PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

A CHARACTERISTIC PROPERTY OF THE SPHERE

L. TAMÁSSY

It was proven by P. Funk that the only symmetrical star-shaped body, all of whose intersections with planes through its midpoint have surface area π, is the unit sphere. In this paper the same conclusion is deduced from a materially weaker hypothesis and an application of the result is given.

We consider the set Π of those planes through the midpoint of a symmetrical body in the three-space, whose normals make an angle $<\eta$ with a fixed plane Γ. η is an arbitrary positive constant.

THEOREM. *The only symmetrical star-shaped body with smooth boundary, which is intersected by elements of Π in figures of surface-area π, is the unit sphere.*

Star-shapeness means, that each closed ray from the midpoint out meets the boundary of the body at most at one point.

The property expressed in the theorem obviously holds for the unit sphere.

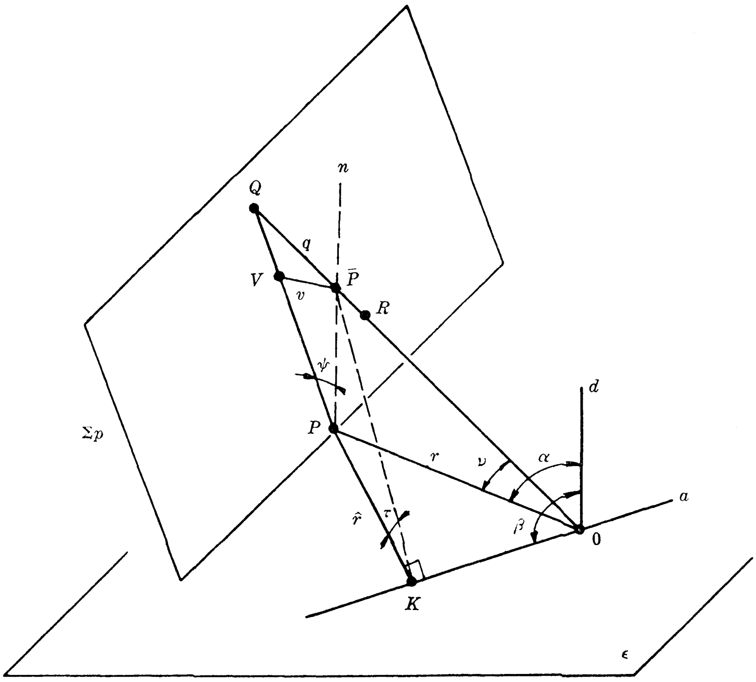
Let B be a symmetric starshaped body with smooth boundary and center 0. Let $\vec{n}$ be a unit vector. We denote the plane through 0 and perpendicular to $\vec{n}$ by $P(\vec{n})$, and the surface-area of the intersection of $P(\vec{n})$ and of B by $A(\vec{n})$: $A(\vec{n}) = \text{Area } (B \cap P(\vec{n}))$. Let 0 be the origin of a cartesian coordinate system and Γ the (y, z) plane.

We suppose $A(\vec{n}) = \pi$ for $\sphericalangle(\vec{n}, \Gamma) < \eta$, and we start to prove the uniqueness of B.

Let P be a point of the boundary ∂B of B in the (x, y) plane ε. We denote the tangent plane to ∂B in P by Σ_p. Let a be a line in ε through 0 and σ a plane through a (see the figure). We draw the normal n to ε through P and we denote the intersection of n and σ by $\bar{P}$; the intersection of ∂B and $\bar{P}0$ by R; and the intersection of Σ_p and $0R$ by Q. K denotes the foot of the normal from P to a. We draw the perpendicular to n through $\bar{P}$ in the plane of 0 and n; let V be the point of intersection of this line and of QP. Finally we denote

$$\overline{Q\bar{P}} = q; \overline{\bar{P}0} = r; \overline{\bar{P}K} = \hat{r}; \sphericalangle QP\bar{P} = \psi; \sphericalangle PK\bar{P} = \tau; \sphericalangle P0\bar{P} = \nu$$

and the angles measured in the positive sense from an arbitrary direction d to $0P$ (resp. to a) by α (resp. β). τ and ν are regarded positive if $\bar{P}$ lies in the positive halfspace of ε, otherwise they are negative. q is positive if n separates Q and 0 (in the plane of n, 0),



otherwise it is negative. ψ has the same sign as q, and r and $\hat{r}$ are not negative. We note that Q and R are uniquely determined by α, β and τ for a given body B.

