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GEOMETRIES ON SURFACES

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Among the topological geometries, two classes have so far attracted special attention, namely the locally compact, 2-dimensional projective and affine planes. Such a plane has a pointset which is homeomorphic to the pointset of the real arguesian plane, hence is a 2-dimensional manifold. In this paper, all the 2-manifolds that can carry topological geometries with connected lines will be determined.

THEOREM. *Let M be a surface carrying a topological geometry such that any two distinct points are on a unique line. Then either*

(1) *M is an open disk, and each line is open, i.e., homeomorphic to the real line R, or*

(2) *M is a compact surface of characteristic* 1, *and each line is closed, i.e., homeomorphic to a circle, or*

(3) *M is a Moebius strip, and through each point there pass closed lines and at least one open line.*

The proof follows immediately by combining statements 1.10 and 2.3, 6, 9, 10, 13 below. From 2.9 and 2.11 we get

COROLLARY. *M is orientable if and only if the space $\mathfrak{L}$ of lines is a Moebius strip, and M is nonorientable if and only if $\mathfrak{L}$ is compact.*

1. Let $\mathfrak{L}$ be a family of subsets of a nontrivial topological space M, and assume that each two distinct "points" $p, q \in M$ are joined by a unique "line" $L = p \cup q \in \mathfrak{L}$. The system $E = (M, \mathfrak{L})$ is called [3] a "plane" whenever $\mathfrak{L}$ can be provided with a topology (necessarily unique) for which E becomes a topological geometry in the following sense:

(a) $p \cup q$ depends continuously on (p, q), and

(b) the set $\mathfrak{D}$ of pairs of intersecting lines is open in $\mathfrak{L} \times \mathfrak{L}$, and intersection is a continuous map from $\mathfrak{D}$ onto M.

Condition (b) is equivalent to

(b′) If $U \subseteqq M$ is open, then $\{(K, L); K \cap L \in U\}$ is open in $\mathfrak{L} \times \mathfrak{L}$.

We shall be concerned with "flat" planes only, i.e., with those planes in which the underlying space M is a 2-dimensional manifold or "surface". The line space $\mathfrak{L}$ of a flat plane is also a surface, and the incident point-line pairs or "flags" form a 3-dimensional closed submanifold F of $M \times \mathfrak{L}$; each line is closed in M and is locally homeomorphic to the real line [3, 2.3].

If B is a connected open subset of M, then the system

$$E_B = (B, \mathfrak{L}_B = \{L \cap B \neq \emptyset; L \in \mathfrak{L}\})$$

is again a flat plane. This shows that, in general, the lines of a geometry will not be connected sets; it is therefore reasonable to determine only those flat planes that have connected lines, i.e. planes satisfying

(c) every line is homeomorphic to the real line or to the circle. Each line can then be provided with two opposite orientations, and the space $\mathfrak{L}^+$ of oriented lines becomes a two-fold covering of $\mathfrak{L}$. The lines homeomorphic to R will be called "open", and the others compact or "closed".

LEMMA 1.1. *A compact disk $D \subseteqq M$ cannot contain any line.*

Proof. A line $K \subseteqq D$ would have to be a closed line. Hence K would separate some points $a \in D$ and $b \in M$, and then $a \cup b$ would intersect K twice.

DEFINITION 1.2. A set $C \subseteqq M$ is convex if $L \cap C$ is connected for every line L. If the boundary $\operatorname{bd} C$ consists of finitely many, resp. of 4 or 3 line segments, C will be called a polygon resp. a quadrangle or a triangle.

LEMMA 1.3. *Each point $p \in M$ has a neighborhood basis consisting of quadrangles (triangles).*

Proof. Let W be a neighborhood of p. Choose points a_1, a_2 such that $p \notin a_1 \cup a_2$, and let J_k be a sufficiently small connected neighborhood of p on $a_k \cup p$. Then

$$U = \{(x_1 \cup a_2) \cap (x_2 \cap a_1); x_k \in J_k\} \subseteqq W$$

is a neighborhood of p homeomorphic to $J_1 \times J_2$, which is easily seen to be a convex quadrangle.

COROLLARY 1.4. *The space $\mathfrak{L}_p^+$ of oriented lines through p is homeomorphic to a circle.*

Convergence in M and in $\mathfrak{L}$ are related by

LEMMA 1.5. $L_n \xrightarrow[n]{} L$ *if and only if* $L = \{x; \vee x_n \in L_n x_n \xrightarrow[n]{} x\}$.

This follows immediately from the continuity of join and intersection, (a) and (b).

A set is termed "bounded" if it has compact closure.

