S(A \cap J(N))} \leqq \varliminf_{N \to \infty} \frac{1}{N} = 0 .$$

(ii) If $S \backslash A \subset J(N)$, then $A \cup J(M) = S$ for $M \geqq N$. Thus $d(A \cup J(M)) = d(S) = 1$ and the result follows.

To conclude this section we prove that δ generalizes the usual asymptotic density.

THEOREM 2.12. *In the case* $n = 1$, $\delta(A)$ *is the usual asymptotic density of* A.

Proof. It is assumed the reader is familiar with the usual notation for this case. By 2.6,

$$\delta(A) \leqq \varliminf_{i \to \infty} \frac{A(i)}{i}$$

and, by 2.7, there is a sequence of integers n_i such that $n_i \to \infty$ as $i \to \infty$ and

$$\delta(A) = \lim_{i \to \infty} \frac{A(n_i)}{n_i} \geq \varliminf_{i \to \infty} \frac{A(i)}{i} .$$

3. **Some addition theorems.** Let A and B be subsets of S and define $A + B$ to be the set $\{\boldsymbol{a} + \boldsymbol{b} \mid \boldsymbol{a} \in A, \boldsymbol{b} \in B\}$. If A is a singleton $\{\boldsymbol{x}\}$, then write $A + B$ as $\boldsymbol{x} + B$. Futhermore, if $A \subset U(\boldsymbol{y})$, then define $A - \boldsymbol{y}$ to be the set $\{\boldsymbol{x} \mid \boldsymbol{x} \in S, \boldsymbol{x} + \boldsymbol{y} \in A\}$. Addition of elements in S is done coordinatewise.

LEMMA 3.1. *If* $\mathbf{0} \in A \cap B$ *and* $A(L(\boldsymbol{x})) + B(L(\boldsymbol{x})) \geqq S(L(\boldsymbol{x}))$, *then* $\boldsymbol{x} \in A + B$.

Proof. This is done in the proof of Theorem 1 in [6].

THEOREM 3.2. *If* $\mathbf{0} \in A \cap B$ *and* $\delta(A) + \delta(B) > 1$, *then*

$$S \backslash (A + B) \subset J(N)$$

for some N. *This last condition implies that* $\delta(A + B) = 1$.

Proof. The last statement is just Theorem 2.11 (ii).

Let $e = \delta(A) + \delta(B) - 1$. From the definition of asymptotic density it follows that, for some integer N_0,

$$d(A \cup J(N_0)) + d(B \cup J(N_0)) > 1 + \frac{e}{2} .$$

Since $L(\boldsymbol{x}) \in \mathscr{K}$ for each $\boldsymbol{x} \in S \backslash \mathbf{0}$, it follows from the last expression that

$$[A \cup J(N_0)](L(\boldsymbol{x})) + [B \cup J(N_0)](L(\boldsymbol{x})) > S(L(\boldsymbol{x})) + \frac{e}{2} S(L(\boldsymbol{x})) .$$

Let M be so large that $g(M, N_0, n) > 4/e$. By Lemma 2.2, if $\boldsymbol{x} \notin J(M)$, then

$$\frac{S(L(\boldsymbol{x}))}{s(L(\boldsymbol{x}) \cap J(N_0))} > g(M, N_0, n) > \frac{4}{e} .$$

Hence, for $\boldsymbol{x} \notin J(M)$,

$$\begin{aligned} A(L(\boldsymbol{x})) + B(L(\boldsymbol{x})) &\geqq [A \cup J(N_0)]((L(\boldsymbol{x})) + [B \cup J(N_0)](L(\boldsymbol{x})) \\ &\qquad - 2S(L(\boldsymbol{x}) \cap J(N_0)) \\ &\geqq S(L(\boldsymbol{x})) + \frac{e}{2} S(L(\boldsymbol{x})) - 2 \cdot \frac{e}{4} S(L(\boldsymbol{x})) \\ &= S(L(\boldsymbol{x})) . \end{aligned}$$

Therefore, by Lemma 3.1, $\boldsymbol{x} \in A + B$ so that $S \backslash (A + B) \subset J(M)$.

