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ADDITIONAL RESULTS ON MODULES OVER POLYDISC ALGEBRAS

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This paper deals with a class $\mathscr{K}_N$ of domains in Stein manifolds and with certain algebras of holomorphic functions naturally associated with them.

The class $\mathscr{K}_N$ consists of those relatively compact domains $\varDelta$ in N-dimensional Stein manifolds such that for some neighborhood Ω of $\bar{\varDelta}$ and some neighborhood W of $\bar{U}^N$, the closure of $U^N = \{(z_1, \cdots, z_N) \in C^N: |z_1|, \cdots, |z_N| < 1\}$, the unit polydisc in C^N, there exists a proper holomorphic map $\Phi: \Omega \rightarrow W$ which is nonsingular at every point of $\Phi^{-1}(T^N)$, T^N the distinguished boundary of U^N, and which has, in addition, the property that $\varDelta = \Phi^{-1}(U^N)$. The collection of all such maps Φ is denoted by $\mathscr{M}(\varDelta, U^N)$, and if $\varDelta, \varDelta' \in \mathscr{K}_N$, $\mathscr{M}(\varDelta, \varDelta')$ denotes the set of all maps Ψ: $\varDelta \rightarrow \varDelta'$ such that if $\Phi \in \mathscr{M}(\varDelta', U^N)$, then $\Phi \circ \Psi \in \mathscr{M}(\varDelta, U^N)$. For $\varDelta \in \mathscr{K}_N$ let $\mathscr{A}(\varDelta) = \{f \in \mathscr{C}(\bar{\varDelta})$: f is holomorphic in $\varDelta\}$, and let $H^\infty(\varDelta) = \{f$: f is holomorphic and bounded in $\varDelta\}$. If $\Phi \in \mathscr{M}(\varDelta, \varDelta')$, then $\mathscr{A}(\varDelta)$ is a module over its subalgebra $\Phi^*\mathscr{A}(\varDelta') = \{f \circ \Phi$: $f \in \mathscr{A}(\varDelta')\}$, and this paper treats the structure of $\mathscr{A}(\varDelta)$ as a $\Phi^*\mathscr{A}(\varDelta')$-module. The first section of the paper presents an example to show that $\mathscr{A}(\varDelta)$ need not be free over $\Phi^*\mathscr{A}(\varDelta')$, and the second section shows that it is a finitely generated, projective $\Phi^*\mathscr{A}(\varDelta')$-module. The final section establishes certain conditions sufficient for the freeness of $\mathscr{A}(\varDelta)$. Parallel results obtain for $H^\infty(\varDelta)$ as a $\Phi H^\infty(\varDelta')$-module.

These results supplement results obtained in [7]. In that paper some of these questions were treated for the special case that $\varDelta = U^N$. For example, it was shown there that if $\Phi \in \mathscr{M}(\varDelta, U^N)$, then $\mathscr{A}(\varDelta)$ is a free module over $\Phi^*\mathscr{A}(U^N)$. We refer to this paper for some of the elementary properties of the elements of $\mathscr{K}_N$ and of $\mathscr{M}(\varDelta', \varDelta)$.

Given $\varDelta', \varDelta \in \mathscr{K}_N$, and $\Psi \in \mathscr{M}(\varDelta; \varDelta)$, the triple $(\varDelta; \Psi \mid \varDelta', \varDelta)$ is an analytic cover in the sense of [5] and consequently has a well defined multiplicity λ: λ is that integer such that for all points $\mathfrak{z} \in \varDelta'$ off a variety, the set $\Psi^{-1}(\mathfrak{z})$ consists of λ points.

If M is a complex manifold, $\mathscr{S}$ a sheaf on M, and $\mathfrak{z}$ a point of M, then $\mathscr{S}_\mathfrak{z}$ denotes the stalk of $\mathscr{S}$ at $\mathfrak{z}$ and $\mathscr{O}_M$ denotes the sheaf of germs of functions holomorphic on M. We will usually write $\mathscr{O}_\mathfrak{z}$ instead of $(\mathscr{O}_M)_\mathfrak{z}$. If K is a subset of M, $\mathscr{O}(K)$ denotes the sections of $\mathscr{O}_M$ over K.

1. An example.

If $\varDelta, \varDelta' \in \mathscr{K}_1$ and $\Psi \in \mathscr{M}(\varDelta', \varDelta)$, then [1] $\mathscr{A}(\varDelta')$

is a free module over $\Psi^*\mathscr{A}(\Delta)$ whose rank is the multiplicity of Ψ. In higher dimensions the analogous result is not true as the following example shows.

We will show that for $N = 4, 5, 6, \cdots$, there exist $\Delta, \Delta' \in \mathscr{K}_N$ and $\Psi \in \mathscr{M}(\Delta', \Delta)$ such that $\mathscr{A}(\Delta')$ admits no set of generators over $\Psi^*\mathscr{A}(\Delta)$ consisting of λ elements, λ the multiplicity of Ψ. In our example Ψ will be two-to-one and a local homeomorphism at each point of Δ. Denote by $P_N(C)$ and $P_N(R)$ respectively N-dimensional complex and real projective space. In $P_N(C)$, $N \geqq 4$, consider the manifold V consisting of those points with homogeneous coordinates $(z_0, \cdots, z_N)$ such that $z_0^2 + \cdots + z_N^2 \neq 0$. In the case $N = 2$, this manifold was considered in another connection by Forster [3]. The manifold V is connected since it is the complement in $P_N(C)$ of a variety, and, as Forster remarked, it is a Stein manifold. The space $P_N(R)$ is contained in a natural way in V: $P_N(R)$ is the set of all points which admit real homogeneous coordinates, and, moreover, $P_N(R)$ is a deformation retract of V. This was the fact which made V useful for Forster, and it is the essential point in the present example. A deformation of V onto $P_N(R)$ can be given explicitly as follows [3]. If $\mathfrak{z} \in V$, let $(z_0, \cdots, z_N)$ be homogeneous coordinates for $\mathfrak{z}$ such that $z_0^2 + \cdots + z_N^2 > 0$. Given $t \in [0, 1]$, define $H_t(\mathfrak{z})$ to be the point with homogeneous coordinates $(x_0 + ity_0, \cdots, x_N + ity_N)$ if $z_j = x_j + iy_j$. Thus, V and $P_N(R)$ are of the same homotopy type and in particular they have the same fundamental group, Z_2[1]. Consequently, if $\tilde{V}$ denotes the universal covering manifold of V and η: $\tilde{V} \to V$ the natural projection, then η is a local homeomorphism and each fiber $\eta^{-1}(p)$ consists of exactly two points. The manifold $\tilde{V}$ admits a complex structure with respect to which η is a holomorphic map, and when $\tilde{V}$ is endowed with this complex structure, it becomes a Stein manifold. (For the fact that $\tilde{V}$ is Stein, see [8]).