Let now σ turn about a into ε. Then τ and ν tend to zero and $R(\nu)$ moves in a curve in the plane of 0 and n having the tangent QP at P. Thus we must have

$$(1) \qquad \lim_{\nu \to 0} \frac{\overline{0Q} - \overline{0R}}{\nu} = 0 .$$

Since $\nu \to 0$ follows from $\tau \to 0$ and inversely, and since $|\tau| \geqq |\nu|$, we get from (1)

$$\lim_{\tau \to 0} \frac{\overline{0Q} - \overline{0R}}{\tau} = 0 .$$

Multiplying by $\overline{0Q} + \overline{0R}$ and emphasizing the dependence of Q and R on α, β, τ, we get

$$(2) \qquad \lim_{\tau \to 0} \frac{\overline{0Q}^2(\alpha, \beta, \tau) - \overline{0R}^2(\alpha, \beta, \tau)}{\tau} = 0 .$$

Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \pi$ be a subdivision of the interval $(0, \pi)$ of α. We consider (2) for the different $\alpha_i (i = 1, 2, \cdots, n)$, and we put $\Delta\bar{\alpha}_i = \sphericalangle Q(\alpha_{i-1}, \beta, \tau) 0 Q(\alpha_i, \beta, \tau)$. Now, multiplying by $\Delta\bar{\alpha}_i$, summing for i form 1 to n, and then carrying out an appropriate limit process, we obtain

$$(3) \qquad \lim_{\tau\to 0} \frac{\int_0^\pi \overline{0Q}^2 d\bar{\alpha} - \int_0^\pi \overline{0Q}^2 d\bar{\alpha}}{\tau} = 0 \,.$$

where the second integral of the numerator gives the surface-measure of the intersection of B and σ and, thus, its value equals π according to our condition, provided that $|\tau| < \eta$. (In this case the normal of σ makes with Γ a smaller angle than η).

Furthermore

$$\overline{0Q} = \overline{0\bar{P}} + \overline{\bar{P}Q} = r + q + o(\tau)$$

where o denotes the small ordo, introduced by Landau.[1] According to an elementary computation,

$$\varDelta\bar{\alpha} = \varDelta\alpha + o(\tau)\varDelta\alpha$$

and

$$(4) \qquad \begin{aligned}\int_0^\pi \overline{0Q}^2 d\bar{\alpha} &= \int_0^\pi r^2 d\alpha + 2\int_0^\pi rq\, d\alpha + \int_0^\pi q^2 d\bar{\alpha} + \int_0^\pi (r^2 + 2rq) o(\tau) d\alpha \\ &+ \int_0^\pi [2r + 2q + o(\tau)] o(\tau) d\bar{\alpha} \,.\end{aligned}$$

Substituting (4) into (3) and changing the order of the integration and of the limit process in some terms we obtain

$$(5) \qquad \begin{aligned}&\lim_{\tau\to 0} \frac{\int_0^\pi r^2 d\alpha - \int_0^\pi \overline{0R}^2 d\bar{\alpha}}{\tau} + 2\int_0^\pi \left[\lim_{\tau\to 0} \frac{rq}{\tau}\right] d\alpha \\ &+ \int_0^\pi \left[\lim_{\tau\to 0} \frac{q^2}{\tau}\right] d\bar{\alpha} + \int_0^\pi \lim_{\tau\to 0} \frac{[r^2 + 2rq] o(\tau)}{\tau}\, d\bar{\alpha} \\ &+ \int_0^\pi \lim_{\tau\to 0} \frac{[2r + 2q + o(\tau)] o(\tau)}{\tau}\, d\bar{\alpha} = 0 \,.\end{aligned}$$

The first term vanishes since $\int_0^\pi r^2 d\alpha = \text{Area}\,(B \cap \varepsilon) = \pi$. The two last terms vanish because of $\lim_{\tau\to 0} \{o(\tau)/\tau\} = 0$.