The following theorem shows that the asymptotic density of a set is invariant under translation.

THEOREM 3.3. $\delta(\boldsymbol{x} + A) = \delta(A)$ *for each* $A \subset S$ *and* $\boldsymbol{x} \in S$.

Proof. If $\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x} + A = A$ and the theorem is trivial. Hence it is assumed that $\boldsymbol{x} \neq \mathbf{0}$. Furthermore, since $\boldsymbol{x} + A$ and $\boldsymbol{x} + (A \backslash \mathbf{0})$ differ in at most one point, it may be assumed that $\mathbf{0} \in A$.

It is first shown that $\delta(\boldsymbol{x} + A) \geqq \delta(A)$. Let $N = \max \{x_1, \cdots, x_n\}$. It is sufficient to prove that, for each $M \geqq N$,

$$d((\boldsymbol{x} + A) \cup J(M)) \geqq d(A \cup J(M - N)) .$$

Let $D_M = (\boldsymbol{x} + A) \cup J(M)$ and $E_M = A \cup J(M)$ and let $G \in \mathscr{K}$. If $\boldsymbol{x} \notin G \backslash G^*$, then $G \subset J(N)$. To see this let $z \in G \backslash J(N)$. Then $z_i > N \geqq x_i (i = 1, \cdots, n)$ and so $\boldsymbol{x} \in L(z) \subset G$. Since $\boldsymbol{x} \neq z$, $\boldsymbol{x} \in G \backslash G^*$, contradiction. Thus, in the case $\boldsymbol{x} \notin G \backslash G^*$, for $M \geqq N$,

$$\frac{D_M(G)}{S(G)} = 1 \geqq d(E_{M-N}) .$$

Hence, suppose that $\boldsymbol{x} \in G \backslash G^*$. It is easily shown that

$$G' = [G \cap U(\boldsymbol{x})] - \boldsymbol{x} \in \mathscr{K}.$$

Now, for $M \geqq N$,

$$\begin{aligned} D_M(G) &= D_M(G \backslash U(\boldsymbol{x})) + D_M(G \cap U(\boldsymbol{x})) \\ &= S(G \backslash U(\boldsymbol{x})) + ([D_M \cap U(\boldsymbol{x})] - \boldsymbol{x})(G') + 1 \\ &\geqq S(G \backslash U(\boldsymbol{x})) + E_{M-N}(G') + 1 . \end{aligned}$$

The second equality follows from the fact that, first, $G \backslash U(\boldsymbol{x}) \subset J(M) \subset D_M$, and, second, if $\boldsymbol{x} \in Z \subset U(\boldsymbol{x})$ and $\boldsymbol{x} \in H \subset U(\boldsymbol{x})$ with H finite, then $Z(H) = (Z - \boldsymbol{x})(H - \boldsymbol{x}) + 1$. Above we have $Z = D_M \cap U(\boldsymbol{x})$ and $H = G \cap U(\boldsymbol{x})$. The last inequality follows for the fact that

$$E_{M-N} \subset [D_M \cap U(\boldsymbol{x})] - \boldsymbol{x} .$$

To see this let $z \in E_{M-N}$ and suppose first that $z \in A$. Then

$$z + \boldsymbol{x} \in (\boldsymbol{x} + A) \cap U(\boldsymbol{x}) \subset D_M \cap U(\boldsymbol{x})$$

and so $z = (z + \boldsymbol{x}) - \boldsymbol{x} \in [D_M \cap U(\boldsymbol{x})] - \boldsymbol{x}$. If $z \in J(M - N)$, then $z + \boldsymbol{x} \in J(M) \cap U(\boldsymbol{x}) \subset D_M \cap U(\boldsymbol{x})$ so again $z \in [D_M \cap U(\boldsymbol{x})] - \boldsymbol{x}$. Here we have used the fact that if $\boldsymbol{w} \in J(i)$, $\boldsymbol{v} \in L((j, \cdots, j))$, then $\boldsymbol{w} + \boldsymbol{v} \in J(i + j)$.

