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A RADON-NIKODYM THEOREM FOR VECTOR AND OPERATOR VALUED MEASURES

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The main result of this paper is a Radon-Nikodým theorem for measures taking values in a separable Hilbert space and on the bounded operators of such a space. The integral used for the representation is a Gelfand-Pettis integral, which in this case is also equivalent to the Bochner integral.

1.1. Basic definitions. We will consider the following objects: a measure space $(\Omega, \mathscr{A}, \mu)$, where $\mathscr{A}$ is a σ-algebra of subsets of Ω and μ is a σ-finite nonnegative measure; a separable Hilbert space H and the space $B(H)$ of bounded linear operators from H into H, and also the objects which we define below.

1.2. DEFINITION. By *vector function* and *operator function* we will understand functions defined on Ω and taking values in H and $B(H)$ respectively. A vector function $x(\omega)$ is *measurable* if for each y in H, the function $(y, x(\omega))$ is measurable. An operator function $A(\omega)$ is *measurable* if for each x, y in H, the function $(A(\omega)x, y)$ is measurable. Obviously $A(\omega)$ is measurable if and only if $A(\omega)x$ is a measurable vector function for each x in H.

1.3. LEMMA. *If $x(\omega)$ is a measurable vector function, then $\|x(\omega)\|$ is measurable. If $A(\omega)$ is a measurable operator function, then $\|A(\omega)\|$ is measurable.*

Proof. Let $x(\omega)$ be measurable and let $\{e_1 e_2, \cdots\}$ denote an orthonormal basis for H. Then $(x(\omega), e_n)$ is measurable for each n and so $\|x(\omega)\|^2 \sum_{n=1}^{\infty} |(x(\omega), e_n)|^2$ is measurable. Now let $A(\omega)$ be measurable and let S_0 be a countable dense subset of the unit ball in H. Then $\|A(\omega)\| = \sup\{\|A(\omega)x\|: x \in S_0\}$ is measurable.

1.4. DEFINITION. A measurable vector function $x(\omega)$ is *integrable* if $\|x(\omega)\|$ is integrable (i.e., it belongs to $L_1(\mu)$). A measurable operator function $A(\omega)$ is *integrable* if $\|A(\omega)\|$ is integrable.

Let $x(\omega)$ be integrable and let $y \in H$. Then $|(y, x(\omega))| \leqq \|y\| \cdot \|x(\omega)\|$ and $(y, x(\omega))$ is integrable. $\int (y, x(\omega)) d\mu(\omega)$ is a linear functional bounded by $\int \|x(\omega)\| d\mu(\omega)$ and there is a unique vector $z \in H$ such that $\int (y, x(\omega)) d\mu(\omega) = (y, z)$. The vector z is by definition the integral

$\int x(\omega)d\mu(\omega)$; we already proved that $\left\|\int x(\omega)d\mu(\omega)\right\| \leqq \int \| x(\omega) \| \, d\mu(\omega)$. The integral is obviously linear. For each

$$x \in H, \; \| A(\omega)x \| \leqq \| A(\omega) \| \cdot \| x \|$$

so that $A(\omega)x$ is an integrable vector function. Since

$$\left\|\int A(\omega)x d\mu(\omega)\right\| \leqq \int \| A(\omega)x \| \, d\mu(\omega) \leqq \int \| A(\omega) \| \, d\mu(\omega) \cdot \| x \| \, ,$$

$\int A(\omega)x d\mu(\omega)$ defines a bounded linear operator on x. This operator is by definition the integral of $A(\omega)$, so that $\int A(\omega)x d\mu(\omega) = \left(\int A(\omega)d\mu(\omega)\right)x$ for each $x \in H$. Obviously $\left\|\int A(\omega)d\mu(\omega)\right\| \leqq \int \| A(\omega) \| \, d\mu(\omega)$ and the integral is linear.

2.1. Indefinite integrals and the Radon-Nikodým theorem. If $x(\omega)$ is a measurable vector function and $E \in \mathscr{A}$, $\chi_E(\omega)x(\omega)$ is also measurable and if $x(\omega)$ is integrable, so is $\chi_E(\omega)x(\omega)$. Similarly, if $A(\omega)$ is an operator function, $\chi_E(\omega)A(\omega)$ will be measurable or integrable if $A(\omega)$ has the same property. Thus, if $x(\omega)$ and $A(\omega)$ are integrable, $\int_E x(\omega)d\mu(\omega) \equiv \int \chi_E(\omega)x(\omega)d\mu(\omega)$ and

$$\int_E A(\omega)d\mu(\omega) \equiv \int \chi_E(\omega)A(\omega)d\mu(\omega)$$

will exist for all $E \in \mathscr{A}$.