We show that $\lim_{\tau\to 0} (q^2/\tau) = 0$. From the triangle $PQ\bar{P}_\varDelta$ we have $q : rtg\nu = \sin\psi : \sin(\pi/2 - \psi - \nu)$. Thus, $q = rtg\nu \ \sin\psi/\cos(\psi + \nu)$, and, because of $|\tau| \gtreqqless |\nu|$

[1] $\lim_{\tau\to 0} \frac{\theta(\tau)}{\tau} = 0 \,.$

$$\left|\frac{q^2}{\tau}\right| \leqq \left|\frac{q^2}{\nu}\right| = r^2\left|\frac{tg\nu}{\nu}\right|\left|tg\nu\right|\frac{\sin^2\psi}{\cos^2(\psi+\nu)} \longrightarrow 0 .$$

if $\tau \longrightarrow 0$ and, thus $\nu \longrightarrow 0$.

Now, we transform the only remaining term $\int_0^\pi [\lim_{\tau\to 0}(rq/\tau)]d\alpha$ of (5). An elementary computation shows that $q = v + o(\tau)$, where $v = \overline{VP}$. (The sign of v equals that of q.) So (5) gives

$$\int_0^\pi \left[\lim_{\tau\to 0}\frac{rv}{\tau}\right]d\alpha = 0 . \tag{6}$$

Since Q is a function of α, β, τ and P a function of α, v is a function of α, β, τ. If $a \perp 0P$, then $\beta = \alpha \pm (\pi/2)$ $0 \leqq \beta < \pi.$, and then $\nu = \tau$. We put $v(\alpha, \alpha \pm \pi/2, T) \equiv v_0^*(\alpha)$. T being a fixed positive value of τ which satisfies the condition $T' < \eta$. The subscript "0" denotes the perpendicularity of a and $0P$, and the asterisk denotes $\tau = T$. Thus, $v^*: v_0^* = \overline{P\bar{P}}^*: \overline{P\bar{P}}_0^* = \hat{r}\, tg\, T: r\, tg\, T$ and $v^* = v_0^*(\hat{r}/r)$. Furthermore, $v^*: v = \hat{r}\, tgT: \hat{r}\, tg\tau$. Hence $v^* = v(tg\, T/tg\, \tau)$. Comparing this to the previous expression of v^* and taking in account the relation $\hat{r} = r\,|\sin(\alpha-\beta)|$, we have

$$v = v_0^*\frac{tg\,\tau}{tg\,T}\,|\sin(\alpha-\beta)|\,.$$

Substituting this into (6), and taking in account that r and v_0^* are functions of the α alone, we get

$$\int_0^\pi r(\alpha)v_0^*(\alpha)\left[\lim_{\tau\to 0}\frac{tg\,\tau}{\tau}\right]\frac{1}{tg\,T}\,|\sin(\alpha-\beta)|\,d\alpha = 0 .$$

Performing the limit process and reducing by the constant $1/tg\,T$, we obtain

$$\int_0^\pi r(\alpha)v_0^*(\alpha)\,|\sin(\alpha-\beta)|\,d\alpha = 0 . \tag{7}$$

B must be shaped so that this equation is fulfilled for any β between 0 and π.

We show that (7) can hold only for $v_0^*(\alpha) \equiv 0$. Put $r(\alpha)v_0^*(\alpha) \equiv \varphi(\alpha)$. Then (7) has the form

$$\int_0^\pi \varphi(\alpha)\,|\sin(\alpha-\beta)|\,d\alpha = 0 \qquad \text{for all } \beta \in (0,\pi) . \tag{8}$$

Let us suppose for the moment that $\varphi(\alpha)$ has a finite number of sign changes between 0 and π and is not identically zero. Let these be $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then

$$(9) \qquad \int_0^\pi \varphi(\alpha) \sum_{i=1}^{r} [a_i \,|\sin(\alpha - \alpha_i)| + b_i \,|\sin(\alpha - \beta_i)|] d\alpha = 0$$

for any a_i, b_i, β_i, since (9) is a linear combination of (8) for some special β which satisfy $\alpha_j < \beta_j < \alpha_{j+1} (j = 1, 2, \cdots, r-1)$; $\beta_r = \pi$. We consider the functions