In a similar manner, we obtain

$$S(G) = S(G \backslash U(\boldsymbol{x})) + S(G') + 1 .$$

Thus,

$$\begin{aligned} \frac{D_M(G)}{S(G)} &\geqq \frac{E_{M-N}(G') + S(G \backslash U(\boldsymbol{x})) + 1}{S(G') + S(G \backslash U(\boldsymbol{x})) + 1} \\ &\geqq \frac{E_{M-N}(G')}{S(G')} \geqq d(E_{M-N}) . \end{aligned}$$

Hence it follows that $d(D_M) \geqq d(E_{M-N})$.

It remains to show that $\delta(\boldsymbol{x} + A) \leqq \delta(A)$. Clearly, for each $G \in \mathscr{K}$, $(\boldsymbol{x} + A)(G) \leqq A(G) + 1$. Let $(G_i) \in \mathscr{S}$ such that $\lim_{i \to \infty} A(G_i)/S(G_i) = \delta(A)$. Then

$$\delta(x + A) \leqq \lim \frac{(x + A)(G_i)}{S(G_i)} \leqq \lim \frac{A(G_i) + 1}{S(G_i)} = \delta(A) .$$

Let $A \subsetneqq S$ and define the generalized Erdös density of A to be

$$d_1(A) = \text{glb}\left\{\frac{A(F)}{S(F) + 1}\middle| F \in \mathscr{K}, A(F) < S(F)\right\} .$$

This density has been studied by Kvarda in [7] where the following important result is proved.

THEOREM 3.4. (Kvarda). *Let A, $B \subset S$, $0 \in A \cap B$ and $F \in \mathscr{K}$ such that $(A + B)(F) < S(F)$ and for each $b \in B \cap F$ there exists $g \in F \backslash (A + B)$ with $b \in L(g)$. Then*

$$(A + B)(F) \geqq d_1(A) \cdot (S(F) + 1) + B(F) .$$

In particular, the hypotheses of 3.4 will be satisfied if F is taken so that $F^* \subset S \backslash (A + B)$. Using 3.4 the following "mixed" density result is proved.

THEOREM 3.5. *If $0 \in A \cap B$ and $A \neq S$, then* $\delta(A + B) \geqq \min \{1, d_1(A) + \delta(B)\}$.

Proof. If $\delta(A + B) = 1$, the theorem is obvious. Hence assume that $\delta(A + B) < 1$ and denote $A + B$ by C and $d_1(A)$ by α_1. For any $F \in \mathscr{K}$ with $F^* \subset S \backslash C$, by 3.4,

$$\frac{C(F)}{S(F)} \geqq \alpha_1 + \frac{B(F)}{S(F)} .$$

Thus, for any $N \geqq 0$,

$$\begin{aligned} \frac{C(F) + S(F \cap J(N))}{S(F)} &\geqq \alpha_1 + \frac{B(F) + S(F \cap J(N))}{S(F)} \\ &\geqq \alpha_1 + \frac{[B \cup J(N)](F)}{S(F)} \\ &\geqq \alpha_1 + d(B \cup J(N)) . \end{aligned}$$

By 2.7, there exists a sequence $(F_i) \in \mathscr{S}$ such that $F_i^* \subset S \backslash C$ and $\lim_{i\to\infty} C(F_i)/S(F_i) = \delta(C)$.
Then, for each $N \geqq 0$,

$$\begin{aligned} \delta(C) = \lim_{i\to\infty} \frac{C(F_i)}{S(F_i)} &= \lim_{i\to\infty} \frac{C(F_i) + S(F_i \cap J(N))}{S(F_i)} \\ &\geqq \alpha_1 + d(B \cup J(N)) . \end{aligned}$$

Letting $N \to \infty$ we obtain

$$\delta(C) \geqq \alpha_1 + \delta(B)$$

and the theorem is proved.

As an example of an application of Theorem 2.10 the following theorem is proved.

THEOREM 3.6. *If $A \cap B \subset J(N)$ for some N, then $\delta(A \cup B) \geqq \delta(A) + \delta(B)$. In particular, if $\mathbf{0} \in A \cap B \subset J(N)$, then $\delta(A + B) \geqq \delta(A) + \delta(B)$.*

Proof. The second statement follows easily from the first since $A + B \supset A \cup B$ if $\mathbf{0} \in A \cap B$.

