PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

MULTIPLIER ALGEBRAS OF BIORTHOGONAL SYSTEMS

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Let $\{e_i, E_i\}$ be a total biorthogonal system in a linear topological space X. The multiplier algebra of X with respect to $\{e_i, E_i\}$ written $M(X)$ is the set of all scalar sequences $(t^{(i)})$ such that for each $x \in X$ there is $y \in X$ with

$$E_i(y) = t^{(i)} E_i(x) .$$

The form of $M(X)$ is determined when $\{e_i, E_i\}$ is a norming complete biorthogonal system in a Banach space or a basis in a complete barreled space. It is shown that a sequence space is the multiplier algebra for a basis in a Banach space if and only if it is a γ-perfect BK-algebra.

A *biorthogonal system* is a double sequence $\{e_i, E_i\}$ with each e_i in a locally convex space X and each E_i a continuous linear functional on X (i.e., in X^*) which satisfies the relationship

$$E_i(e_j) = \delta_{ij} \text{ (Kronecker } \delta) \quad i, j = 1, 2, \cdots .$$

The biorthogonal system is *total* if $\{E_i\}$ is total on X; that is, $E_i(x) = 0$ for each i implies $x = 0$. If $\{e_i, E_i\}$ is a total biorthogonal system then the space X can be identified with the space of all sequences $(E_i(x))$ by means of the natural correspondence x to $(E_i(x))$. Under this correspondence e_i becomes the ith coordinate vector, the sequence which has a one in the ith coordinate and 0's elsewhere and E_i becomes the ith coordinate functional, the functional whose value on the sequence $(x^{(1)}, x^{(2)}, \cdots)$ is $x^{(i)}$. This identification will be assumed whenever a total biorthogonal system is considered.

DEFINITION 1.1. Let $\{e_i, E_i\}$ be a total biorthogonal system in a locally convex space X. A scalar sequence

$$t = (t^{(1)}, t^{(2)}, \cdots)$$

is a *multiplier of X with respect to* $\{e_i, E_i\}$ if for each $x \in X$ there is $y \in X$ for which

$$E_i(y) = t^{(i)} E_i(x) \qquad i = 1, 2, \cdots .$$

The set of all such t is written $M(X; e_i, E_i)$ or simply $M(X)$ and called the *multiplier algebra of X (with respect to* $\{e_i, E_i\}$).

In other words $M(X)$ is the set of all t such that

$$tx \in X \text{ whenever } x \in X$$

where X is now considered a sequence space and multiplication of sequences is defined coordinatewise. It is now obvious that $M(X)$ forms a linear algebra of operators from X into X; namely the operators which are diagonal with respect to $\{e_i, E_i\}$. Multiplication in this algebra is defined coordinatewise. The purpose of this paper is to study the properties of the space $M(X)$ and the possible forms which it can assume with varying hypotheses on X or on $\{e_i, E_i\}$. Results of this type were obtained by Yamazaki in [11] and [12] for $\{e_i\}$ a basis in a Banach space. The concept of imultiplier space is implicitly treated in [4].

Throughout this paper it is immaterial whether the scalar field considered consists of the real or complex numbers.

2. Sequence spaces: notation and basic facts. A set of scalar sequences which is closed under coordinatewise addition and scalar multiplication is a *sequence space*; if it is closed under coordinatewise multiplication as well it will be called a *sequence algebra*. The ith coordinate vector is written e_i; the ith coordinate functional, E_i. If each E_i is continuous on a locally convex sequence space (algebra) X and $e_i \in X$ for each i then X is called a *K-space* (algebra). If in addition X is an F-space (complete metric linear space) X will be called an FK-space or FK-algebra as the case may be. If X is a Banach space (algebra), X will be called a BK-space (algebra). Note that in an FK-algebra X the functions $x \to tx$ and $x \to xt$ are continuous in x for fixed t by the continuity of the coordinate functionals and the closed graph theorem. This is enough to conclude that a BK-algebra is a Banach algebra without identity. See p. 860 and 861 of [3].

The following are well known sequence spaces. For additional discussion see Chapter IV of [2], [5] or p. 289 of [10]:

ω sometimes called s is the set of all scalar sequences. Endowed with the topology of coordinatewise convergence it is an FK-algebra.