LEMMA 1.6. *A set $\mathfrak{S}$ of lines is bounded in $\mathfrak{L}$ if and only if there is a bounded set $C \subseteqq M$ intersecting every line $S \in \mathfrak{S}$.*

Proof. Let $S_n \in \mathfrak{S}$ and $x_n \in S_n \cap C$. We may assume $x_n \xrightarrow[n]{} x$. Choose a triangle with boundary T containing x in its interior; then for each sufficiently large n there exists a point $t_n \in S_n \cap T$. The t_n accumulate at some $t \in T$, and a subsequence of the S_n converges to $t \cup x$. For the converse note that M is a union of a countable family of compact subsets.

LEMMA 1.7. *Let K be a closed line, and $B \subseteqq M$ a bounded open set containing K. Then $\mathfrak{B} = \{L \in \mathsf{L}; L \subseteqq B\}$ is an open neighborhood of K in $\mathfrak{L}$ consisting of closed lines only.*

Proof. Since bounded lines are closed, $\mathfrak{B}$ cannot contain any open line. If $\mathfrak{B}$ were not a neighborhood of K, we would find a sequence of lines $L_n \xrightarrow[n]{} K$ with $L_n \cap B \neq \emptyset$ and $L_n \notin \mathfrak{B}$. By (c) each line L_n would contain a point x_n of the compact boundary bd B, in contradiction to 1.5.

COROLLARY 1.8. *The space $\mathfrak{K}$ of closed lines is open in $\mathfrak{L}$.*

LEMMA 1.9. *A closed line K intersects every line $L \in \mathfrak{L}$.*

Proof. For $a \notin K$ the map $\alpha = (x \mapsto a \cup x)$ maps K into the set $\mathfrak{L}_a$ of lines through a. Since $\mathfrak{L}_a$ is homeomorphic to a circle by 1.4, and α is one-to-one, α is even a bijection.

COROLLARY 1.10. *If M is a compact surface, then $(M, \mathfrak{L})$ is a projective plane, and M is homeomorphic to the real projective plane.*

Proof. Any two different lines intersect by 1.9, and this is the first assertion. The second statement follows now from [2].

PROPOSITION 1.11. If a geometry contains a closed line K, then each point lies on a closed line.

Proof. Assume that each line through s is homeomorphic to R, and let $s \cup z^+$ denote the line $s \cup x$ oriented from s to x. Then $(x \mapsto s \cup x^+): K \to \mathfrak{L}_s^+$ would be a homeomorphism from K onto a set of oriented lines through s containing exactly one of any two lines of opposite orientation. This contradicts 1.4.

2. From now on we shall assume for the sake of simplicity that

M is a surface of finite connectivity, i.e., that M can be embedded in a compact surface $\bar{M}$ by adding a finite number of "endpoints" $e_1, \cdots, e_r$.

LEMMA 2.1. *Let $g: R \to M$ represent an oriented open line $L = g(R)$. Then L has unique endpoints*

$$e^{\pm}(L) = \lim_{t\to\pm\infty} g(t)$$

in $\bar{M}$.

Proof. Choose small disjoint open neighborhoods U_k of e_k in $\bar{M}$. Since each line is closed in M, there exists an $s \in R$ such that the connected set $\{g(t); t \geqq s\}$ is contained in $\bigcup_{k=1}^{r} U_k$ and hence in one of the U_k.

LEMMA 2.2. *The oriented lines L having a given endpoint $e^+(L) = e$ form a closed subset of $\mathfrak{L}^+$.*

Proof. Let $L_n \xrightarrow[n]{} L$ with $e^+(L_n) = e$, and choose a small neighborhood U of e in $\bar{M}$. Then all the L_n intersect the compact boundary $\mathrm{bd}\, U \subseteqq M$. Hence $L \cap \mathrm{bd}\, U \neq \varnothing$ by 1.5, and $e^+(L) = e$.

LEMMA 2.3. *For each $a \in M$ and each endpoint e there is an oriented line $L \in \mathsf{L}_a^+$ with $e^+(L) = e$.*

Proof. Let $b_n \xrightarrow[n]{} e$ in $\bar{M}$, and $a \cup b_n \xrightarrow[n]{} L$.

COROLLARY 2.4. *Let $(M, \mathfrak{L})$ be a flat plane with connected lines. Then M is compact if and only if each line is closed.*

PROPOSITION 2.5. A surface M carrying a plane geometry with connected lines has at most one endpoint.