Let $C = A \cup B$. If $M \geqq N$, then $A \cap B \subset J(M)$ and so for any $F \in \mathscr{K}$

$$\begin{aligned}[C \cup J(M)](F) &= A(F \backslash J(M)) + B(F \backslash J(M)) + S(F \cap J(M)) \\ &= [A \cup J(M)](F) + [B \cup J(M)](F) - S(F \cap J(M)) .\end{aligned}$$

Thus, for $M \geqq N$, $F \in \mathscr{K}$,

$$\begin{aligned}\frac{[C \cup J(M)](F) + S(F \cap J(M))}{S(F)} &= \frac{[A \cup J(M)](F)}{S(F)} + \frac{[B \cup J(M)](F)}{S(F)} \\ &\geqq d(A \cup J(M)) + d(B \cup J(M)) .\end{aligned}$$

Hence, by definition of d^M,

$$d^M(C \cup J(M) \geqq d(A \cup J(M)) + d(B \cup J(M)) .$$

Letting $M \to \infty$, we obtain by 2.10,

$$\delta(C) \geqq \delta(A) + \delta(B) .$$

The proofs of the following two results are left for the reader. The results are to be compared with Buck's measure theoretic development of asymptotic density in [1].

THEOREM 3.7. *Let $A \subset S$ and let k, j be positive integers. $1 \leqq j \leqq n$. Suppose that, for each $\boldsymbol{x} \in A$, $k \mid x_j$. Define the set*

$$B = \left\{\left(x_1, \cdots, \frac{x_j}{k}, \cdots, x_n\right) \middle| (x_1, \cdots, x_n) \in A\right\} .$$

Then $\delta(B) = k\delta(A)$.

THEOREM 3.8. *Let $\{a_i j + b_i \mid j = 0, 1, 2, \cdots\} = P(a_i, b_i)$ be n arithmetic progressions ($a_i > 0$, $b_i \geqq 0 (i = 1, \cdots, n)$) Let*

$$A = P(a_1, b_1) \times \cdots \times P(a_n, b_n) \subset S .$$

Then

$$\delta(A) = \prod_{i=1}^{n} \frac{1}{a_i} .$$

4. Extensions to the infinite dimensional case. For this section the following notation is adopted. The set of all n-tuples of nonnegative integers will be denoted by I_n. The set of all infinite sequences $\boldsymbol{x} = (x_1, x_2, \cdots)$ of nonnegative integers with the property that only finitely many terms are different from 0 will be denoted by Q. For $\boldsymbol{x} \in Q$, let $k(\boldsymbol{x})$ be the largest index k such that $x_k \neq 0$. For $A \subset Q$, n a positive integer, let

$$A_n = \{x_1, \cdots, x_n) \mid \boldsymbol{x} = (x_1, x_2, \cdots) \in A, k(\boldsymbol{x}) \leqq n\} .$$

The asymptotic density of a set $A \subset Q$ is defined to be

$$\delta(A) = \underline{\lim}_{n \to \infty} \delta(A_n) .$$

Although K-density can easily be extended to Q (see [5]), there does not seem to be any direct way of obtaining a good definition of asymptotic density from it as was done in the finite dimensional case. In particular, it is not clear how one should define the $J(N)$ (if, indeed, this approach is at all possible). The definition given here, however, seems worthy enough as the following results indicate.

THEOREM 4.1. (*Extension of* 3.2). *If* $A, B \subset Q$, $\mathbf{0} \in A \cap B$, *and* $\delta(A) + \delta(B) > 1$, *then there exists an integer* M *and a sequence of nonnegative integers* $N_M, N_{M+1}, \cdots$ *such that, if* $\boldsymbol{x} \in Q \backslash (A + B)$ *and* $k(\boldsymbol{x}) \geqq M$, *then* $x_i \leqq N_{k(\boldsymbol{x})}$ *for some* $i, 1 \leqq i \leqq k(\boldsymbol{x})$. *This condition implies that* $\delta(A + B) = 1$.