Let $\varphi(E)$ denote the integral over E of a vector or operator function. Then φ is σ-additive in norm, that is, if $\{E_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathscr{A}$, then $\varphi(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \varphi(E_n)$ in norm. Also φ is absolutely continuous with respect to $\mu(\varphi \ll \mu)$ in the sense that $(\mu E) = 0$ implies $\varphi(E) = 0$. Finally if $E \in \mathscr{A}$ and $\{E_n\}_{n=1}^{\infty}$ is a disjoint sequence of sets in $\mathscr{A}$ such that $E = \bigcup_{n=1}^{\infty} E_n$, then we must have $\sum_{n=1}^{\infty} \| \varphi(E_n) \| < \infty$. We will denote this property saying that is σ-*bounded* on E.

2.2. Lemma. *Let X be a normed space and φ a σ-additive function from $\mathscr{A}$ into X. Then there is a nonnegative measure ν on $\mathscr{A}$ such that for each $E \in \mathscr{A}$, $\| \varphi(E) \| \leqq \nu(E)$, and $\nu(E)$ is finite if and only if φ is σ-bounded on E. Furthermore if $\varphi \ll \mu$, then $\nu \ll \mu$. (Obviously in any case $\varphi \ll \nu$).*

Proof. Let $\mathscr{P} = \{E_1, \cdots, E_n\}$ be a (measurable) partition of $E \in \mathscr{A}$ and let $| \mathscr{P} |$ denote the number $\sum_{i=1}^{n} \| \varphi(E_i) \|$. Temporarily we will say that E is *unbounded* if for each $K > 0$ there is a partition $\mathscr{P}$ of E with $| \mathscr{P} | > K$. Assume that φ is σ-bounded on E, but that E is unbounded. We claim that E contains disjoint measurable subsets $E_0, E_1, \cdots, E_n$, $n \geqq 1$ with E_0 unbounded and $\sum_{i=1}^{n} \| \varphi(E_i) \| > 1$. Otherwise each partition of E contains precisely one unbounded set and for positive integer n there is a partition $\mathscr{P}_n$ with $|\mathscr{P}| \geqq n + 1$, containing the unbounded set F_n for which we must have $\| \varphi(F_n) \| \geqq n$. If necessary, by refining these partitions we may obtain that $F_{n+1} \supseteqq F_n$ for each n. Since $F_n = F \cup \bigcup_{k=1}^{\infty} (F_k \backslash F_{k+1})$, where $F = \bigcap_{k=1}^{\infty} F_k$, and φ is σ-additive in norm, we have

$$n \leqq \| \varphi(F_n) \| \leqq \| \varphi(F) \| + \sum_{k=n}^{\infty} \| \varphi(F_k \backslash F_{k+1}) \|$$

which is impossible since $\sum_{k=1}^{\infty} \| \varphi(F_k \backslash F_{k+1}) \|$ is convergent, E being σ-bounded. Having proved our claim, we arrive at a new contradiction, since then we may construct a disjoint sequence $\{E_n\}_{n=1}^{\infty}$ measurable of subsets of E with $\sum_{n=1}^{\infty} \| \varphi(E_n) \| = \infty$. Thus a σ-bounded set E is not unbounded, i.e., there is a constant $K_E > 0$ such that $\sum_{n=1}^{\infty} \| \varphi(E_n) \| < K_E$ for each disjoint sequence $\{E_n\}_{n=1}$ of measurable subsets of E.

Now we define ν on $\mathscr{A}$ by $\nu(E) = \sup \{\sum_{n=1}^{\infty} \| \varphi(E_n) \| : \{E_n\}_{n=1}^{\infty} \subset \mathscr{A}$, disjoint and $\bigcup_{n=1}^{\infty} E_n = E\}$. Obviously $\| \varphi(E) \| \leqq \nu(E)$, $\nu(E) < \infty$ if and only if φ is σ-bounded on E, and $\varphi \ll \mu$ implies $\nu \ll \mu$. We only need to prove that ν is σ-additive. Suppose that $E = \bigcup_{n=1}^{\infty} E_n$ where the E_n are disjoint and measurable. For any $\varepsilon > 0$ there is a disjoint sequence $(G_m)_{m=1}^{\infty}$ of measurable subsets of E such that $E = \bigcup_{m=1}^{\infty} G_m$ and $\nu(E) \leqq \sum_{m=1}^{\infty} \| \varphi(G_m) \| + \varepsilon$ (if $\nu(E) = \infty$, E is not σ-bounded and the G_m may taken such that $\sum_{m=1}^{\infty} \| \varphi(G_m) \| = \infty$). Since

$$\varphi(G_m) = \sum_{n=1}^{\infty} \varphi(G_m \cap E_n) ,$$

we have $\| \varphi(G_m) \| \leqq \sum_{n=1}^{\infty} \| \varphi(G_m \cap E_n) \|$ and therefore

$$\nu(E) \leqq \sum_{m,n} \| \varphi(G_m \cap E_n) \| + \varepsilon \leqq \sum_{n=1}^{\infty} \nu(E_n) + \varepsilon .$$

On the other hand, for each positive n there is a disjoint sequence $\{G_{nm}\}_{m=1}^{\infty}$ of measurable sets such that $\bigcup_{m=1}^{\infty} G_{nm} = F_n$ and

$$\nu(E_n) \leqq \sum_{m=1}^{\infty} \| \varphi(G_{nm}) \| + 2^{-n}\varepsilon .$$

Then $\sum_{n=1}^{\infty} \nu(E_n) \leqq \sum_{n,m} \| \varphi(G_{nm}) \| + \varepsilon \leqq \nu(E) + \varepsilon$. Since ε was arbitrary, we obtain $\nu(E) = \sum_{n=1}^{\infty} \nu(E_n)$.