We set $S = \eta^{-1}(P_N(R))$, and we shall show that S is, topologically, the N-sphere. Since η is a local homeomorphism, S is evidently an N-dimensional manifold. A priori it is not clear that S is connected; let S_0 be a component of S. Then η carries S_0 onto $P_N(R)$, and with the projection η, S_0 is a covering space of $P_N(R)$. Let $p_0 \in P_N(R)$ and let $s_0 \in \eta^{-1}(p_0) \cap S_0$. Let i: $(P_N(R), p_0) \to (V, p_0)$ and j: $(S_0, s_0) \to (\tilde{V}, s_0)$ be inclusion maps. We then have induced homomorphisms of the fundamental groups

$$\begin{aligned} i_*&: \pi_1(P_N(R), p_0) \to \pi_1(V, p_0), \\ j_*&: \pi_1(S_0, s_0) \to \pi_1(\tilde{V}, s_0), \\ \eta_*&: \pi_1(\tilde{V}, s_0) \to \pi_1(V, p_0), \\ (\eta \mid S_0)_*&: \pi_1(S_0, s_0) \to \pi_1(P_N(R), p_0) . \end{aligned}$$

[1] The integers mod 2.

There is the commutativity relation $\eta_* j_* = i_*(\eta \mid S_0)_*$. Since $\tilde{V}$ is the universal covering space of V, $\eta_* = 0$, and since $P_N(R)$ is a deformation retract of V, i_* is an isomorphism. Consequently $(\eta \mid S_0)_* = 0$. From the uniqueness of covering spaces corresponding to a given subgroup of the fundamental group (see, e.g. [6, Th. 6.6.16]) it follows that S_0 is homeomorphic to the universal covering space of $P_N(R)$, i.e., to the N-sphere and that $\eta \mid S_0$ is two-to-one. Since η is two-to-one on $\tilde{V}$, we have that $S = S_0$ and consequently that S is an N-sphere.

Next we show the existence of a $\varDelta \in \mathscr{K}_N$ which contains $P_N(R)$ and which is contained in V. For this purpose, define a map $\varPhi$: $V \to \boldsymbol{C}^N$ by setting

$$\varPhi(\mathfrak{z}) = \left(\frac{z_1^2}{z_0^2 + \cdots + z_N^2}, \cdots, \frac{z_N^2}{z_0^2 + \cdots + z_N^2}\right)$$

if $\mathfrak{z} \in V$ has homogeneous coordinates $(z_0, \cdots, z_N)$. The $\varPhi$ so defined is proper. If not, there is a sequence $\{\zeta^{(k)}\}_{k=1}^{\infty}$ in $\boldsymbol{C}^N$ which converges to $\zeta^{(0)} \in \boldsymbol{C}^N$ such that for some choice of points $\mathfrak{z}_k \in \varPhi^{-1}(\zeta^{(k)})$, $\mathfrak{z}_k \to \mathfrak{z}_0 \in P_N(C)\backslash V$. Let $\mathfrak{z}_0$ have homogeneous coordinates $(z_0^{(0)}, \cdots, z_N^{(0)})$. Then $\Sigma(z_j^{(0)})^2 = 0$. We can choose homogeneous coordinates $(z_0^{(k)}, \cdots, z_N^{(k)})$ for $\mathfrak{z}_k$ so that for fixed j, $0 \leqq j \leqq N$, $z_j^{(k)} \to z_j^{(0)}$. Let $\zeta^{(0)} = (\zeta_1^{(0)}, \cdots, \zeta_N^{(0)})$. Since $\varPhi(\mathfrak{z}_k) \to \zeta^{(0)}$, we have, for $1 \leqq j \leqq N$,

$$\zeta_j^{(0)} = \lim_{k\to\infty} (z_j^{(k)})^2((z_0{}^{(k)})^2 + \cdots + (z_0^{(k)})^2)^{-1} ,$$

and since $\mathfrak{z}_k \to \mathfrak{z}_0$ and $\mathfrak{z}_0 \notin V$, this implies that $z_j^{(k)} \to 0$. From $z_j^{(k)} \to z_j^{(0)}$, we conclude that $z_j^{(0)} = 0$ for $1 \leqq j \leqq N$. The fact that $\Sigma(z_j^{(0)})^2 = 0$ implies that $z_0^{(0)} = 0$, so $(z_0^{(0)}, \cdots, z_N^{(0)}) = (0, \cdots, 0)$ which is impossible. Thus $\varPhi$ is proper. A short calculation shows that with the exception of the points in the variety

$$E = \{(\zeta_1, \cdots, \zeta_N) \in \boldsymbol{C}^N\colon \quad \zeta_1 \cdots \zeta_N = 0\} ,$$

every point of $\boldsymbol{C}^N$ has exactly 2^N preimages under $\varPhi$ so the multiplicity of $\varPhi$ is 2^N. This remark also indicates that $\varPhi$ is regular at each point of the sets $\varPhi^{-1}(\{(\zeta_1, \cdots, \zeta_N)\colon |\zeta_1| = \cdots = |\zeta_N| = R\})$. If $\mathfrak{z} \in V$ has real homogeneous coordinates, the definition of $\varPhi(\mathfrak{z})$ shows that $\varPhi(\mathfrak{z})$ lies in $\bar{U}^N$, so if $\varepsilon > 0$, then the set

$$\varDelta_\varepsilon = \varPhi^{-1}\{(z_1, \cdots, z_N) \in \boldsymbol{C}^N\colon \quad |z_j| < 1 + \varepsilon \text{ for } j = 1, \cdots, N\}$$

is an element of $\mathscr{K}_N$ with the desired property.