Proof. From $\delta(A) + \delta(B) > 1$, it follows that there exists an $M > 0$ such that, if $n \geqq M$, then $\delta(A_n) + \delta(B_n) > 1$. Thus, by 3.2, for $n \geqq M$, there exists an integer N_n such that $I_n \backslash (A_n + B_n) \subset J(N_n)$. Observe that, for each n, $A_n + B_n = (A + B)_n$. Hence, if $\boldsymbol{x} \in Q \backslash (A + B)$ and $k(\boldsymbol{x}) = k \geqq M$, then $(x_1, \cdots, x_k) \in I_k \backslash (A + B)_k = I_k \backslash (A_k + B_k) \subset J(N_k)$ so that there is an $i, 1 \leqq i \leqq k$, such that $x_i \leqq N_k$.

To prove the last statement observe that, for $n \geqq M$, $\delta(A_n + B_n) = 1$. Thus

$$\delta(A + B) = \underline{\lim}_{n \to \infty} \delta((A + B)_n) = \underline{\lim}_{n \to \infty} \delta(A_n + B_n) = 1 .$$

THEOREM 4.2. (*Extension of* 3.3). *If* $A \subset Q$ *and* $\boldsymbol{x} \in Q$, *then* $\delta(\boldsymbol{x} + A) = \delta(A)$.

Proof. Define $\boldsymbol{y}_n = (x_1, \cdots, x_n)$ where $\boldsymbol{x} = (x_1, x_2, \cdots)$ and observe that, for $n \geqq k(\boldsymbol{x})$, we have $(\boldsymbol{x} + A)_n = \boldsymbol{y}_n + A_n$. Hence, using 3.3, $\delta(\boldsymbol{x} + A) = \underline{\lim}_{n\to\infty} \delta((\boldsymbol{x} + A)_n) = \underline{\lim}_{n\to\infty} \delta(\boldsymbol{y}_n + A_n) = \underline{\lim}_{n\to\infty} \delta(A_n) = \delta(A)$.

We proceed to extend Theorem 3.5. For $\boldsymbol{x} \in Q$, let $L(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \in Q, y_i \leqq x_i (i = 1, 2, \cdots)\}$. Then let $\mathscr{K}$ be the class of all nonempty finite subsets F of Q with at least one nonzero element satisfying the condition: $\boldsymbol{x} \in F \Rightarrow L(\boldsymbol{x}) \subset F$.

For $A \subset Q$, define

$$d_1(A) = \operatorname{glb}\left\{\frac{A(F)}{Q(F) + 1} \middle| F \in \mathscr{K}, A(F) < Q(F)\right\} .$$

THEOREM 4.3. *For* $A \subsetneqq Q$,

$$d_1(A) = \lim_{n\to\infty} d_1(A_n) .$$

Proof. Denote by $\mathscr{K}_n$ the class $\mathscr{K}$ defined in I_n. Since $A \neq Q$. there exists N such that, for all $n \geqq N$, $A \neq I_n$. Let $F \in \mathscr{K}_n$ such that $A_n(F) < I_n(F)$. Define $F' \in \mathscr{K}$ by

$$F' = \{\boldsymbol{x} \mid \boldsymbol{x} \in Q, k(\boldsymbol{x}) \leqq n, (x_1, \cdots, x_n) \in F\} .$$

Then $A_n(F) = A(F')$ and $I_n(F) = Q(F')$ so that

$$\frac{A_n(F)}{I_n(F) + 1} = \frac{A(F')}{Q(F') + 1} \geqq d_1(A) .$$

Thus $d_1(A_n) \geqq d_1(A)$ for all $n \geqq N$.

Next, note that $d_1(A_n)$ forms a nonincreasing sequence. To see this, let $F \in \mathscr{K}_n$ such that $A_n(F) < I_n(F)$. Then

$$F_0 = \{(x_1, \cdots, x_n, 0) \mid (x_1, \cdots, x_n) \in F\} \in \mathscr{K}_{n+1}$$

and

$$\frac{A_n(F)}{I_n(F) + 1} = \frac{A_{n+1}(F_0)}{I_{n+1}(F_0) + 1} \geqq d_1(A_{n+1})$$

so that $d_1(A_n) \geqq d_1(A_{n+1})$. Hence $\lim_{n\to\infty} d_1(A_n)$ exists and is $\geqq d_1(A)$.