φ is the linear span of $\{e_i\}$ in ω, i.e., the space of all finitely nonzero sequences.

l^1 is the set of all sequences t such that

$$\| t \| = \sum_{i=1}^{\infty} | t^{(i)} | < \infty$$

which is a BK-space with this norm.

m is the set of all sequences t such that

$$\| t \| = \sup_i | t^{(i)} | < \infty$$

which is a BK algebra with this norm.

bs is the set of all sequences t such that

$$\| t \| = \sup_n | \sum_{i=1}^{n} t^{(i)} | < \infty$$

which is a BK space with this norm.

cs is the closed linear span of $\{e_i\}$ in bs; it consists of all sequences t such that $\sum_{i=1}^{\infty} t^{(i)}$ converges.

bv is the set of all sequences t such that

$$\| t \| = \lim_n | t_n | + \sum_{i=1}^{\infty} | t_i - t_{i+1} | < \infty$$

which is a BK-algebra with this norm. See §3 of [4] or p. 3 of [11]. Yamazaki denoted bv by w.

NOTATION 2.1. (a) Let $t, u \in \omega$ be such that $ut \in cs$; the sum $\sum_{i=1}^{\infty} u^{(i)} t^{(i)}$ is denoted by (u, t).

(b) For $A \subseteqq \omega$
A^{α}, the α-dual of A is $\{t: ut \in l^1, u \in A\}$
A^{β}, the β-dual of A is $\{t: ut \in cs, u \in A\}$
A^{γ}, the γ-dual of A is $\{t: ut \in bs, u \in A\}$.

(c) For A and $B \subseteqq \omega$
$AB = \{uv: u \in A, v \in B\}$.

(d) For $t \in \omega$ and $A \subseteqq \omega$
$t^{-1}A = \{u \in \omega: tu \in A\}$.

(e) For $A \subseteqq \omega$
$A^{\varphi} = \{t \in \varphi: | (t, u) | \leqq 1, u \in A\}$.

(f) For $A \subseteqq \varphi$
$A^{\omega} = \{t \in \omega: | (t, u) | \leqq 1, u \in A\}$.

(g) For X aK-space, X^{δ} is the space of all sequences $(f(e_i))$ as f ranges over X^*. Note that for $t \in X^{\delta}$ with $t^{(i)} = f(e_i)$, $E_i(t) = t^{(i)} = f(e_i)$.

Gamma-perfect BK-spaces can be constructed by means of sequential norms. A sequential norm (s.n.) is a function P from ω into R^* which is an extended norm and in addition satisfies the condition

$$P(x) = \sup_n P\left(\sum_{i=1}^{n} x^{(i)} e_i\right) x \in \omega \; .$$

If

$$0 < \inf_n P(e_n) \leqq \sup_n P(e_n) < \infty$$

P is a proper sequential norm (p.s.n.). For P a s.n, S_P is the set of all $x \in \omega$ for which $P(x) < \infty$ endowed with the topology determined by $P\varepsilon$. The closed linear span of $\{e_1, e_2, \cdots\}$ in S_P is denoted by S_P^0. The following proposition contains information about sequential norms which was derived in [7] and [8] and which we shall use in §6.

PROPOSITION 2.2. (a) *If P is a* s.n. *S_P is a γ-perfect BK-space. If X is a γ-perfect BK-space there is an* s.n. *P such that $X = S_P$.*

(b) *If P is a* s.n. *the function P' given by*

$$P'(x) = \sup\left\{\sup_n \left|\sum_{i=1}^{n} x^{(i)}y^{(i)}\right| : P(y) \leqq 1\right\}$$

is a s.n. *and $P'' = P$. If P is a* p.s.n. *so is P'.*

(c) $(S_P^0)^\delta = S_{P'}$ *and* $(S_{P'}^0)^\delta = S_P$.

(d) *An* s.n. *P is a* p.s.n. *if and only if $l^1 \subseteqq S_P \subseteqq m$.*

3. Preliminary results.

PROPOSITION 3.1. *If $\{e_i, E_i\}$ is a total biorthogonal system in X*

$$M(X) = \cap \{y^{-1}X : y \in S\} \tag{3-1}$$

where S is any absorbing subset of X.