Let $P_N(R) \subset \varDelta \subset V$, $\varDelta \in \mathscr{K}_N$, and let $\varDelta' = \eta^{-1}(\varDelta)$. The mapping η: $\tilde{V} \to V$ is a covering map and so is certainly an element of $\mathscr{M}(\varDelta', \varDelta)$. Assume that $\mathscr{A}(\varDelta')$ is generated as a module over $\eta^*\mathscr{A}(\varDelta)$ by two elements, F_1 and F_2, so that if $f \in \mathscr{A}(\varDelta')$ then for some $g_1, g_2 \in \mathscr{A}(\varDelta)$

we have that $f(\mathfrak{z}) = F_1(\mathfrak{z})g_1(\eta(\mathfrak{z})) + F_2(\mathfrak{z})g_2(\eta(\mathfrak{z}))$. In particular, the functions F_1 and F_2 must separate points on each of the fibers $\eta^{-1}(\mathfrak{z})$, $\mathfrak{z} \in P_N(R)$. Let S^N be the standard N-sphere in R^{N+1} and denote by $\xi: S^N \rightarrow P_N(R)$ the usual covering map which identifies antipodal points. If τ: $S^N \rightarrow S$ is a homeomorphism such that $\xi = \eta \circ \tau$, then the mapping $S^N \rightarrow C^2$ given by $\mathfrak{z} \rightarrow (F_1(\tau(\mathfrak{z})), F_2(\tau(\mathfrak{z})))$ is continuous and separates antipodal points. Since C^2 is, topologically, R^4, and since we have assumed $N \geqq 4$, we have obtained a contradiction to the Borsuk-Ulam Theorem [6, Corollary 4.3.7]. Consequently, $\mathscr{A}(\varDelta')$ is not generated by two elements over $\eta^*\mathscr{A}(\varDelta)$. This argument shows, in fact, that any set of generators for $\mathscr{A}(\varDelta')$ must contain more than $N/2$ generators.

2. $\mathscr{A}(\varDelta')$ **as a module over** $\varPsi^*\mathscr{A}(\varDelta)$. Complementing the previous example, we have the following result.

THEOREM 2.1. *If* $\varDelta', \varDelta \in \mathscr{K}_N$ *and* $\varPsi \in \mathscr{M}(\varDelta', \varDelta)$, *then* $\mathscr{A}(\varDelta')$ *and* $H^\infty(\varDelta')$ *are finitely generated projective modules over* $\varPsi^*\mathscr{A}(\varDelta)$ *and* $\varPsi^*H^\infty(\varDelta)$ *respectively.*

Proof. It is easy to see that $\mathscr{A}(\varDelta')$ is finitely generated over $\varPsi^*\mathscr{A}(\varDelta)$ and that a similar result obtains concerning $H^\infty(\varDelta')$. Let $\varPsi \in \mathscr{M}(\varDelta, U^N)$. Then $\varPhi \circ \varPsi \in \mathscr{M}(\varDelta', U^N)$, and consequently $\mathscr{A}(\varDelta')$ is finitely generated over $(\varPhi \circ \varPsi)^*\mathscr{A}(U^N)$: We know by [7, Th. I. 4] that for some $B_1, \cdots, B_m \in \mathscr{O}(\bar{\varDelta}')$, each $f \in \mathscr{A}(\varDelta)$ is of the form

$$f = \sum B_j f_j \circ \varPhi \circ \varPsi$$

for some choice of f_j in $\mathscr{A}(U^N)$. Since $f \circ \varPhi$ is in $\mathscr{A}(\varDelta)$, this shows that $\mathscr{A}(\varDelta')$ is finitely generated over $\varPsi^*\mathscr{A}(\varDelta)$. The case of $H^\infty(\varDelta')$ can be treated in the same way. Somewhat more is required to show that these modules are projective.

Since $\varPsi \in \mathscr{M}(\varDelta', \varDelta)$, there is a neighborhood $\varOmega'$ of $\bar{\varDelta}'$ which is mapped properly onto a neighborhood $\varOmega$ of $\bar{\varDelta}$ by $\varPsi$. We know [7, Lemma 1. 2] that the direct image sheaf $\varPsi * \mathscr{O}_{\varOmega'}$ is locally free of rank λ, λ the multiplicity of $\varPsi$ on $\varOmega$. Let W be a relatively compact open set which is a Stein manifold and which satisfies $\varOmega \supset \bar{W} \supset W \supset \bar{\varDelta}$. By Cartan's Theorem A and compactness, there exist

$$\tilde{F}_1, \cdots, \tilde{F}_q \in \Gamma(\varOmega, \varPsi * \mathscr{O}_\varOmega)$$

such that if $\mathfrak{z} \in \bar{W}$, then the germs $(\tilde{F}_j)_\mathfrak{z}$ generate $(\varPsi * \mathscr{O}_{\varOmega'})_\mathfrak{z}$.[2] Thus, if H is the sheaf homomorphism $\mathscr{O}^q \rightarrow \varPsi_* \mathscr{O}_{\varOmega'}$ given by $H(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_q) = \sum \boldsymbol{f}_j(\tilde{F}_j)_\mathfrak{z}$ for all $\boldsymbol{f}_j \in \mathscr{O}_\mathfrak{z}$ then

[2] By a result of Forster and Ramspott [4, Satz 2], we can take $q = \lambda + [N/2]$.

$$(1) \qquad \mathcal{O}^q \xrightarrow{H} \Psi_* \mathcal{O}_{\Omega'} \longrightarrow 0$$

is exact over W.