Now let $F \in \mathscr{K}$ such that $A(F) < Q(F)$. Let $n = \max\{k(\boldsymbol{x}) \mid \boldsymbol{x} \in F\}$ and set $F' = \{(x_1, \cdots, x_n) \mid (x_1, x_2, \cdots) \in F\}$. It follows that $F' \in \mathscr{K}_n$, $A_n(F') = A(F)$ and $I_n(F') = Q(F)$. Thus

$$\frac{A(F)}{Q(F) + 1} = \frac{A_n(F')}{I_n(F') + 1} \geqq d_1(A_n) \geqq \lim_{n\to\infty} d_1(A_n) .$$

Hence $d_1(A) \geqq \lim_{n\to\infty} d_1(A_n)$ and the theorem is proved.

THEOREM 4.4. (*Extension of* 3.5). *If* $A, B \subset Q$, $\mathbf{0} \in A \cap B$ *and* $A \neq Q$, *then*

$$\delta(A + B) \geqq \min\{1, d_1(A) + \delta(B)\} .$$

Proof. By 3.5, for all sufficiently large n,

$$\delta((A + B)_n) = \delta(A_n + B_n) \geqq \min\{1, d_1(A_n) + \delta(B)\} .$$

Thus

$$\begin{aligned}\delta(A + B) &= \varliminf_{n\to\infty} \delta((A + B)_n) \\ &\geqq \varliminf_{n\to\infty} \min\{1, d_1(A_n) + \delta(B_n)\} \\ &= \min\{1, \varliminf_{n\to\infty} (d_1(A_n) + \delta(B_n))\} \\ &\geqq \min\{1, \varliminf_{n\to\infty} d_1(A_n) + \varliminf_{n\to\infty} \delta(B_n)\} \\ &= \min\{1, d_1(A) + \delta(B)\} .\end{aligned}$$

The proof of the following extension of 3.6 is omitted. Note that for $A, B \subset Q$, $(A \cap B)_n = A_n \cap B_n$.

THEOREM 4.5. *If* $A, B \subset Q$ *and for all sufficiently large* n *there exists an integer* N_n *such that* $(A \cap B)_n \subset J(N_n)$, *then*

$$\delta(A \cup B) \geqq \delta(A) + \delta(B) .$$

5. **Natural density.** In this section we return to consideration only of the (finite) n-dimensional case, and to the notation of § 1 – § 3.

For a set $A \subset S$, define the upper K-density of A to be

$$\bar{d}(A) = \text{lub}\left\{\frac{A(F)}{S(F)} \,\middle|\, F \in \mathscr{K}\right\} .$$

and the upper asymptotic density of A to be

$$\bar{\delta}(A) = \lim_{N\to\infty} \bar{d}(A \backslash J(N)) .$$

Since $\bar{d}(A\backslash J(N))$ forms a nonincreasing sequence it follows that $\bar{\delta}(A)$ always exists.

If $\delta(A) = \bar{\delta}(A)$, then we say that the natural density of A exists and write $\nu(A) = \delta(A) = \bar{\delta}(A)$.

R. C. Buck [2] has defined asymptotic density, upper asymptotic density and natural density for subsets of a measure space X. Briefly, the procedure is this: Take a countable increasing sequence $K(i)$ of subsets of X which covers X and a sequence μ_i of measures defined on the same class of sets which includes the sets $K(i)$. The following properties are assumed: (i) $\mu_i(X) = 1$ for all i; (ii) $\mu_i\ (K(j)) \to 0$ as $i \to \infty$ (fixed j); (iii) for each i there exists $\alpha(i)$ such that, if $A \cap K(\alpha(i)) = \phi$, then $\mu_i(A) = 0$. Then define the asymptotic density of A to be $\underline{D}(A) = \underline{\lim}_{i\to\infty} \mu_i(A)$, and the upper asymptotic density to be $\bar{D}(A) = \overline{\lim}_{i\to\infty} \mu_i(A)$, and the natural density $D(A)$ as usual.