2.3. LEMMA. *Let $f(\omega)$ and $r(\omega)$ be integrable functions, the first complex and the second nonnegative, such that for each $E \in \mathscr{A}$, $\left| \int_E f(\omega) d\mu(\omega) \right| \leqq \int_E r(\omega) d\mu(\omega)$. Then $|f(\omega)| \leqq r(\omega)$ almost everywhere.*

Proof. If the lemma is false, there is a positive integer n such that $\mu(\{\omega \in \Omega: |f(\omega)| > r(\omega) + 1/n\}) > 0$ since then $\{\omega \in \Omega: |f(\omega)| > r(\omega)\}$ has positive measure. Also, for some open circle S of radius $1/2n$ on the complex plane we must have $0 < \mu(F) < \infty$, where F denotes a subset of $\{\omega: |f(\omega)| > r(\omega) + 1/n\} \cap \{\omega: f(\omega) \in S\}$. Let z_0 be center of S. Then for each $\omega \in F$, $|f(\omega) - z_0| < 1/2n$ and $|f(\omega)| > r(\omega) + 1/n$. Integrating the identity $f(\omega) = z_0 - (z_0 - f(\omega))$ over F and taking absolute values we obtain

$$\left|\int_F f(\omega)d\mu(\omega)\right| \geqq \left|\int_F z_0 d\mu(\omega)\right| - \left|\int_F (z_0 - f(\omega))d\mu(\omega)\right|$$
$$\geqq |z_0| \mu(F) - 1/2n\, \mu(F) > r(\omega)\mu(F)$$

for all $\omega \in F$, since $r(\omega) < |f(\omega)| - 1/n < 1/2n \div |z_0| - 1/n$. Integrating again over F and dividing by $\mu(F)$ we obtain

$$\left|\int_F f(\omega)d\mu(\omega)\right| > \int_F r(\omega)d\mu(\omega) ,$$

which contradicts our hypothesis.

2.4. THEOREM. *Let φ be a measure defined on $\mathscr{A}$ and taking values in H or $B(H)$. If φ is σ-additive in norm, σ-bounded and absolutely continuous with respect to μ then φ is the indefinite integral with respect to μ of an integrable vector function or operator function which is unique almost everywhere.*

Proof. We consider first the case in which φ takes values in H. Since for each $z \in H$, $(x, \varphi(E))$ is a complex, finite measure, absolutely continuous with respect to μ, the Radon-Nikodým theorem says that there is a complex integrable function $f_\omega(x)$ (with respect to ω) such that

$$(x,\varphi(E)) = \int_E f_\omega(x)d\mu(\omega) \tag{1}$$

and the function $f_\omega(x)$ differs from another with the same properties at most in a μ-null set. If α, β are complex and $x, y \in H$, it is clear that $f_\omega(\alpha x + \beta y) = \alpha f_\omega(x) + \beta f_\omega(y)$ except in a μ-null set. Also

$$\left|\int_E f_\omega(x)d\mu(\omega)\right| = |(x, \varphi(E))| \leqq \|\varphi(E)\| \cdot \|x\| \leqq \nu(E)\,\|x\| ,$$

where ν is the measure defined in Lemma 2.2. Since $\nu \ll \mu$ and ν is finite, there is a nonnegative, finite and integrable function r_ω such that $\nu(E) = \int_E r_\omega d\mu(\omega)$. From the inequality

$$\left|\int_E f_\omega(x)d\mu(\omega)\right| \leqq \int_E r_\omega \|x\| \, d\mu(\omega)$$

for each $E \in \mathscr{A}$, by Lemma 2.3. we conclude that $|f_\omega(x)| \leqq r_\omega \|x\|$ for almost all ω.

The next steps of the proof lead to the construction for each $x \in H$ of a particular function $f_\omega(x)$, which for each ω will be a continuous linear functional in x. Let $\{e_1, e_2, \cdots\}$ be an orthonormal base for H and let H_0 be the set of linear combinations with rational complex coefficients of the base vectors.

Step 1. We choose finite functions $\tilde{f}_\omega(e_k)$ such that $(e_k, \varphi(E) = \int_E \tilde{f}_\omega(e_k) d\mu(\omega)$ for each $E \in \mathscr{A}$.

Step 2. We define $\tilde{f}_\omega$ on H_0 by linearity.

Step 3. We choose a nonnegative, finite function r_ω such that $\nu(E) = \int_E r_\omega d\mu(\omega)$ for each $E \in \mathscr{A}$.

Step 4. Since H_0 is countable and for each $x \in H_0$, $|\tilde{f}_\omega(x)| \leqq r_\omega \|x\|$ for almost all ω, we choose a μ-null set N such that $|\tilde{f}_\omega(x(| \leqq r_\omega \|x\|$ for all $x \in H_0$ and $\omega \in \Omega \backslash N$.