We now need a lemma which is surely well known but for which we are unable to provide a reference. (Remarks of the referee have enabled us to abbreviate our discussion of this lemma).

LEMMA 2.2. *If* $\mathcal{S}_1 \xrightarrow{H} \mathcal{S}_2 \to 0$ *is an exact sequence of locally free sheaves over a complex manifold* M, *then* $\ker H$ *is locally free.*

Proof. The sheaves $\mathcal{S}_1$ and $\mathcal{S}_2$ are locally free and the question is local, so we can suppose $\mathcal{S}_1 = \mathcal{O}^p$, $\mathcal{S}_2 = \mathcal{O}^q$. Since $\ker H$ is coherent, it suffices to prove that for each $\mathfrak{z} \in M$, the stalk $(\ker H)_\mathfrak{z}$ is a free $\mathcal{O}_\mathfrak{z}$-module. (See Lemma 1.2 of [7]). We have the exact sequence

$$0 \longrightarrow \ker H \longrightarrow \mathcal{O}^p \longrightarrow \mathcal{O}^q \longrightarrow 0$$

so we can invoke [2, I. 2.5, Proposition 5 and II. 5.2, Corollary 2] to conclude that $(\ker H)_\mathfrak{z}$ is a projective $\mathcal{O}_\mathfrak{z}$-module. Since projective modules over local rings are free, we can conclude that $(\ker H)_\mathfrak{z}$ is free as desired.

To continue with the proof of the theorem, we apply the lemma to the sequence

$$(2) \qquad 0 \longrightarrow \ker H \longrightarrow \mathcal{O}^q \xrightarrow{H} \Psi * \mathcal{O}_{\Omega'} \longrightarrow 0$$

over W obtained from (1), and we find that this is an exact sequence of locally free sheaves. Consequently [4, Th. VIII C 7], the sequence (2) splits, and in particular there is a sheaf isomorphism L of $\Psi_*\mathcal{O}_{\Omega'}$ into $\mathcal{O}^q$ such that $H \circ L$ is the identity on $\Psi_* \mathcal{O}_{\Omega'}$ and $L \circ H$ is a projection of $\mathcal{O}^q$ onto the range of L. Apply this to the spaces of sections over $\bar{\Delta}$ and pull the resulting statement back to $\bar{\Delta}'$ by way of the map Ψ. We find that there are exact sequences of $\Psi^*\mathcal{O}(\bar{\Delta})$-module homomorphisms

$$0 \longrightarrow \mathcal{O}(\bar{\Delta}') \xrightarrow{L'} (\Psi^*\mathcal{O}(\bar{\Delta}))^q$$

and

$$(\Psi^*\mathcal{O}(\bar{\Delta}))^q \xrightarrow{H'} \mathcal{O}(\bar{\Delta}') \longrightarrow 0 .$$

Here $H'(f_1 \circ \Psi, \cdots, f_q \circ \Psi) = \sum F_j f_j \circ \Psi$ where $F_j \in \mathcal{O}(\bar{\Delta}')$ corresponds to the section $\tilde{F}_j$ of $\Psi_*\mathcal{O}_{\Omega'}$, $L' \circ H'$ projects $\Psi^*\mathcal{O}(\bar{\Delta})$ onto the range of L', and $H' \circ L'$ is the identity on $\mathcal{O}(\bar{\Delta}')$. This establishes $\mathcal{O}(\bar{\Delta}')$ as a finitely generated, projective $\Psi^*\mathcal{O}(\bar{\Delta})$-module.

To treat $\mathcal{A}(\Delta')$ and $H^\infty(\Delta')$, note that since L is a sheaf isomorphism, L' extents to an isomorphism L'' of $\mathcal{O}(\Delta')$ into $(\Psi^*\mathcal{O}(\Delta))^q$ and that H' extends to a homomorphism H'' of $(\Psi^* \mathcal{O}(\Delta))^q$ to $\mathcal{O}(\Delta')$. The form of H shows that H'' carries $(\Psi^*\mathscr{A}(\Delta))^q$ into $\mathscr{A}(\Delta')$ and $(\Psi^*H^\infty(\Delta))^q$ into $H^\infty(\Delta')$. In fact H'' carries $(\Psi^*\mathscr{A}(\Delta))^q$ and $(\Psi^*H^\infty(\Delta))^q$ onto $\mathscr{A}(\Delta')$ and $H^\infty(\Delta')$ respectively. As we noted at the beginning of the proof, if $f \in \mathscr{A}(\Delta')$, then $f = \sum B_j f_j \circ \Psi$ for some $f_j \in \mathscr{A}(\Delta)$ and some $B_j \in \mathcal{O}(\bar{\Delta}')$. We have $B_j = H'(\tilde{B}_j)$ for some $\tilde{B}_j \in (\Psi^* \mathcal{O}(\bar{\Delta}))^q$. On $(\Psi^*\mathscr{A}(\Delta))^q$, H'' acts as a $\Psi^*\mathscr{A}(\Delta)$-module homomorphism, so we have $f = H''(\sum \tilde{B}_j f_j \circ \Psi)$. The case that f lies in $H^\infty(\Delta')$ may be treated in a similar way.

Also, L'' carries $\mathscr{A}(\Delta')$ into $(\Psi^*\mathscr{A}(\Delta))^q$ and $H^\infty(\Delta')$ into $(\Psi^*H^\infty(\Delta))^q$. If $f \in \mathscr{A}(\Delta')$, we write $f = \sum B_j f_j \circ \Psi$ as above. Then $L''f = \sum f_j \circ L''B_j$. We have $L''B_j \in (\Psi^*\mathcal{O}(\bar{\Delta}))^q \subset (\Psi^*\mathscr{A}(\Delta))^q$. Thus $L''f$ is in $(\Psi^*\mathscr{A}(\Delta))^q$ as asserted. The H^∞ case follows in the same way.