It seems surprising that for $n \geqq 2$, $X = S$, it always happens that δ is different from $\underline{D}$ no matter how the measures μ_i are chosen. Moreover, to prove this fact we only use property (i) of the preceding paragraph.

THEOREM 5.1. *If the dimension $n \geqq 2$, and if μ_i is a sequence of measures on S such that $\mu_i(S) = 1$ for all i, then there is a set $A \subset S$ such that $\delta(A) \neq \underline{\lim}_{i\to\infty} \mu_i(A)$.*

Proof. We must assume that $\mu_i(A)$ is defined for each $A \subset S$ and $i \geqq 1$. (It is evident that limited representations of δ in terms the μ_i may be obtained, if the class of μ_i-measurable sets is restricted). It follows that, since S is countable, each μ_i has the form

$$(*)\mu_i(A) = \sum_{x \in A} \mu_i(\{x\}). \quad (A \subset S) .$$

Two cases are distinguished:

Case I. For each M, $\overline{\lim}_{i\to\infty} \mu_i(J(M)) = 0$. Let $j(0) = 1$ and, for $M \geqq 1$, let $j(M)$ be so large that $j(M) > j(M-1)$ and, if $i \geqq j(M)$, then $\mu_i(J(M)) < 2^{-M}$. Now by (*), for each $M \geqq 0$, there exists a finite set $H_M \subset S$ such that $\mu_i(H_M) > 1 - 2^{-M}$ for all i with $j(M) \leqq i < j(M+1)$. Let $A = \cup_{M=1}^{\infty} (H_M \backslash J(M))$. By Theorem 2.11(i), $\delta(A) = 0$. For each $i, j(M) \leqq i < j(M+1)$ we have

$$\begin{aligned}\mu_i(A) \geqq \mu_i(H_M \backslash J(M)) &= \mu_i(H_M) - \mu_i(H_M \cap J(M)) \\ &> 1 - 2^{-M} - 2^{-M} = 1 - 2^{-(M-1)} .\end{aligned}$$

Hence $\underline{\lim}_{i\to\infty} \mu_i(A) \geqq \underline{\lim}_{M\to\infty} 1 - 2^{-(M-1)} = 1$.

Case II. There exists M such that $\overline{\lim}_{i\to\infty} \mu_i(J(M)) = k > 0$. Here Let $A = S\backslash J(M)$. By 2.11(ii), $\delta(A) = 1$. However, for infinitely many i,

$$\mu_i(A) = \mu_i(S) - \mu_i(J(M)) \leqq 1 - k + \varepsilon_i\ (\varepsilon_i \to 0) ,$$

and so $\underline{\lim}_{i\to\infty} \mu_i(A) \leqq 1 - k < 1$. This completes the proof.

Observe that, for a finite set $F \subset S$, $\mu(A) = A(F)/S(F)$ defines a measure on S. Let $(F_i) \in \mathscr{S}$, and define μ_i by

$$\mu_i(A) = \frac{A(F_i)}{S(F_i)} .$$

Then μ_i is a sequence of measure each defined on every subset of S and satisfying $\mu_i(S) = 1$. By the previous theorem, there is a set $A \subset S$ such that $\delta(A) \neq \underline{\lim}_{i\to\infty} \mu_i(A) = \underline{\lim}_{i\to\infty} A(F_i)/S(F_i)$. This shows that there is no "universal" sequence $(F_i) \in \mathscr{S}$, i.e., one such that $\delta(A) = \underline{\lim}_{i\to\infty} A(F_i)/S(F_i)$ for all $A \subset S$. It also shows that the density in [3] is different from δ.