Proof. Let R denote the set on the right of (3–1). If $t \in R$ and $x \in X$ there is $a > 0$ such that $ax \in S$. Thus $atx \in X$ so that $tx \in X$ which implies $t \in M(X)$. If $t \in M(X)$ then $tx \in X$ for every $x \in X$ so in particular for every $x \in S$.

A complete biorthogonal system is a total biorthogonal system $\{e_i, E_i\}$ on X such that sp $\{e_i\}$ $(= \varphi)$, the linear span of $\{e_i\}$ is dense in X.

PROPOSITION 3.2. *Let $\{e_i, E_i\}$ be a complete biorthogonal system in X.*

(a) *For each $t \in M(X)$, the mapping $x \to tx$ is a closed linear operator from X into X.*

(b) *The set $M_c(X)$ of all $t \in M(X)$ for which $x \to tx$ is continuous is a closed sub-algebra of $\mathscr{L}(X)$ where $\mathscr{L}(X)$ has any topology containing the topology of simple convergence. Here $\mathscr{L}(X)$ denotes the space of continuous operators from X into X.*

Proof. (a) Obvious.

(b) Define $E_i \otimes e_j$ on $\mathscr{L}(X)$ by

$$E_i \otimes e_j(T) = E_i(Te_j) .$$

Then $E_i \otimes e_j$ is a continuous linear functional on $\mathscr{L}(X)$ given the topology of simple convergence. Therefore

$$S = \cap \{[E_i \otimes e_j]^{-1}(0) : i \neq j\}$$

is closed in this topology and every topology containing it.

The following statement generalizes Theorem 1 of [11].

COROLLARY 3.3. *If $\{e_i, E_i\}$ is a complete biorthogonal system in a Banach space X then there is a topology on $M(X)$ which makes it a BK algebra.*

Proof. In this case $M_c(X) = M(X)$ since the mapping $x \rightarrow tx$ is closed. Thus by 3.2 $M(X)$ is a Banach algebra with the norm

$$\| t \| = \sup\{\| tx \| : \| x \| \leqq 1\} .$$

Each E_i is continuous on $M(X)$ since

$$E_i(T) = E_i \otimes e_i(T_t)$$

where $T_t(x) = tx$; and $E_i \otimes e_i$ is continuous on $\mathscr{L}(X)$.

PROPOSITION 3.4. *If $\{e_i, E_i\}$ is a total biorthogonal system in a linear topological space X then*

$$M_c(X; e_i, E_i) \subseteqq M_c(X_0; e_i, E_i)$$

where X_0 is the closed linear span of $\{e_i\}$ in X.

Proof. If $t \in M_c(X)$ then $tx \in \varphi$ for $x \in \varphi$. Since φ is dense in X_0 and t is continuous $tx \in X_0$ for $x \in X_0$ so that $t \in M_c(X_0)$.

If $\{e_i, E_i\}$ is a complete biorthogonal system on X, X^* is isomorphic to X^δ under the correspondence of f in X^* to $(f(e_i))$ in X^δ and $\{e_i, E_i\}$ is a total biorthogonal system on X^δ.

PROPOSITION 3.5. *If $\{e_i, E_i\}$ is a complete biorthogonal system in a locally convex space X then*

$$M_c(X; e_i, E_i) \subseteqq M(X^\delta; e_i, E_i) .$$

Proof. If $t \in M_c$ for $f \in X^*$ let $f_t(x) = f(tx)$, $x \in X$. Then $f_t \in X^*$ and $f_t(e_i) = t^{(i)}f(e_i)$ for each i so that $ty \in X^\delta$ for $y \in X^\delta$.

4. Multiplier algebras of a norming biorthogonal system in a Banach space. A biorthogonal system $\{e_i, E_i\}$ in a normed space X is called *norming* if the topology of X is determined by a norm of the type

$$\| x \| = \sup \{| f(x) | : f \in A\}$$

where A is a subset of the linear span of $\{E_i\}$ in X^*. An equivalent condition is that the above norm be given by

(4-1) $$||x|| = \sup\{|(x, t)| : t \in A\}$$

where A is a subset of φ.