$$\begin{aligned}
f_1(\alpha) &= -a_1 \sin(\alpha - \alpha_1) - b_1 \sin(\alpha - \beta_1) - a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
\tilde{f}_1(\alpha) &= +a_1 \sin(\alpha - \alpha_1) - b_1 \sin(\alpha - \beta_1) - a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
f_2(\alpha) &= +a_1 \sin(\alpha - \alpha_1) + b_1 \sin(\alpha - \beta_1) - a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
\tilde{f}_2(\alpha) &= +a_1 \sin(\alpha - \alpha_1) + b_1 \sin(\alpha - \beta_1) + a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
&\vdots \\
\tilde{f}_r(\alpha) &= +a_1 \sin(\alpha - \alpha_1) + b_1 \sin(\alpha - \beta_1) + a_2 \sin(\alpha - \alpha_2) \\
&\quad + b_2 \sin(\alpha - \beta_2) + \cdots + b_r \sin(\alpha - \beta_r) \,.
\end{aligned}$$

We denote by $\Phi(\alpha)$ the sum under the integral sign in (9). Then

$$\begin{aligned}
f_i(\alpha) &\equiv \Phi(\alpha) \qquad \beta_{i-1} \leqq \alpha \leqq \alpha_i \qquad (\beta_0 = 0) \\
\tilde{f}_i(\alpha) &\equiv \Phi(\alpha) \qquad \alpha_i \leqq \alpha \leqq \beta_i \qquad (i = 1, 2, \cdots, r) \,.
\end{aligned}$$

and

$$(10) \qquad f_i(\alpha_i) = \tilde{f}_i(\alpha_i)$$

as well as

$$\tilde{f}_j(\beta_j) = f_{j+1}(\beta_j) \qquad (j = 1, 2, \cdots, r-1) \,.$$

Each of the $f_i(\alpha)$ and of the $\tilde{f}_i(\alpha)$ has the form $A \sin \alpha + B \cos \alpha$. Thus each of them has only one zero between 0 and π defined by $tg\,\alpha = B/A$, and $f_i(\alpha)$ esp. $\tilde{f}_i(\alpha)$ changes sign at this point.

Making use of the $3n - 1$ parameters a_i, b_i, β_j we can achieve

$$(11) \qquad f_i(\alpha_i) = 0 \,.$$

and

$$(12) \qquad \operatorname{sgn} \frac{df_i}{d\alpha}(\alpha_i) = \operatorname{sgn} \frac{d\tilde{f}_i}{d\alpha}(\alpha_i) \,.$$

Then $\tilde{f}_i(\alpha_i) = 0$ is a consequence of (10). Thus the $\alpha_i (i = 1, 2, \cdots, r)$

are all the zeros and sign changes of $\Phi(\alpha)$ between 0 and π. $\varphi(\alpha)$ is continuous, since both $r(\alpha)$ and $v_0^*(\alpha)$ are so. Because of (12) and the continuity of $f(\alpha)$, $\Phi(\alpha)$ and $\varphi(\alpha)$ have the same sign over $(0, \pi)$, and they do not vanish identically. This implies a contradiction to (9). Therefore, $\varphi(\alpha)$ cannot have a finite number of sign changes. Hence, the sign changes of $\varphi(\alpha)$ have some accumulation points. Let us suppose that these accumulation points are not everywhere dense. Since $\varphi(\alpha)$ is continuous, cutting out the accumulation points (each with a suitable small neighborhood) changes the integral (8) by less than an arbitrary small number. Performing the previous process on the remaining domain of α we arrive again at a contradiction. Hence, the sing changes of $\varphi(\alpha)$ must be everywhere dense. This is impossible unless $\varphi(\alpha)$ vanishes identically:

$$\varphi(\alpha) = r(\alpha)v_0^*(\alpha) \equiv 0 . \tag{13}$$

If $r(\alpha)$ were zero for any α, then B would reduce to the point 0 by the starshapeness and the symmetry of B, and we would have Area $(B \cap \varepsilon) = 0 \neq \pi$. Thus we get $r(\alpha) \neq 0$, and from (13) $v_0^*(\alpha) \equiv 0$. This means that ε is perpendicular to the tangent planes to ∂B at the points of $\partial B \cap \varepsilon$.

Until now we used $A(\vec{n})$ for planes whose angle with ε is smaller than η. Then we have used $A(\vec{n})$ only for those $\vec{n}$ for which $(\vec{n}, z) < \eta$, z being the coordinate axis perpendicular to ε. The endpoints of these vectors $\vec{n}$ fill a small circle C on the unit sphere around "north pole" (resp. around its south pole).