Step 5. We define $f_\omega(x)$ for $\omega \in \Omega$ and $x \in H_0$ by $f_\omega(x) = \tilde{f}_\omega(x)$ if $\omega \in \Omega \backslash N$ and $f_\omega(x) = 0$ if $\omega \in N$. The functions we have defined have the following properties:

(a) $(x, \varphi(E)) = \int_E f_\omega(x) d\mu(\omega)$, for each $x \in H_0$ ane $E \in \mathscr{A}$.

(b) $|f_\omega(x)| \leqq r_\omega \|x\|$ for each $x \in H_0$ and $\omega \in \Omega$,

(c) if α, β are rational complex numbers and $x, y \in H_0$, then $f_\omega(\alpha x + \beta y) = \alpha f_\omega(x) + \beta f_\omega(y)$, for all $\omega \in \Omega$.

Step 6. Let $x \in H$ and $\{x_n\}_{n=1}^\infty$ be a sequence in H_0 converging to x. For each $\omega \in \Omega$, $|f_\omega(x_n) - f_\omega(x_m)| = |f_\omega(x_n - x_m)| \leqq r_\omega \|x_n - x_m\|$. Therefore $\lim_{n\to\infty} f_\omega(x_n)$ exists and obviously it is independent of the particular sequence $\{x_n\}_{n=1}^\infty$. We define $f_\omega(x) = \lim_{n\to\infty} f_\omega(x_n)$. From the continuity of the norm we obtain $|f_\omega(x)| \leqq r_\omega \|x\|$. Also $(x, \varphi(E)) = \lim_{n\to\infty}(x_n, \varphi(E)) = \lim_{n\to\infty} \int_E f_\omega(x_n) d\mu(\omega) = \int_E f_\omega(x) d\mu(\omega)$, the last equality being valid by the dominated convergence theorem. Finally, if α, β are arbitrary complex numbers and x, y are any two vectors in H, there are sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ of rational complex numbers and sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ of vectors in H_0 such that $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Then $f_\omega(\alpha x + \beta y) = \lim_{n\to\infty} f_\omega(\alpha_n x_n + \beta_n y_n) = \lim_{n\to\infty}(\alpha_n f_\omega(x_n) + \beta_n f_\omega(y_n)) = \alpha f_\omega(x) + \beta f_\omega(y)$.

Thus for each ω, $f_\omega(x)$ is a continuous linear functional and by the Riesz theorem there is a unique vector $x(\omega)$ such that $f_w(x) = (x, x(\omega))$ for each $x \in H$. Since $f_\omega(x)$ is measurable, $x(\omega)$ is measurable and since $\|x(\omega)\| = \|f_\omega\| \leqq r_\omega$, $x(\omega)$ is also integrable. From the equation $(x, \varphi(E)) = \int_E (x, x(\omega)) d\mu(\omega) = (x, \int_E x(\omega) d\mu(\omega))$ we obtain $\varphi(E) = \int_E x(\omega) d\mu(\omega)$. The uniqueness almost everywhere of the vector function $x(\omega)$ is trivial.

The proof for the case when φ takes values in $B(H)$ follows along the same lines. Now we obtain $(\varphi(E)x, y) = \int_E f_\omega(x, y)d\mu(\omega)$, where $f_\omega(x, y)$ is for all $x, y \in H$ an integrable function and for each $\omega \in \Omega$ is a bilinear functional in x, y, bounded by some Radon-Nikodým derivative r_ω of the measure ν. By a corollary of the Riesz theorem, $f_\omega(x, y) = (A(\omega)x, y)$ for some linear operator $A(\omega)$, with $\|A(\omega)\| = \|f_\omega\| \leqq r_\omega$ and as before we obtain $\varphi(E) = \int_E A(\omega)d\mu(\omega)$ for each $E \in \mathscr{A}$. The uniqueness a.e. of $A(\omega)$ is again trivial.

2.5. REMARK. From the proof of Theorem 2.4., we have that $\|x(\omega)\| \leqq r_\omega$ (a.e.), where $r_\omega = d\nu/d\mu$ (a.e.). It is easy to see that $\|x(\omega)\|$ is actually equal to r_ω (a.e.). In fact, from $\|\varphi(E)\| \leqq \nu(E)$ and the definition of $\nu(E)$, we obtain $\int_E \|x(\omega)\| d\mu \geqq \nu(E)$ since

$$\sum_{n=1}^{\infty} \|\varphi(E)\| \leqq \sum_{n=1}^{\infty} \int_E \|x(\omega)\| d\mu = \int_{E_n} \|x(\omega)\| d\mu$$

for each countable partition of E. Also $\int_E \|x(\omega)\| d\mu \leqq \int_E r_\omega d\mu = \nu(E)$ and therefore $\|x(\omega)\| = r_\omega$ (a.e.). If we write $x(\omega) = d\varphi/d\mu$, $r_\omega = d\nu/d\mu$, we have $\|d\varphi/d\mu\| = d\nu/d\mu$. Of course, the same formula holds for operator valued measures.