If $\{e_i, E_i\}$ is a complete biorthogonal system which is norming on X and the norm of X is given by (4-1) it may be assumed that A consists of all sequences t in φ for which

(4-2) $$|(t, x)| \leqq ||x|| \qquad x \in X .$$

Denote by $\hat{X}$ the space of all $x \in \omega$ for which

(4-3) $$||x|| = \sup\{|(x, t)| : t \in A\} < \infty .$$

The function defined in (4-3) is a norm since $a_n^{-1} e^n \in A$ for $n = 1, 2, \cdots$ where a_n is the norm of E_n as a member of X^*. With this norm $\hat{X}$ is a BK-space in which X is the closed linear span of $\{e_i\}$.

PROPOSITION 4.1. *The space X^δ consists of all $y \in \omega$ for which*

$$||y||' = \sup\{|(y, x)| : x \in A^\varphi\} < \infty .$$

The correspondence

(4-4) $$f \text{ to } (f(e_i)) \qquad f \in X^*$$

is an isometry from X^* onto $(X^\delta, ||\,||')$. The correspondence

(4-5) $$g \text{ to } (g(e_i)) \qquad g \in (X_0^\delta)^*$$

is an isometry from $(X_0^\delta)^*$ onto $\hat{X}$, where X_0^δ denotes the closed linear span of $\{e_i\}$ in X^δ.

Proof. The correspondence in (4-4) is clearly well defined and linear.

If $f \in X^*$ and $x \in A^\varphi$ then $x \in X$ and $||x|| \leqq 1$. Thus

$$\begin{aligned} |(f(e_i), x)| &= |\sum_i x^{(i)} f(e_i)| \\ &= |f(x)| \leqq ||f|| \end{aligned}$$

so that

$$||(f(e_i))||' \leqq ||f|| .$$

If $y \in X^\delta$ define f on $\{\varphi, ||\,||\}$ by

$$f(x) = (y, x) .$$

Then f is a bounded linear functional on $\{\varphi, \|\ \|\}$ for which

$$|f(x)| \leqq \|y\|' x \in \varphi, \|x\| \leqq 1$$

because

$$A^{\varphi} = \{x \in \varphi : \|x\| \leqq 1\} .$$

Since φ is dense in X, f can be continuously extended to X with

$$\|f\| \leqq \|y\|' .$$

For this extended f

$$(f(e_i)) = (y, e_i) = y_i \qquad i = 1, 2, \cdots .$$

Therefore, the correspondence in (4–4) is an isometry.

That (4–5) is an isometry from $(X_0^{\circ})^*$ onto $\hat{X}$ will follow from an analogous argument if it is shown that

$$A^{\varphi\varphi} = A .$$

When A has the form given by (4–2).
That $A^{\varphi\varphi} \supseteqq A$ is clear, if $z \in A^{\varphi\varphi}$ then

$$|(z, x)| \leqq 1 \qquad x \in A^{\varphi}$$

but A^{φ} is dense in the unit ball of X so that

$$|(z, x)| \leqq 1 \qquad x \in X, \|x\| \leqq 1 .$$

Thus if

$$f(x) = (z, x) \qquad x \in X$$

we have $\|f\| \leqq 1$ and

$$f(e_i) = z_i \qquad i = 1, 2, \cdots$$

so that $z \in A$. It is here that the assumption that $\{e_i, E_i\}$ is norming was used.

THEOREM 4.2. *If $\{e_i, E_i\}$ is a norming complete biorthogonal system in the Banach space X then $M(X)$ is of the form*

$$\bigcup_{n=1}^{\infty} n(AA^{\omega})^{\omega} \tag{4–6}$$

where A is a coordinatewise bounded subset of φ which contains a multiple of e_i for each i.

Proof. Let A be given by (4–2) and let Z denote the sequence space (4–6).

By 3.4, 3.5 and the fact that $M_c(Y) = M(Y)$ for Y a Banach space we have

$$M(X) \subseteqq M(X^\delta) \subseteqq M(X_0^\delta) \subseteqq M(\hat{X}) \subseteqq M(X)$$

so that $M(X)$ and $M(\hat{X})$ are equal. It will be shown that $M(\hat{X}) = Z$. Suppose $t \in M(\hat{X})$, then there is k such that

$$\| tx \| \leqq k \| x \| \qquad x \in \hat{X} .$$

If $s \in A$ and $x \in A^\omega$

$$| (sx, t) | = | (s, tx) | \leqq k$$

so that

$$t \in k(AA^\omega)^\omega \subseteqq Z .$$

If $z \in Z$ and $x \in \hat{X}$, $x/\| x \| \in A^\omega$ so if n is such that $z \in n(AA^\omega)^\omega$

$$| (t, zx) | = | (tx, z) | \leqq n \| x \|$$

for $t \in A$. Therefore $zx \in \hat{X}$.