Now we want to show that $\Sigma_p \perp 0P$. Instead of ε we choose a plane ε' through $0P$ whose angle with ε is $\omega (0 < \omega < T)$. Let us perform the foregoing consideration for this ε' replacing T by a T' such that $0 < T' < T - \omega$. These considerations lead to $\Sigma_p \perp \varepsilon'$. But from $\Sigma_p \perp \varepsilon, \varepsilon'$ it follows that Σ_p is perpendicular to the intersection of ε and ε', i.e., $\Sigma_p \perp 0P$. In this part of the proof we have used $A(\vec{n})$ again just for the $\vec{n}$ whose endpoints fill a circle C' on the unit sphere with a center inside C (because of $\omega < T$), and with a radius such that $C' \subset C$ (because of $T' < T - \omega$). Hence, we were able to proove $\Sigma_p \perp 0P; P \in \partial B \cap \varepsilon$, using $A(\vec{n})$ merely for $\vec{n}$ belonging to C.

Let us consider a pencil of planes through the x axis. Let us perform our whole proof for all the planes of the pencil. This leads us to the result that $\Sigma_p \perp 0P$ for each $P \in \partial B$. But a continuous surface with this property cannot be other than a sphere whose radius must be 1 because of $A(\vec{n}_0) = \pi$, $\vec{n}_0 \perp \varepsilon$. During this last step of the proof we used $A(\vec{n})$ for all planes of the pencil in a small circular neighborhood of the normal of the plane of the pencil on the unit sphere. The union of these circles forms a small strip around the great circle in the (y, z) plane of the unit sphere, as stated in the theorem.

We show an immediate application of the theorem. In a Minkowskian geometry M_n with indicatrix S, the surface area $|F|$ of a plane figure F is defined as the euclidean surface area $\lambda(F)$ of this figure F, multiplied by an appropriate factor. This factor is π divided by the euclidean surface-area of the intersection of the interior B of S and of the plane σ through the midpoint 0 of S and parallel to F:

$$|F| = \frac{\pi}{\lambda(B \cap \sigma)} \lambda(F)^2 . \tag{14}$$

The first factor on the right hand side of (14) is a function of the plane position only: $\pi/m(\sigma)$. H. Busemann [2] investigated affine measures where this factor is not deduced from a convex symmetric body B but it is rather arbitrarily given. From the theorem of this paper follows again an answer to the question treated by H. Busemann, whether or not an arbitrary continuous $m(\sigma)$ can always be derived from a symmetric star-shaped body B as surface area of its intersections with planes through its midpoint.[3] Namely, taking a differentiable function $m(\sigma)$ over the positions of planes in 3-space, (a) having value π on the planes, whose normals make an angle with the y, z plane smaller than η, and (b) having values differing from π elsewhere, we get a function which cannot be derived from any symmetrical body in the above sense (i.e., so that $m(\sigma) \equiv$ Area $(B \cap \sigma)$). Namely the body B ought to be a sphere because of (a), according to our theorem; on the other hand it cannot be a sphere, since no sphere satisfies (a) and (b) simultaneously. Thus this $m(\sigma)$ presents a simple and concrete negative example. Negative examples for higher dimensional spaces are any functions over the position of planes in n-space that are differentiable extensions of the previous function.

I wish to thank Professor Heinrich Guggenheimer for valuable conversations concerning this matter.

REFERENCES

1. W. Blaschke, *Kreis und Kugel*, New York, 1949.
2. H. Busemann, *Areas in affine spaces* I, II, Rend. Circ. Mat. Palermo **9** (1960), 81-90.
3. ———, *Areas in affine spaces* III—*The integral geometry of affine area*, Rend. Circ. Mat. Palermo **9** (1960), 226-242.
4. P. Funk, *Über Flächen mit lauter geschlossenen geodätischen Linien*, Math. Ann. **74** (1913), 278-300.

[2] See H. Rund [**5**], p. 39.

[3] H. Busemann [**3**], see also W. Blaschke [**1**] pp. 154, 155.

5. H. Rund, *The differential geometry of Finsler spaces*, Berlin, 1959.

Received April 26, 1968. This research was accomplished while the author was receiving a Ford Foundation Grant.

POLYTECHNIC INSTITUTE OF BROOKLYN, NEW YORK
AND
UNIVERSITY OF DEBRECEN, HUNGARY