2.6. If $x(\omega)$ is a measurable function which is not necessarily integrable, we may still integrate it on those sets in $\mathscr{A}$ where $\|x(\omega)\|$ is integrable. In fact, since $\|x(\omega)\|$ is everywhere finite and μ is σ-finite, there is a countable covering of Ω consisting of such sets. On each of these sets the indefinite integral is σ-bounded. Reciprocally, if there is a countable covering of Ω by measurable sets Ω_n and a vector (or operator) valued measure φ defined on the measurable subsets of each Ω_n, which is σ-additive and σ-bounded on each Ω_n, then φ is the indefinite integral of some unique (a.e.) $\mathscr{A}$-measurable vector (or operator) function, and this function will be integrable if and only if the (unique) extension of φ to all of $\mathscr{A}$, is σ-additive in norm and σ-bounded.

2.7. A COUNTEREXAMPLE. We may exhibit a vector (or operator) measure φ which is σ-additive on $\mathscr{A}$, absolutely continuous with respect to some non-negative measure μ, but σ-bounded only on sets of μ-measure zero. In fact there is a vector measure γ defined on the Borel subsets of $[0, 1]$, such that for each Borel set E, $\|\gamma(E)\| = \sqrt{\lambda(E)}$, where λ is the Lebesgue measure of E (so that $\gamma \ll \lambda$), and furthermore, if $E_1 \cap E_2 = \varnothing$ then $(\gamma(E_1), \gamma(E_2)) = 0$, i.e., $\gamma(E_1)$ and $\gamma(E_2)$ are orthogonal. It is easy to see that such a measure is σ-additive in norm, absolutely continuous with respect to λ, and if $\gamma(E) \neq 0$ (or equivalently, $\lambda(E) \neq 0$), then γ is not σ-bounded on E.

In fact, let $\mathscr{B}$ denote the Borel sets on $[0, 1]$ and let $\{E_k\}_{k=1}^{\infty}$ be a disjoint sequence in $\mathscr{B}$, $\bigcup_{k=1}^{\infty} E_k = E$. Then $\| \gamma(E) = \sum_{k=1}^{n} \gamma(E_k) \| = \| \gamma(E) - \gamma(\bigcup_{k=1}^{n} E_k) \| = \| \gamma(\bigcup_{k=n+1}^{\infty} E_k) \| = \sqrt{\lambda(\bigcup_{k=n+1}^{\infty} E_k)} \rightarrow 0$ as $n \rightarrow \infty$ and therefore $\gamma(E) = \sum_{k=1}^{\infty} \gamma(E_k)$.

Now let $\gamma(E) \neq 0$. Consider the sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, 1]$ defined by $t_n = \inf \{t \colon \lambda(E \cap [0, t]) > 6\lambda(E)/\pi^2 \sum_{k=1}^{n} 1/k^2\}$ for $n \geqq 1$ and $t_0 = 0$. We define $E_n = E \cap [t_{n-1}, t_n]$ so that $\{E_n\}_{n=1}^{\infty}$ is a disjoint sequence in and $\bigcup_{n=1}^{\infty} E_n \subseteqq E$. Also $\lambda(E_n) = 6\lambda(E)/\pi^2 n^2$ and therefore

$$\| \gamma(E_n) \| = \frac{\sqrt{6\lambda(E)}}{\pi} \cdot \frac{1}{\pi} ,$$

so that $\sum_{n=1}^{\infty} \| \gamma(E_n) \|$ diverges, although $\sum_{n=1}^{\infty} \gamma(E_n)$ is obviously convergent and equal to $\gamma(E)$. (Let $E_0 = E \backslash \bigcup_{n=1}^{\infty} E_n$, then $\lambda(E_0) = 0$ and therefore $\gamma(E) = 0$).

2.8. *Construction of* γ. We construct first inductively a sequence of sets $\{A_n\}_{n=1}^{\infty}$ in H having the following properties:

(i) A_n consists of 2^n mutually orthogonal vectors $a_n^1, a_n^2, \cdots, a_n^{2^n}$ each of length $2^{-n/2}$.

(ii) For each $n \geqq 0$ and $1 \leqq p \leqq 2^n$, $a_n^p = a_{n+1}^{2p-1} + a_{n+1}^{2p}$.

We start choosing a unit vector which we denote by a_0^1 and call $A_0 = a_0^1$. Having constructed $A_0, A_1, \cdots, A_n$, we construct A_{n+1} in the following way. Choose 2^n vectors $b_1, b_2, \cdots, b_{2^n}$, each of length $2^{-n/2}$, orthogonal with respect to each other and to $a_n^1, a_n^2, \cdots, a_n^{2^n}$. Now define $a_{n+1}^{2p-1} = 1/2(a_n^p + b_p)$, $a_{n+1}^{2p} = 1/2(a_n^p - b_p)$, $p = 1, 2, \cdots, 2^n$ and then $A_{n+1} = \{a_{n+1}^1, a_{n+1}^2, \cdots, a_{n+1}^{2^{n+1}}\}$. Obviously a sequence $\{A_n\}_{n=1}^{\infty}$ constructed in this way satisfies (i-ii).