The operator $L'' \circ H''$ acts on $(\Psi^*\mathscr{A}(\Delta))^q$ as a projection with range the range of L'' on $\mathscr{A}(\Delta')$. To see this, note first that the range of $L'' \circ H''$ is $L''(\mathscr{A}(\Delta'))$, for H'' carries $(\Psi^*\mathscr{A}(\Delta))^q$ onto $\mathscr{A}(\Delta')$. If $f \in \mathscr{A}(\Delta')$, then since $H'' \circ L''$ is the identity, we find that $L'' \circ H''(L''f) = L''f$, so $L'' \circ H''$ is a projection. Since L'' takes $\mathscr{A}(\Delta')$ isomorphically into $(\Psi^*\mathscr{A}(\Delta))^q$, we have proved that $\mathscr{A}(\Delta')$ is a projective $\Psi^*\mathscr{A}(\Delta)$-module. In the same way, it follows that $H^\infty(\Delta')$ is a projective $\Psi^*H^\infty(\Delta)$-module, and the proof of the theorem is concluded.

3. Criteria for the freeness of $\mathscr{A}(\Delta')$ over $\Psi^*\mathscr{A}(\Delta)$. There are certain cases in which $\mathscr{A}(\Delta')$ is necessarily free over $\Psi^*\mathscr{A}(\Delta)$. To introduce some of these we need to consider products. Suppose $\Delta_1 \in \mathscr{K}_{N_1}$, $\Delta_2 \in \mathscr{K}_{N_2}$, and let $\Phi_j \in \mathscr{M}(\Delta_j, U^{N_j})$. If Ω_j is a neighborhood of $\bar{\Delta}_j$ which is mapped properly into the neighborhood W_j of $\bar{U}^{N_j}$, $j = 1, 2$, then the map Ψ: $\Omega_1 \times \Omega_2 \to W_1 \times W_2$ given by

$$\Psi(\mathfrak{z}_1, \mathfrak{z}_2) = (\Phi_1(\mathfrak{z}_1), \Phi_2(\mathfrak{z}_2)) \in W_1 \times W_2$$

is proper, and $\Delta_1 \times \Delta_2 = \Psi^{-1}(U^{N1+N2})$. Moreover, Ψ is a local homeomorphism at each point of $\Psi^{-1}(T^{N1+N2})$. Thus $\Delta_1 \times \Delta_2 \in \mathscr{K}_{N1+N2}$, and $\Psi \in \mathscr{M}(\Delta_1 \times \Delta_2, U^{N1+N2})$. Similarly, if we are given Δ_1' and Δ_2' in $\mathscr{K}_{N1}$ and $\mathscr{K}_{N2}$ respectively and if $\Phi_j \in \mathscr{M}(\Delta_j', \Delta_j)$, $j = 1, 2$, then the map $\Phi_1 \times \Phi_2$ from $\Delta_1' \times \Delta_2'$ to $\Delta_1 \times \Delta_2$ defined by $\Phi_1 \times \Phi_2(\mathfrak{z}_1, \mathfrak{z}_2) = (\Phi_1(\mathfrak{z}_1), \Phi_2(\mathfrak{z}_2))$ is an element of $\mathscr{M}(\Delta_1' \times \Delta_2', \Delta_1 \times \Delta_2)$. If Φ_j has multiplicity λ_j, then $\Phi_1 \times \Phi_2$ has multiplicity $\lambda_1\lambda_2$.

THEOREM 3.1. *If for $j = 1, 2$, $\Delta_j, \Delta_j' \in \mathscr{K}_{N_j}$, if $\Phi_j \in \mathscr{M}(\Delta_j', \Delta_j)$ is of multiplicity λ_j, and if $\mathscr{A}(\Delta_j')$ is free of rank λ_j over $\Psi^*\mathscr{A}(\Delta_j)$, then $\mathscr{A}(\Delta_1' \times \Delta_1')$ is free of rank $\lambda_1\lambda_2$ over $(\Phi_1 \times \Phi_2)^*\mathscr{A}(\Delta_1 \times \Delta_2)$.*

Before giving the proof of this theorem, let us mention that by Theorem 2.3 of [7], if $\Phi \in \mathscr{M}(U^N, U^N)$, then in an obvious extension of the above notation, $\Phi = \varphi_1 \times \cdots \times \varphi_N$ where each $\varphi_j \in \mathscr{M}(U, U)$ is a finite Blaschke product.

Proof. Let $\{F_1^{(1)}, \cdots, F_{\lambda_1}^{(1)}\}$ be a free basis for $\mathscr{A}(\Delta_1')$ over $\Phi_1^* \mathscr{A}(\Delta_1)$ and let $\{F_1^{(2)}, \cdots, F_{\lambda_2}^{(2)}\}$ be one for $\mathscr{A}(\Delta_2')$ over $\Phi_2^* \mathscr{A}(\Delta_2)$. We assert that the set $\{F_j^{(1)} F_k^{(2)}: \ 1 \leqq j \leqq \lambda_1, 1 \leqq k \leqq \lambda_2\}$ is a free basis for $\mathscr{A}(\Delta_1' \times \Delta_2')$ over $(\Phi_1 \times \Phi_2)^* \mathscr{A}(\Delta_1 \times \Delta_2)$.