We proceed to prove an equivalent form of the definition of ν. By using methods similar to those in Theorems 2.6 and 2.7 it is easy to prove the "duals" of these theorems for $\bar{\delta}$. Namely, it can be proved:

THEOREM 5.2. *If* $(F_i) \in \mathscr{S}$ *and* $A \subset S$, *then*

$$\bar{\delta}(A) \geqq \overline{\lim_{i\to\infty}} \frac{A(F_i)}{S(F_i)} .$$

THEOREM 5.3. *For* $A \subset S$, *there exists* $(F_i) \in \mathscr{S}$ *such that*

$$\bar{\delta}(A) = \lim_{i\to\infty} \frac{A(F_i)}{S(F_i)} .$$

THEOREM 5.4. *The natural density of* $A \subset S$ *exists if and only if, for each* $(F_i) \in \mathscr{S}$, *the quotients* $A(F_i)/S(F_i)$ *form a convergent sequence. In this case*

$$\nu(A) = \lim_{i\to\infty} \frac{A(F_i)}{S(F_i)}$$

for each sequence $(F_i) \in \mathscr{S}$.

Proof. If $\nu(A)$ *exists, then, for each sequence* $(F_i) \in \mathscr{S}$, *by* 2.6 *and* 5.2,

$$\nu(A) = \delta(A) \leqq \underline{\lim_{i\to\infty}} \frac{A(F_i)}{S(F_i)} \leqq \overline{\lim_{i\to\infty}} \frac{A(F_i)}{S(F_i)} \leqq \bar{\delta}(A) = \nu(A) .$$

Suppose $A(F_i)/S(F_i)$ is convergent for each $(F_i) \in \mathscr{S}$. All the limits must be the same, for, if (F_i) and (G_i) are two sequences in $\mathscr{S}$ such that $A(F_i)/S(F_i)$ and $A(G_i)/S(G_i)$ converge to different limits, then the sequence (H_i), defined by

$$H_i = \begin{cases} F_i & \text{for } i \text{ odd} \\ G_i & \text{for } i \text{ even} \end{cases}$$

is in $\mathscr{S}$ and $\lim_{i\to\infty} A(H_i)/S(H_i)$ does not exist. Thus by 2.7 and 5.3, there exist (F_i) and (G_i) in $\mathscr{S}$ such that

$$\delta(A) = \lim_{i\to\infty} \frac{A(F_i)}{S(F_i)} = \lim_{i\to\infty} \frac{A(G_i)}{S(G_i)} = \bar{\delta}(A) .$$

Hence $\nu(A)$ exists and the last statement of the theorem is obvious.

This paper is concluded by nothing that ν is a finitely additive set function.

THEOREM 5.5. *Let $A_1, \cdots, A_m$ be sets with natural density such that, for each pair i, j with $i \neq j$ there is an N_{ij} such that $A_i \cap A_j \subset J(N_{ij})$. Then $A = A_1 \cup \cdots \cup A_m$ has natural density and*

$$\nu(A) = \sum_{i=1}^{m} \nu(A_i) .$$

Proof. Let $N = \max_{i,j} \{N_{ij}\}$. Clearly $B_i = A_i \backslash J(N)$ has natural density and $\nu(B_i) = \nu(A_i)$. The B_i are disjoint. Thus, for any sequence $(F_i) \in \mathscr{S}$,

$$\frac{A_1(F_i) + \cdots + A_m(F_i)}{S(F_i)} \geqq \frac{A(F_i)}{S(F_i)} \geqq \frac{B_1(F_i) + \cdots + B_m(F_i)}{S(F_i)}$$

where both ends converge to $\sum_{j=1}^{m} \nu(A_j)$ as $i \to \infty$. By 5.4 the theorem is proved.

REFERENCES

1. R.C. Buck, *The measure-theoretic approach to density*, Amer. J. Math. **68** (1946), 560-580.

2. ———, *Generalized asymptotic density*, Amer, J. Math. **75** (1953), 335-346.

3. J. Christopher, *The asymptotic density of some k-dimensional sets*, Amer. Math. Monthly **63** (1956), 399-401.

4. A.R. Freedman, *An inequality for the sum of sets of vectors in n-dimensional space*, Pacific J. Math. **19** (1966), 265-267.

5. A.R. Freedman, *Some generalizations in additive number theory*, J. reine angew. Math. (in press)

6. B. Kvarda, *On densities of sets of lattice points*, Pacific J. Math. **13** (1963), 611-615.

7. B. Kvarda, *An inequality for the number of elements in a sum of two sets of lattice points*, Pacific J. Math. **15** (1965) 545-550.

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