Now we begin the construction of our measure. *A basic interval* of order n will be an interval of the form $[p - 1/2^n, p/2^n]$ where n and p are integers and $n \geqq 0$, $1 \leqq p \leqq 2^n$. $\mathscr{F}$ and $\mathscr{G}$ will denote respectively the class of all finite unions and the class of all countable unions of basic intervals and $\mathscr{B}$ will denote the Borel sets of $[0, 1)$. A set in $\mathscr{F}$ (or in $\mathscr{G}$) can always be expressed as a finite (or countable) union of disjoint basic intervals. For a set in $\mathscr{F}$ this is obvious and for a set in $\mathscr{G}$ a simple inductive process will give us the required decomposition. It is clear that $\mathscr{F}$ is an algebra, that is, it is closed with respect to finite unions and complementation. $\mathscr{G}$ is closed with respect to countable unions and finite intersections. The latter follows from the identity $(\bigcup_{j=1}^{\infty} F_i) \cap (\bigcup_{j=1}^{\infty} H_j) = \bigcup_{i=1}^{\infty}(F_i \cap H_i)$, where $\{F_i\}_{i=1}^{\infty}$ and $\{H_i\}_{i=1}^{\infty}$ are nondecreasing sequences of sets in $\mathscr{F}$.

If V is the basic interval $[p - 1/2^n, p/2^n)$, we define $\gamma(V) = a_n^p$. If $V_1 = [2p - 2/2^{n+1}, 2p - 1/2^{n+1})$ and $V_2 = [2p - 1/2^{n+1}, 2p/2^{n+1})$, so that $V = V_1 \cup V_2$, by (ii) we have that $\gamma(V) = \gamma(V_1) + \gamma(V_2)$. By induction we obtain that if $V_1, V_2, \cdots, V_{2^m}$ denote the 2^m basic subintervals of V of order $n + m$, then $\gamma(V) = \sum_{n=1}^{\infty} \gamma(V_i)$. Finally if $V_1, V_2, \cdots, V_n$ are disjoint basic intervals, not necessarily of the same order, such that $V = \bigcup_{i=1}^n V_i$ and $n + m$ is the highest order among the V_i, we decompose each V_i in basic subintervals of order $n + m$, say $V_i = \bigcup_i W_j^{(i)}$, so that $\gamma(V_i) = \sum_i \gamma(W_j^{(i)})$ and we obtain

$$\sum_{i=1}^{n} \gamma(V_i) = \sum_i \sum_j \gamma(W_j^{(i)}) = \gamma(V) .$$

Thus γ is additive on the basic intervals.

If $F \in \mathscr{F}$ and $F = \bigcup_{i=1}^n V_i$, where the V_i are disjoint basic intervals, we define $\gamma(F) = \sum_{i=1}^n \gamma(V_i)$. From the additivity of γ on the basic intervals it follows immediately that $\gamma(F)$ is well defined, i.e., it doesn't depend upon the particular decomposition of F and that γ is additive on $\mathscr{F}$.

If $V = [p - 1/2^n, p/2^n)$, $\| \gamma(V) \|^2 = \| a_n^p \|^2 = \| 2^{-n} = \lambda(V)$, where λ denotes Lebesgue measure. If V_1 and V_2 are disjoint basic intervals, $\gamma_2(V_1)$ and $\gamma(V_2)$ are mutually orthogonal, which implies that $\| \gamma(F) \|^2 = \| \sum_{i=1}^n \gamma(V_i) \|^2 = \sum_{i=1}^n \| \gamma(V_i) \|^2 = \sum_{i=1}^n \lambda(V_i) = \lambda(F)$, where $F \in \mathscr{F}$, $F = \bigcup_{i=1}^n V_i$ and V_i are disjoint basic intervals.

Suppose now that $V = \bigcup_{i=1}^{\infty} V_i$, where the V_i are disjoint basic intervals and V is also a basic interval. Then $V \backslash \bigcup_{i=1}^n V_i \in \mathscr{F}$ for each $n \geqq 1$ and therefore $\| \gamma(V) - \sum_{i=1}^n \lambda(V_i) \| = \| \gamma(V \backslash \bigcup_{i=1}^n V_i) \| = \sqrt{\lambda(V \backslash \bigcup_{i=1}^n V_i)} \rightarrow 0$ as $n \rightarrow \infty$, which implies that $\gamma(V) = \sum_{i=1}^{\infty} \gamma(V_i)$, i.e., γ is σ-additive on the basic intervals.