Question. If $\{e_i, E_i\}$ is a complete biorthogonal system in the Banach space X and $M(X)$ has the form (4–6) is $\{e_i, E_i\}$ norming?

5. **Multiplier algebras of bases.** A biorthogonal system $\{e_i, E_i\}$ in a linear topological space X is a (Schauder) *basis* for X if

(5–1) $$x = \sum_{i=1}^{\infty} E_i(x) e_i \qquad x \in X .$$

It is an *unconditional basis* if the convergence in (5–1) is unconditional. If X is a l.c.s. $\{e_i, E_i\}$ is an *absolute basis* if the convergence in (5–1) is absolute, i.e., if

$$\sum_{i=1}^{\infty} | E_i(x) | p(e_i) < \infty$$

for each $x \in X$ and each continuous seminorm p on X. It is clear that an absolute basis is unconditional.

The proofs of 5.1, 5.2, and 5.3 are omitted since these statements are essentially known. See p. 205 of [10] and Propositions 4 and 5 of [1].

LEMMA 5.1. *For $\{e_i, E_i\}$ a complete biorthogonal system in a barreled* l.c.s. *X it is always true that $X^\gamma \subseteqq X^\delta$.*

LEMMA 5.2. *For $\{e_i, E_i\}$ a basis in a locally convex space X, $X^\delta \subseteqq X^\beta$.*

PROPOSITION 5.3. *For $\{e_i, E_i\}$ a complete biorthogonal system in a barreled locally convex space X the following are equivalent.*

(a) *$\{e_i, E_i\}$ is a basis for X.*

(b) *$X^\delta = X^\beta$.*

(c) *$X^\delta = X^\gamma$.*

THEOREM 5.4. *If $\{e_i, E_i\}$ is a basis of a complete barreled space X then*

$$M_c(X) = M(X) = (XX^\beta)^\beta = (XX^\gamma)^\gamma .$$

Proof. Suppose $t \in M(X)$, $x \in X$ and $y \in X^\beta$. Then $tx \in X$ so that $txy \in cs$ by 5.1 which implies $t \in (XX^\beta)^\beta$.

Let P denote the family of all continuous seminorms on X. Let $\hat{X}$ be the linear space of all $x \in \omega$ such that

$$p'(x) = \sup_n p\Big(\sum_{i=1}^n x^{(i)} e_i\Big) < \infty , p \in P .$$

Since X is barreled, p' restricted to X is continuous and since φ is dense in X, $p'(x) \geqq p(x)$ for $x \in X$. Thus X is the closed linear span of $\{e_i\}$ in the space $\hat{X}$ with the topology determined by the seminorms $\{p' \colon p \in P\}$.

For $t \in (XX^\gamma)^\gamma$ define

$$p_t(x) = \sup_n p\Big(\sum_{i=1}^n t^{(i)} x^{(i)} e_i\Big) .$$

Since $t \in (XX^\gamma)^\gamma$, $\{\sum_{i=1}^n t^{(i)} x^{(i)} e_i \colon n = 1, 2, \cdots\}$ is a weakly bounded, thus a strongly bounded subset of X so that

$$P_t(x) < \infty \qquad x \in X ,$$

and p_t is a continuous seminorm on X. If $x \in \hat{X}$, $tx \in \hat{X}$ and

$$p'(tx) = p'_t(x)$$

so that $t \in M_c(\hat{X})$. By 3.4, $(XX^\gamma)^\gamma \subseteqq M_c(X)$. Thus

$$M_c(X) \subseteqq M(X) \subseteqq (XX^\gamma)^\beta \subseteqq (XX^\gamma)^\gamma \subseteqq M_c(X)$$

which establishes the result.