Define $E_1(\mathfrak{z})$ for $\mathfrak{z} \in \bar{\Delta}_1'$ to be the set $\Phi_1^{-1}(\Phi_1(\mathfrak{z}))$, and define $E_2(\mathfrak{z})$ for $\mathfrak{z} \in \bar{\Delta}_2'$ in an analogous way. In general $E_1(\mathfrak{z})$ will consist of λ_1 points. Since $\{F_k^{(1)}\}$ is a free basis for $\mathscr{A}(\Delta_1')$ over $\Phi_1^* \mathscr{A}(\Delta_1)$, each $f \in \mathscr{A}(\Delta_1')$ has a unique expression in the form

$$f(\mathfrak{z}) = F_1^{(1)}(\mathfrak{z}) f_1(\Phi_1(\mathfrak{z})) + \cdots + F_{\lambda_1}^{(1)}(\mathfrak{z}) f_{\lambda_1}(\Phi_1(\mathfrak{z}))$$

with $f_1, \cdots, f_{\lambda_1} \in \mathscr{A}(\Delta_1)$. The functions $f_j \circ \Phi_1$ can be computed explicitly by Cramer's rule: $f_j(\Phi_1(\mathfrak{z})) = D_j(\mathfrak{z}) D^{-1}(\mathfrak{z})$ where, for $\mathfrak{z} \in \bar{\Delta}'$ such that $E_1(\mathfrak{z})$ consists of λ_1 distinct points, say $E_1(\mathfrak{z}) = \{\mathfrak{z}_1, \cdots, \mathfrak{z}_{\lambda_1}\}$, we have

$$D(\mathfrak{z}) = \det (F_j^{(1)}(\mathfrak{z}_k))_{1 \leqq j,k \leqq \lambda_1} ,$$

and $D_j(\mathfrak{z})$ is obtained from $D(\mathfrak{z})$ by replacing the j^{th} column by the column vector $(f(\mathfrak{z}_1), \cdots, f(\mathfrak{z}_{\lambda_1}))$.

If we are given an element $G \in \mathscr{A}(\Delta_1' \times \Delta_2')$, then for fixed $w \in \bar{\Delta}_2'$, the preceding remarks may be applied to the element $G(\cdot, w)$ of $\mathscr{A}(\Delta_1')$:

$$G(\mathfrak{z}, w) = \sum_{j=1}^{\lambda_1} F_j^{(1)}(\mathfrak{z}) f_j(\Phi_1(\mathfrak{z}), w)$$

where, for fixed w, $f_j(\cdot, w) \in \mathscr{A}(\Delta_1)$. The expression for f_j as a certain quotient of determinants shows that $f_j(\Phi_1(\mathfrak{z}), w)$ is in fact an element of $\mathscr{A}(\Delta_1' \times \Delta_2')$. Thus, for fixed $\mathfrak{z}$, we can write

$$f_j(\Phi_1(\mathfrak{z}), w) = \sum_{k=1}^{\lambda_2} F_k^{(2)} w g_{j,k}(\mathfrak{z}, \Phi_2(w)) .$$

Again, we can compute the functions $g_{j,k}$ explicitly: If $E_2(w) = \{w_1, \cdots, w_{\lambda_2}\}$, then $g_{j,k}(\mathfrak{z}, \Phi_2(w)) = \tilde{D}_k(\mathfrak{z}, \Phi_2(w)) \tilde{D}(\mathfrak{z}, \Phi_2(w))^{-1}$ where, as before,

$$\tilde{D}(\mathfrak{z}, \Phi_2(w)) = \det (F_k^{(2)}(w_m))_{1 \leqq k,m \leqq \lambda_2}$$

and $\tilde{D}_k(\mathfrak{z}, \Phi_2(w))$ is obtained by replacing the k^{th} column of $\tilde{D}$ by the column vector $(f_j(\Phi_1(\mathfrak{z}), w_1), \cdots, f_j(\Phi_1(\mathfrak{z}), w_{\lambda_2}))$. This representation for $g_{j,k}$ shows that for fixed w, $g_{j,k}(\mathfrak{z}, \Phi_2(w))$ is, as a function of $\mathfrak{z}$, constant on the set $E_1(\mathfrak{z})$. Thus, we can write $g_{j,k}(\mathfrak{z}, \Phi_2(w)) = h_{j,k}(\Phi_1(\mathfrak{z}), \Phi_2(w))$ for some suitably chosen $h_{j,k} \in \mathscr{A}(\Delta_1 \times \Delta_2)$. We now have the representation

$$G(\mathfrak{z}, w) = \Sigma F_j^{(1)}(\mathfrak{z}) F_k^{(2)}(w) h_{j,k}(\Phi_1(\mathfrak{z}), \Phi_2(w))$$

for G. Thus, $\{F_j^{(1)}F_k^{(2)}\}$ is a set of generators for $\mathscr{A}(\Delta_1' \times \Delta_2')$ over $(\Phi_1 \times \Phi_2)^* \mathscr{A}(\Delta_1 \times \Delta_2)$.

That $F_j^{(1)}F_k^{(2)}$ are free generators is now clear: If there were a nontrivial relation

$$0 = \Sigma F_j^{(1)}(\mathfrak{z})F_k^{(2)}(w)h_{j,k}(\Phi_1(\mathfrak{z}), \Phi_2(w)) .$$

Then for some fixed choice of w we could regard this as a nontrivial relation among the functions $F_1^{(1)}, \cdots, F_{\lambda_1}^{(1)}$. But since $\{F_j^{(1)}\}$ is a free basis for $\mathscr{A}(\Delta_1')$, no such relation can exist, and the theorem is proved.

It is clear that a similar result obtains for products of more than two elements of $\mathscr{K}_N$ and that an analogous theorem holds for bounded functions.

We saw in [7] that if $\Delta \in \mathscr{K}_N$ and $\Phi \in \mathscr{M}(\Delta, U^N)$, then $\mathscr{A}(\Delta)$ is a free module over $\Phi^*\mathscr{A}(U^N)$. The essential ingredient of this proof is the fact that for some neighborhood Ω of $\bar{\Delta}$ and some neighborhood W of U^N, the sheaf $\Phi_*\mathscr{O}_\Omega$ is a free sheaf over W. The relation between the freeness of $\mathscr{A}(\Delta)$ over $\Phi^*\mathscr{A}(U^N)$ and the freeness of the sheaf $\Phi_*\mathscr{O}_\Omega$ is one which persists in more general settings.