Now we define γ on $\mathscr{G}$ by $\gamma(G) = \sum_{i=1}^{\infty} \gamma(V_i)$, where $G = \bigcup_{i=1}^{\infty} V_i$ and the V_i are disjoint basic intervals. First we observe that since the vector $\gamma(V_i)$ are pairwise orthogonal and $\sum_{i=1}^{\infty} \| \gamma(V_i) \|^2 = \sum_{i=1}^{\infty} \gamma(V_i) = \lambda(G) \leqq 1$, the series $\sum_{i=1}^{\infty} \gamma(V_i)$ converges and $\| \gamma(G) \|^2 = \lambda(G)$. If $G = \bigcup_{i=1}^{\infty} V_i = \bigcup_{j=1}^{\infty} W_j$ are two decompositions of G into disjoint basic subintervals, $\sum_{i=1}^{\infty} \gamma(V_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \gamma(V_i \cap W_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \gamma(V_i \cap W_j) = \sum_{j=1}^{\infty} \gamma(W_j)$ (the sums commute because the vectors are orthogonal) so that $\gamma(G)$ is well defined. If $\{F_i\}_{i=1}^{\infty}$ is a nondecreasing sequence in $\mathscr{F}$ with $G = \bigcup_{n=1}^{\infty} F_n$, then $\gamma(G) \lim_{n \rightarrow \infty} \gamma(F_n)$. In fact there is a sequence $\{V_i\}_{i=1}^{\infty}$ of disjoint basic intervals such that $F_n = \bigcup_{i=1}^{r_n} V_i$, where $r_1 \leqq r_2 \leqq \cdots$ are integers with $\lim_{n \rightarrow \infty} r_n = \infty$, so that $\gamma(G) = \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \gamma(V_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^{r_n} \gamma(V_i) = \lim_{r \rightarrow \infty} \gamma(F_n)$. Suppose now that G_1 and G_2 are in $\mathscr{G}$ and that $\{F_n\}_{n=1}^{\infty}$, $\{H_n\}_{n=1}^{\infty}$ are nondecreasing sequences in $\mathscr{F}$ with $G_1 = \bigcup_{n=1}^{\infty} F_n$, $G_2 = \bigcup_{n=1}^{\infty} H_n$. Then we have that $G_1 \cup G_2 = \bigcup_{n=1}^{\infty} (F_n \cup H_n)$, $G_1 \cap G_2 = \bigcup_{n=1}^{\infty} (F_n \cap H_n)$, and taking limits, from the relation $\gamma(F_n \cup H_n) + \gamma(F_n \cap H_n) = \gamma(F_n) + \gamma(H_n)$ we obtain

$\gamma(G_1 \cup G_2) + \gamma(G_1 \cap G_2) = \gamma(G_1) + \gamma(G_2)$, i.e., γ is *modular* in $\mathscr{G}$.

It is clear that $\mathscr{G}$ contains all open sets in $[0, 1)$. Therefore, if $E \in \mathscr{B}$, for each $\varepsilon > 0$, there is some $G \in \mathscr{G}$ such that $G \supseteqq E$ and $\lambda(G \backslash E) < \varepsilon$. Let $G_1, G_2 \in \mathscr{G}$, $G_1 \subseteqq G_2$, $G_1 = \bigcup_{i=1}^{\infty} V_i$, the V_i disjoint basic intervals. Then for each n, $G_2 \backslash \bigcup_{i=1}^{n} V_i \in \mathscr{G}$ and expressing $G_2 \backslash \bigcup_{i=1}^{n} V_i$ as a union of disjoint basic intervals we see that $\gamma(G_2 \backslash \bigcup_{i=1}^{n} V_i)$ $\gamma(G_2) - \sum_{i=1}^{n} \gamma(V_i)$. Therefore

$$\| \gamma(G_2) - \gamma(G_1) \|^2 = \lim_{n\to\infty} \| \gamma(G_2) - \sum_{i=1}^{n} \gamma(V_i) \|^2$$
$$= \lim_{n\to\infty} \| \gamma(G_2 \backslash \bigcup_{i=1}^{n} V_i) \|^2 = \lim_{n\to\infty} \gamma(G_2 \backslash \bigcup_{i=1}^{n} V_i) = \lambda(G_2) - \lambda(G_1) .$$

This implies that if the sequence $\{G_n\}_{n=1}^{\infty}$ of sets in $\mathscr{G}$ is nonincreasing, each G_n contains $G \in \mathscr{B}$ and $\lim_{n\to\infty} \lambda(G_n) = \lambda(E)$, then $\{\gamma(G_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in H. We define $\gamma(E)$ as the limit of this sequence and obviously $\| \gamma(E) \|^2 = \gamma(E)$. In order to prove that $\gamma(E)$ does not depend upon the particular sequence $\{G_n\}_{n=1}^{\infty}$, we take another such sequence, say $\{\tilde{G}_n\}_{n=1}^{\infty}$. Evidently $\lim_{n\to\infty} \lambda(G_m \backslash \tilde{G}_n) = \lim_{n\to\infty} \lambda(\tilde{G}_n \backslash G_n) = 0$ and since

$$\| \gamma(G_n) - \gamma(\tilde{G}_n) \| \leqq \| \gamma(G_n) - \gamma(G_n \cap \tilde{G}_n) \|$$
$$+ \| \gamma(\tilde{G}_n) - \gamma(G_n \cap \tilde{G}_n) \| = \sqrt{\lambda(G_n \backslash \tilde{G}_n)} + \sqrt{\lambda(\tilde{G}_n \backslash G_n)} ,$$