Proof. $\mathfrak{L}_{ik} = \{L \in \mathfrak{L}^+; e^-(L) = e_i, e^+(L) = e_k\}$ is closed in $\mathfrak{L}^+$ by 2.2. Assume $r > 1$, choose small disjoint neighborhoods U_k of e_k in $\bar{M}$, and let $b_{kn} \xrightarrow[n]{} e_k$ in U_k. For $i \neq k$ the lines $b_{in} \cup b_{kn}$ meet the compact complement C of $\bigcup_{k=1}^{r} U_k$, and hence accumulate at a line in $\mathfrak{L}_{ik}$. Since $\mathfrak{L}^+$ is connected, the boundary $\mathrm{bd}\, \mathfrak{L}_{ik} \neq \varnothing$, and there is a sequence of closed lines K_n converging to a line $L \in \mathfrak{L}_{ik}$. For large n a line K_n will contain two arcs joining U_i and U_k; thus a line H intersecting L at a point of C would meet some line K_n twice, a contradiction.

PROPOSITION 2.6. A plane geometry on an orientable surface M

has no compact lines.

Proof. A compact line K on an orientable surface M would have a bounded neighborhood $B \subseteqq M$ which is homeomorphic to a cylinder. By 1.7 the set $\mathfrak{B}$ of all lines $L \subseteqq B$ is an open neighborhood of K in $\mathfrak{L}$, and each $L \in \mathrm{B}$ is closed. Moreover, it follows from 1.1 and 1.9 that a line $L \subseteqq B$ decomposes the cylinder and intersects K. But then K and L would have to intersect twice.

PROPOSITION 2.7. The complement of an open line L is homeomorphic to an open subset of R^2.

Proof. For $a, b \in L$ consider

$$(x \mapsto (a \cup x, b \cup x)) \colon M - L \to (\mathfrak{L}_a - \{L\}) \times (\mathfrak{L}_b - \{L\}) .$$

PROPOSITION 2.8. If M is orientable, then any line L decomposes M into two open half-planes, each of which is homeomorphic to an open disk. Consequently $\bar{M}$ is homeomorphic to a 2-sphere.

Proof. Let $L = a \cup b$, and choose homeomorphisms

$$\alpha \colon \mathfrak{L}_a - \{L\} \to (0, 1) , \qquad \beta \colon \mathfrak{L}_b - \{L\} \to (0, 1)$$

corresponding to the same orientation of M. For $\xi = A^\alpha$ let

$$\delta(\xi) = \inf \{B^\beta; A \cap B = \varnothing\} .$$

Then δ is a monotonically increasing, left-continuous mapping. According to the orientation of M one of the half-planes defined by L will be mapped by

$$x \mapsto ((a \cup x)^\alpha, (b \cup x)^\beta)$$

onto the Jordan domain

$$D = \{(\xi, \eta) \in R^2; 0 < \xi < 1, 0 < \eta < \delta(\xi)\} .$$

COROLLARY 2.9. *Let $(M, \mathfrak{L})$ be a flat plane with connected lines. If M is orientable, then M is homeomorphic to R^2, and $\mathfrak{L}$ is a Moebius strip.*

For the last assertion see [1, § 3(6)] or [3, 2.14].

An argument completely analogous to the proof of 2.8 shows

COROLLARY 2.10. *Each point of a plane geometry on a non-*

orientable surface is incident with a compact line.

THEOREM 2.11. *The line space of a plane geometry* $(M, \mathfrak{L})$ *on a nonorientable surface is a compact surface of characteristic* 1; *the space of oriented lines is homeomorphic to a 2-sphere.*

Proof. If H, K are closed lines through a, then $\mathfrak{L} - \mathfrak{L}_a$ is homeomorphic to $(H - \{a\}) \times (K - \{a\})$.

LEMMA 2.12. *In a plane geometry* $(M, \mathfrak{L})$ *on a nonorientable surface the set of closed lines through a point* a *is connected.*

Proof. Because of 2.5 we may assume that M has exactly one endpoint e. Choose an open line L and two closed lines H and K through a. We have to show that no open line through a is separated from L by H, K. By 2.7 the Jordan curve theorem holds in $M - L$. Hence the union $H \cup K$ decomposes $M - L$ into two connected subsets, only one of which contains points of a small neighborhood U of e. Therefore, a line G which is separated from L by H, K must be disjoint from U and hence is compact.

THEOREM 2.13. *An open nonorientable surface* M *carrying a plane geometry with connected lines is a Moebius strip.*

Proof. The dual $(\mathfrak{L}, M)$ of the geometry defined by the system $\mathfrak{K}$ of closed lines on M is a flat plane by 2.10, 1.8, 1.9, and 2.12. The theorem follows now from 2.9 and 2.11.

Note that in general the lines of a topological geometry on a Moebius strip need not be connected.

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