COROLLARY 5.5. *If $\{e_i, E_i\}$ is an unconditional basis of a complete barreled space X then*

$$M(X) = (XX^\alpha)^\alpha .$$

Proof. Since $\{e_i, E_i\}$ is an unconditional basis $X^\alpha = X^\delta$. If $t \in (XX^\gamma)^\gamma$

and $u \in XX^\gamma$ let $u = xy$ with $x \in X, y \in X^\gamma = X^\delta = X^\alpha$. Let $v^{(i)} =$ sgn $t^{(i)}u^{(i)}$ then $vy \in X^\alpha$ so that $vu \in XX^\gamma$. Hence, $vut \in bs$ so that $ut \in l^1$. Therefore, $(XX^\gamma)^\gamma \subseteqq (XX^\alpha)^\alpha$ from which the conclusion follows.

THEOREM 5.6. *Let $\{e_i, E_i\}$ be an absolute basis of a sequentially complete locally convex space X. If P is any family of continuous seminorms which determines the topology of X then*

$$M(X) = (AA^\alpha)^\alpha$$

where

$$A = \{(p(e_i)): p \in P\} .$$

Proof. The hypotheses of this theorem imply that $X = A^\alpha$. Now $t \in M(X)$ if and only if $tx \in X$ whenever $x \in X$ which happens if and only if $txy \in l^1$ whenever $x \in A^\alpha = X$ and $y \in A$. This will hold if and only if $t \in (AA^\alpha)^\alpha$.

EXAMPLES. $M(\omega) = M(\varphi) = \omega$; $M(cs) = M(bv_0) = bv$; $M(c_0) = M(l^1) = m$.

Let X be the space of all real sequences x for which

$$p_k(x) = \sum_n |x_n| k^n < \infty \qquad k = 1, 2, \cdots .$$

Then X with the seminorms $p_1, p_2, \cdots$ is a nuclear F-space which is equivalent to the space of all infinitely differentiable real functions of period 2π. See §5 of [6]. If $A = (k^n): k = 1, 2, \cdots\}$ it is clear that $X = A^\alpha$. If $x \in A^\alpha, x = xe \in AA^\alpha$ and if $x \in AA^\alpha, x = y(k^n)$ for some k and $y \in A^\alpha$. But

$$\sum_{n=1}^\infty |t_n| k^n h^n = \sum_{n=1}^\infty |t_n| (kh)^n$$

so $x \in A^\alpha$. Thus $M(X) = A^{\alpha\alpha} = X^\delta$.

The following theorem is a version of Theorem 3 and Corollary 2 of [4]. For the definition of B_r-complete see p. 162 of [9].

THEOREM 5.7. *A complete biorthogonal system $\{e_i, E_i\}$ in a space X which is barreled and B_r complete is a basis for X if and only if $M(X) \supseteqq bv$. It is an unconditional basis for X if and only if $M(X) \supseteqq m$.*

6. Gamma-perfect BK-algebras. A proper sequential norm P which satisfies the inequality

$$P(xy) \leqq P(x)P(y) \tag{6-1}$$

will be called an algebraic p.s.n. (a.p.s.n.).

THEOREM 6.1. *The following statements are equivalent for a sequence space* M:

(a) M *is a multiplier algebra for a basis in a Banach space;*

(b) M *is a* γ*-perfect BK-algebra containing* e;

(c) $M = S_P$ *for* P *an* a.p.s.n. *with* $P(e) < \infty$.

Proof. (a) $\Rightarrow$ (b). If M is a multiplier algebra for a basis in a Banach space X it is a BK-algebra containing e by 3.3 and γ-perfect since it is the γ-dual of XX^γ by 5.4.

(b) $\Rightarrow$ (c). Suppose M is a γ-perfect BK-algebra containing e. By 2.2 (a) there is a sequential norm Q such that $M = S_Q$. It is routine to verify that P given by

$$P(x) = \sup\{Q(xy): Q(y) \leqq 1\}$$

is a s.n. equivalent to Q (i.e., $S_P = S_Q$) which satisfies (6–1). It remains to show

$$0 < \inf_n P(e_n) \leqq \sup_n P(e_n) < \infty. \tag{6–2}$$

Since $P(e_n) = P(e_n e_n) \leqq P(e_n)^2$, the left inequality of (6–2) is valid. Since $e \in S_P$, $bs \supseteqq S_P^\gamma$ so that $bv \subseteqq S_P$. That S_P is γ-perfect follows form 2.2 (a). The identity map from bv into S_P is continuous and $\{e_n: n = 1, 2, \cdots\}$ is bounded in S_P. Hence the right inequality in (6–2) is true. Therefore, P is an a.p.s.n.