we have $\lim_{n\to\infty} \| \gamma(G_n) - \gamma(\tilde{G}_n) \| = 0$ and therefore $\lim_{n\to\infty} \gamma(G_n) = \lim_{n\to\infty} \gamma(\tilde{G}_n)$. If $G \in \mathscr{G}$ and $G \supseteqq E$, $E \in \mathscr{B}$, there is a nonincreasing sequence $\{G_n\}_{n=1}^{\infty}$ of sets in $\mathscr{G}$, $G \supseteqq G_n$ and such that $\gamma(E) = \lim_{n\to\infty} \gamma(G_n)$. Then $\| \gamma(G) - \gamma(E) \|^2 = \lim_{n\to\infty} \| \gamma(G) - \gamma(G_n) \|^2 = \lim_{n\to\infty} \lambda(G \backslash G_n) = \lambda(G \backslash E)$.

Our next step is to show that γ is finitely additive in $\mathscr{B}$. Let E_1 and E_2 be disjoint sets in $\mathscr{B}$ and let G_1 and G_2 in $\mathscr{G}$ be such that $G_1 \supseteqq E_1$, $G_2 \supseteqq E_2$, $\| \gamma(G_1) - \gamma(E_1) \| < \varepsilon$ and $\| \gamma(G_2) - \gamma(E_2) \| < \varepsilon$, where $\varepsilon > 0$ is given. Then

$$\| \gamma(G_1 \cup G_2) - \gamma(E_1 \cup E_2) \|$$
$$= \sqrt{\lambda(G_1 \cup G_2) - \lambda(E_1 \cup E_2)} \leqq \sqrt{\lambda(G_1 \backslash E_1) + \lambda(G_2 \backslash E_2)} < \sqrt{2\varepsilon} .$$

Also since γ is modular in $\mathscr{G}$,

$$\| \gamma(G_1 \cup G_2) - \gamma(G_1) - \gamma(G_2) \| = \| \gamma(G_1 \cup G_2) \|$$
$$= \sqrt{\lambda G_1 \cap G_2)} \leqq \sqrt{\lambda(G_1 \backslash E_1) + \lambda(G_2 \backslash E_2)} < \sqrt{2\varepsilon} .$$

Therefore

$$\| \gamma(E_1 \cup E_2) - \gamma(E_1) - \gamma(E_2) \| \leqq \| \gamma(E_1 \cup E_2) - \gamma(G_1 \cup G_2) \|$$
$$+ \| \gamma(G_1 \cup G_2) - \gamma(G_1) - \gamma(G_2) \| + \| \gamma(G_1) - \gamma(E_1) \|$$
$$+ \gamma(G_2) - \gamma(E_2) \| < (2 + 2\sqrt{2})\varepsilon ,$$

which implies that $\gamma(E_1 \cup E_2) = \gamma(E_1) + \gamma(E_2)$.

In 2.7. we proved that γ is countable additive under the assumption that it is finitely additive and $\|\gamma(E)\|^2 = \lambda(E)$ for $E \in \mathscr{B}$. Thus γ is countably additive.

Next, in order to prove the orthogonality property, we observe that since disjoint basic intervals have orthogonal measures, if G_1 and G_2 are disjoint sets in $\mathscr{G}$, $\gamma(G_1)$ and $\gamma(G_2)$ must be orthogonal. If K_1 and K_2 are disjoint compact sets, there are nonincreasing sequences $\{G_n\}_{n=1}^{\infty}$ and $\{\tilde{G}_n\}_{n=1}^{\infty}$ of sets in $\mathscr{G}$ such that $G_n \cap \tilde{G}_m = \emptyset$ for all n and m, and $\lim_{n\to\infty} \gamma(G_n) = \gamma(K_1)$, $\lim_{n\to\infty} \gamma(\tilde{G}_n) = \gamma(K_2)$, which implies that $\gamma(K_1)$ and $\gamma(K_2)$ are orthogonal. Finally if E_1 and E_2 are disjoint sets in $\mathscr{B}$, there are nondecreasing sequences $\{K_n\}_{n=1}^{\infty}$, $\{K_n\}_{n=1}^{\infty}$ of compact subsets of E_1 and E_2 such that $\lambda(E_1) = \lim_{n\to\infty} \lambda(E_n)$, $\lambda(E_2) = \lim_{n\to\infty} \lambda(K_n)$, so that $\gamma(E_1) = \lim_{n\to\infty} \gamma(K_n)$, $\gamma(E_2) = \lim_{n\to\infty} \gamma(K_n)$, and this implies that $\gamma(E_1)$ and $\gamma(E_2)$ are orthogonal. We may extend γ to the Borel subsets of $[0, 1]$ defining $\gamma(\{1\}) = 0$, and even "complete" it, defining $\gamma(E) = 0$ if E is a subset of a Borel set of λ-measure zero.

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