THEOREM 3.2. *If $\Delta, \Delta' \in \mathscr{K}_N$, if $\Phi \in \mathscr{M}(\Delta', \Delta)$, and if $\mathscr{A}(\Delta')$ is free of rank λ, λ the multiplicity of Φ, over $\Phi^*\mathscr{A}(\Delta)$, then for some neighborhood Ω of $\bar{\Delta}'$ on which Φ is defined, the sheaf $\Phi_*\mathscr{O}_\Omega$ is free over $\mathscr{O}_{\Phi(\Omega)}$.*

Before proving the theorem, a simple preliminary observation is needed.

LEMMA 3.3. *If $\Delta, \Delta' \in \mathscr{K}_N$, if $\Phi \in \mathscr{M}(\Delta', \Delta)$, and if $\mathscr{A}(\Delta')$ is free over $\Phi^*\mathscr{A}(\Delta)$, then there exists a set of free generators for $\mathscr{A}(\Delta')$ which consists of functions holomorphic on a neighborhood of $\bar{\Delta}'$.*

Proof. By hypothesis there exists an isomorphism

$$L\colon \quad \Phi^*\mathscr{A}(\Delta)^q \longrightarrow \mathscr{A}(\Delta');$$

it is of the form

$$L(f_1 \circ \Phi, \cdots, f_q \circ \Phi) = \sum_{j=1}^{q} F_j f_j \circ \Phi$$

for some fixed elements $F_1, \cdots, F_q \in \mathscr{A}(\Delta')$. The operator L is continuous and so it has a continuous inverse L^{-1}. If $S\colon\ \Phi^*\mathscr{A}(\Delta)^q \longrightarrow \mathscr{A}(\Delta')$ is near L in the norm topology of $\mathscr{L}(\Phi^*\mathscr{A}(\Delta)^q, \mathscr{A}(\Delta'))$,[3] then $S \circ L^{-1}$

[3] We use $\mathscr{L}(X, Y)$ to denote the continuous linear operators from the Banach space X to the Banach space Y.

is near the identity of $\mathscr{L}(\mathscr{A}(\Delta'), \mathscr{A}(\Delta'))$ and so is invertible. Thus, if S is near L, S is also an isomorphism. Therefore if we choose functions $G_1, \cdots, G_q$ holomorphic on a neighborhood of $\bar{\Delta}'$ so that G_j is uniformly near F_j on $\bar{\Delta}'$ and if we define

$$S: \quad \Phi^*\mathscr{A}(\Delta)^q \to \mathscr{A}(\Delta')$$

by $S(f_1 \circ \Phi, \cdots, f_q \circ \Phi) = \Sigma G_j f_j \circ \Phi$, then S is a $\Phi^*\mathscr{A}(\Delta)$-module isomorphism, i.e., $\{G_1, \cdots, G_q\}$ is a free basis for $\mathscr{A}(\Delta')$ over $\Phi^*\mathscr{A}(\Delta)$. That the desired approximating functions exist is contained in [7, Corollary I. 6].

Proof of Theorem 3.4. Let $F_1, \cdots, F_\lambda \in \mathscr{O}(\bar{\Delta}')$ be a free basis for $\mathscr{A}(\Delta')$ over $\Phi^*\mathscr{A}(\Delta)$, and let Ω be a neighborhood of $\bar{\Delta}$ such that $F_1, \cdots, F_\lambda$ are all holomorphic in $\Omega' = \Phi^{-1}(\Omega)$. We have a homomorphism $\mathscr{F}$: $\mathscr{O}^\lambda_\Omega \to \Phi_*\mathscr{O}_{\Omega'}$ defined by $\mathscr{F}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_\lambda) = \Sigma(\tilde{F}_j)_\mathfrak{z} \boldsymbol{f}_j$ for all

$$(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_\lambda) \in \mathscr{O}^\lambda_\mathfrak{z}, \mathfrak{z} \in \Omega' .$$

Here $\tilde{F}_j$ is the section of $\Phi_*\mathscr{O}_{\Omega'}$ corresponding to F_j, and $(\tilde{F}_j)_\mathfrak{z}$ is its germ at $\mathfrak{z}$. We shall show that $\mathscr{F}$ is a sheaf isomorphism at least when we restrict attention to some, possibly smaller neighborhood of $\bar{\Delta}$.

Consider a point $\mathfrak{z} \in \bar{\Delta}$. We shall show that the stalk map given by $\mathscr{F}$ is an isomorphism in the stalk over $\mathfrak{z}$. Since $\Phi_*\mathscr{O}_{\Omega'}$ is locally free of rank λ, there is an isomorphism L: $\mathscr{O}^\lambda_\mathfrak{z} \to (\Phi_*\mathscr{O}_{\Omega'})_\mathfrak{z}$. Denote by $\boldsymbol{e}_j$ the element $(0, \cdots, 0, 1, 0, \cdots, 0)$ (1 in the j^{th} place) of $\mathscr{O}^\lambda_\mathfrak{z}$, and let $\boldsymbol{g}_j \in (\Phi_*\mathscr{O}_{\Omega'})_\mathfrak{z}$ be the image of $\boldsymbol{e}_j$ under L. There is a neighborhood W of $\mathfrak{z}$ in which all the germs $\boldsymbol{g}_j$ can be represented by sections of $\Phi_*\mathscr{O}_{\Omega'}$; call these sections g_j. Thus, $g_j \in \mathscr{O}(\Phi^{-1}(W))$. If $V \subset W$ is a small neighborhood of $\mathfrak{z}$, then on $(\Phi^{-1}(V))^-$, the functions g_j will admit uniform approximation by functions $G_j \in \mathscr{O}(\bar{\Delta})$. We can choose the approximating functions G_j so that in the expression