(c) $\Rightarrow$ (a). If P is an a.p.s.n. with $P(e) < \infty$, S_P is a BK-algebra with identity so $M(S_P) = S_P$. But since $S_P = (S_{P'}^0)^\delta$ and $S_{P'} = (S_P^0)^\delta$, $M(S_P^0) = M$ and $\{e_i, E_i\}$ is a basis for S_P^0 (2.2).

The following theorem gives a means of constructing γ-perfect BK-algebras with identity which are distinct from bv and m. Let N denote the sequence of positive integers and N_k a subsequence of the form

$$N_k = \{k(1) < k(2) < \cdots\}\,. \tag{6–3}$$

THEOREM 6.2. (a) *Let* $N_1, N_2, \cdots, N_r$ *be a partition of* N *with each* N_k *given by* (6–3). *For each* k *let* P_k *be an* a.p.s.n. *for which* $P_k(e) < \infty$. *Define* P *by*

$$P(x) = \max\{P_k(x^{k(1)}, x^{k(2)}, \cdots): k = 1, 2, \cdots r\}\,.$$

Then P *is an* a.p.s.n. *and* S_P *is a* γ*-perfect BK-algebra containing* e.

(b) *Let* $N_1, N_2, \cdots$ *be an infinite partition of* N *with each* N_k *given by* (6–3). *For each* k *let* P_k *be an* a.p.s.n. *for which*

(6–4) $$\sup_k P_k(e) < \infty .$$

Define P by

$$P(x) = \sup_k \{P_k(x^{k(1)}, x^{k(2)}, \cdots)\} .$$

Then P is an a.p.s.n. *and* S_P *is a* γ*-perfect BK-algebra containing e.*

Proof. The proof of (a) is omitted since it is similar to but less difficult than that of (b).

(b) It is straightforward to verify that P is a norm. That P is a.s.n. follows from the equalities:

$$\begin{aligned}\sup_n P\Big(\sum_{i=1}^{n} x^{(i)} e_i\Big) &= \sup_n \sup_k \Big\{P_k\Big(\sum_{k(i)\leqq n} x^{k(i)} e_i\Big)\Big\} \\ &= \sup_k \Big\{\sup_n P_k\Big(\sum_{k(i)\leqq n} x^{k(i)} e_i\Big)\Big\} \\ &= \sup_k \{P_k(x^{k(1)}, x^{k(2)}, \cdots)\}\end{aligned}$$

since each P_k is an s.n. That S_P is an algebra follows since

$$\begin{aligned}P(xy) &= \sup_k \{P_k(x^{k(i)} y^{k(i)})\} \\ &\leqq \sup_k \{P_k(x^{k(i)}) P_k(y^{k(i)})\} \\ &\leqq P(x)P(y) .\end{aligned}$$

Therefore, S_P is a γ-perfect *BK*-algebra; it contains e because of (6–4).

EXAMPLE. Let M be the set of all sequences x such that

$$P(x) = \sup_k \Big\{\sum_{i=1}^{\infty} | x^{k(i)} - x^{k(i+1)} | + \lim_{k(i)} | x^{k(i)} |\Big\} < \infty$$

for $N_1, N_2, \cdots$ a partition of the integers with each N_k given by (6–3). Then M is a γ-perfect *BK*-algebra containing e but is neither m nor bv. The sequence y with

$$y^{(i)} = \begin{cases} 1 \text{ for } i = k(1) \text{ for each } k \\ 0 \text{ otherwise} \end{cases}$$

is in M but not in bv. The sequence z with

$$z^{(i)} = \begin{cases} 1 \text{ for } i = k(2j-1) \text{ for each } j < k \text{ and all } k \\ 0 \text{ otherwise} \end{cases}$$

is in m but not M.

Question. Are there γ-perfect *BK*-algebras other than those in the smallest class of BK-algebras which contain bv., and m and are closed under the operations described in Theorem 6.2.?

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Received August 5, 1968.

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