$$G_j = \Sigma F_k h_k^{(j)} \circ \Phi \qquad (h_k^{(j)} \in \mathscr{A}(\Delta))$$

the functions $h_k^{(j)}$ lie in $\mathscr{O}(\bar{\Delta})$. The G_j give sections $\tilde{G}_j$ of $\Phi_*\mathscr{O}_{\Omega'}$ which lie in the range of $\mathscr{F}$. Moreover, for any choice of G_j, we obtain a homomorphism L': $\mathscr{O}^\lambda_\mathfrak{z} \to (\Phi_*\mathscr{O}_{\Omega'})_\mathfrak{z}$ by setting $L'(\boldsymbol{e}_j) = (\tilde{G}_j)_\mathfrak{z}$ If the functions G_j approximate the functions g_j sufficiently well, the L' so obtained will be an isomorphism since L is. Fix a choice of the G_j so that L' is an isomorphism.

Using L', we can see that $\mathscr{F}$ is onto (in the stalk over $\mathfrak{z}$), for the range of $\mathscr{F}$ contains $\{\tilde{G}_1, \cdots, \tilde{G}_\lambda\}$ and so it contains the module generated by this set. Since L' is onto, it follows that this module is the whole of $(\Phi_*\mathscr{O}_{\Omega'})_\mathfrak{z}$.

The fact that $\mathscr{F}$ is one-to-one in the stalk over $\mathfrak{z}$ follows from

Cartan's Theorem A. If $\mathscr{T}(f_1, \cdots, f_\lambda) = 0$, $(f_1, \cdots, f_\lambda) \in \mathscr{O}_{\mathfrak{z}}^\lambda$, then $\mathscr{R}$, the sheaf of relations among $\tilde{F}_1, \cdots, \tilde{F}_\lambda$ is nontrivial over $\bar{\Delta}$. Thus by Cartan's Theorem A, there is a nontrivial section of $\mathscr{K}$ over $\bar{\Delta}$, i.e., there exist $h_1, \cdots, h_\lambda \in \mathscr{O}(\bar{\Delta})$ not all of which are the zero function, such that $\Sigma h_j \tilde{F}_j = 0$, i.e., $\Sigma F_j h_j \circ \Phi = 0$. This is impossible since $\{F_1, \cdots, F_\lambda\}$ is a free basis for $\mathscr{A}(\Delta')$ over $\Phi^* \mathscr{A}(\Delta)$.

Thus, for all $\mathfrak{z} \in \Delta$, $\mathscr{T}$ carries $\mathscr{O}_{\mathfrak{z}}^\lambda$ isomorphically onto $(\Phi^* \mathscr{O}_{\Omega'})_{\mathfrak{z}}$. Consequently, the same assertion holds for all $\mathfrak{z}$ in a neighborhood of $\bar{\Delta}$, and the theorem is proved.

Note added in proof. My colleague S.J. Sidney has observed that with a somewhat more careful use of the Borsuk-Ulam theorem, it is possible to show that the example of Section I is valid in dimensions two and three as well as in higher dimensions. Consider, in the notation of that section, a $\Delta \in \mathscr{K}_N$, $N \geqq 2$ with $P_N(R) \subset \Delta \subset V$, and let $\Delta' = \eta^{-1}(\Delta)$. Assume that F and G generate $A(\Delta')$ as a module over $\eta^* \mathscr{A}(\Delta)$. If f is any element of $\mathscr{A}(\Delta')$, we can write $f = f_e + f_0$, $f_e, f_0 \in \mathscr{A}(\Delta')$ where f_e is *even* in the sense that it is constant on the fiber $\eta^{-1}(\mathfrak{z})$, $\mathfrak{z} \in \Delta$, and f_0 is *odd* in that if $\eta^{-1}(\mathfrak{z}) = (\mathfrak{z}', \mathfrak{z}'')$, then $f_0(\mathfrak{z}') = -f_0(\mathfrak{z}'')$. To obtain such a decomposition, write $f_e(\mathfrak{z}) = \frac{1}{2}(f(\mathfrak{z}') + f(\mathfrak{z}''))$ where $\eta^{-1}(\eta(\mathfrak{z})) = \{\mathfrak{z}', \mathfrak{z}''\}$, and define f_0 to be $f - f_e$. It is clear that f_e is, in fact, even, and that f_0 is odd. It is easily verified that this decomposition of f into a sum of even and odd parts is unique.

By hypothesis, if $h \in \mathscr{A}(\Delta')$, we have $h = fF + gG$ for some choice of $f, g \in \eta^* \mathscr{A}(\Delta)$. The functions f and g are both even, and it follows that if $h = h_e + h_0$, then

$$h_e = fF_0 + gG_e$$

and

$$h_0 = fF_0 + gG_0 \ .$$

By suitable choice of $h \in \mathscr{A}(\Delta)$, we can arrange that the pair $(h_e(\mathfrak{z}), h_0(\mathfrak{z}))$ be any point of C^2, so it follows that the

determinant $\mathfrak{d}(\mathfrak{z}) = \begin{vmatrix} F_e(\mathfrak{z}) & G_e(\mathfrak{z}) \\ F_0(\mathfrak{z}) & G_0(\mathfrak{z}) \end{vmatrix}$ is zero for no choice of $\mathfrak{z}$.

However, $\mathfrak{d}$ is an odd function. Thus, continuing with the notation of Section I, $\mathfrak{d} \circ \xi$ is a C-valued function on S^N which is zero-free and odd in that if p and q are antipodal points in S^N, then $\mathfrak{d} \circ \xi(p) = -\mathfrak{d} \circ \xi(q)$. Since C is topologically R^2 and since $N \geqq 2$, the Borsuk-Ulam theorem implies that such an odd function has a zero, and the desired contradiction has been reached.

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