PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

RANK k GRASSMANN PRODUCTS

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The general question concerning the structure of subspaces of a symmetry class of tensors in which every nonzero element has an irreducible representation as a sum of decomposable (or pure) elements of a given length is as yet largely unanswered. This problem relates to the problem of characterizing the linear transformations on such a symmetry class which map the set of tensors of "irreducible length" k into itself; i.e., preserves the *rank* k of the tensors. Another related problem is: "Is it possible to obtain algebraic relations involving the components of a tensor which imply it has rank ("Irreducible length") k, for any positive integer k"?

This paper is concerned mostly with the third question for the $\binom{n}{r}$-dimensional Grassmann Product Space $\wedge^r U$, where U is an n-dimensional vector space over a field F. It includes some discussion of the first question for F *algebraically closed* and $r = 2$.

A vector in $\wedge^r U$ is said to have *rank* k if it can be expressed as the sum of k, and *not less than* k, nonzero pure r-vectors in $\wedge^r U$. We denote the set of such vectors by $C_k^r(U)$. The nonzero pure products in $\wedge^r U$ have rank one.

The results obtained in this paper are as follows: (i) the rank of a vector in $\wedge^r U$ is unchanged if we extend U, (ii) in the Grassmann Algebra $\wedge^0 U + \wedge^1 U + \cdots + \wedge^r U + \cdots$, multiplication of a Grassmann product by a nonzero vector in U either annihilates it or preserves its rank, (iii) we can associate with each vector z in $C_k^r(U)$ a unique subspace $U(z)$ in U, (iv) if $z \in C_k^r(U)$ and $\dim U(z)$ is rk, then z has rank k, (v) $x_1 \wedge y_1 + \cdots + x_s \wedge y_s \in C_s^2(U)$ if and only if $\{x_1, y_1, \cdots, x_s, y_s\}$ is independent. Finally, we discuss the rank two subspaces in $\wedge^2 U$ when $\dim U = 4$. If F is *algebraically closed*, these subspaces are of dimension one. Otherwise, they can be different, as the examples show.

In this paper, $Q(k, t, n)$ will denote the totality of strictly increasing sequences of k integers chosen from $t, t+1, \cdots, n$; $S(k, t, n)$ the totality of sequences of k integers chosen from $t, t+1, \cdots, n$.

Let $x_1, \cdots, x_n$ be a basis of U. For $\omega = (i_1, \cdots, i_r) \in Q(r, 1, n)$, we denote the product $x_{i_1} \wedge \cdots \wedge x_{i_r}$ by $\boldsymbol{x}_\omega$.

Let p be an r-linear alternating function from $\pi_{i=1}^r E \to F$, $E = \{1, \cdots, n\}$.

We will need the following known result.

THEOREM 1. (See [2], p. 289–312.) *Let*

$$z = \sum p(\omega)\boldsymbol{x}_\omega, \ (\omega \in Q(r, 1, n)) \ .$$

Then z is a pure vector if and only if

$$(1) \qquad \sum_{\mu=0}^{r} (-1)^\mu p(\alpha, j_\mu) p(j_0, \cdots, j_{\mu-1}, j_{\mu+1}, \cdots, j_r) = 0$$

for all $\alpha \in S(r-1, 1, n)$ and all $(j_0, \cdots, j_r) \in S(r+1, 1, n)$.

Furthermore, there are $(n-r)$ independent equations in the system of equations (1).

The following lemma will be useful.

LEMMA 2. *Let $z = \sum p(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 1, n); z \in C_k^r(U))$. Let s, m be integers, $0 \leqq s \leqq r, 0 \leqq m \leqq n$, and let*

$$z' = \sum p(1, \cdots, s, \alpha) x_1 \wedge \cdots \wedge x_s \wedge \boldsymbol{x}_\alpha, \qquad (\alpha \in Q(m-s, s+1, m)) \ .$$

Then $z' \in C_l^r(U)$, for some $l, 0 \leqq l \leqq k$.

Proof. We prove first the case $k = 1$.
Let $\omega = (i_1, \cdots, i_r) \in Q(r, 1, n)$. We set

$$p'(i_1, \cdots, i_r) = p(i_1, \cdots, i_r)$$

if $i_1 = 1, \cdots, i_s = s$, and $s+1 \leqq i_{s+1} < \cdots < i_r \leqq m$. Otherwise, $p'(i_1, \cdots, i_r) = 0$. Then $z' = \sum p'(\omega) x_\omega; (\omega \in Q(r, 1, n))$. It is easy to show that the system of equations (1) holds for the p' s; (there are 3 cases to check; viz., $i_t > m$ or $j_t > m$ for some t; not all of the integers $1, \cdots, s$ are present in $i_1, \cdots, i_{r-1}$ or not all of the integers $1, \cdots, s$ are present in $j_0, \cdots, j_r$; and, thirdly, all the integers $1, \cdots, s$ are present in $i_1, \cdots, i_{r-1}$ and in $j_0, \cdots, j_r$ with $i_t \leqq m$ $(t = 1, \cdots, r-1)$ and $j_l \leqq m$ $(l = 0, \cdots, r)$). Thus, by Theorem 1, $z' \in C_1^r(U)$ or is zero.
For $z = z_1 + \cdots + z_k \in C_k^r(U), z_i \in C_1^r(U)$ $(i = 1, \cdots, k)$, we apply the above result to each term z_i, noting that

$$z' = (z_1 \cdots + z_k)' = z_1' + \cdots + z_k' \ .$$

THEOREM 3. *Let $U' \subseteq U$ be a subspace.*
Then $C_k^r(U') \subseteq C_k^r(U)$.

Proof. Let $x_1, \cdots, x_s$ be a basis of U', and let $x_1, \cdots, x_n$ be an extension of this basis to a basis of U. Let

$$y_1 + \cdots + y_k \in C_k^r(U'), y_i \in C_1^r(U') \ .$$

Suppose $y_1 + \cdots + y_k = z_1 + \cdots + z_l \in C_l^r(U), z_i \in C_1^r(U)$. Clearly

$l \leqq k$.

To show $l \geqq k$, let

$$z_j = \sum p^{(j)}(\omega)x_\omega,\ \omega \in Q(r, 1, n), \qquad 1 \leqq j \leqq l\ .$$

Since $y_i \in C_1^r(U')$, $1 \leqq i \leqq k$, then

$$\sum_{j=1}^{l} p^{(j)}(\omega) = 0$$

whenever $\omega = (i_1, \cdots, i_r)$ and $\{i_1, \cdots, i_r\} \not\subseteq \{1, \cdots, s\}$. Hence

$$z'_j = \sum p^{(j)}(\omega)x_\omega,\ \omega \in Q(r, 1, s)\ ,$$

is in $C_1^r(U')$ by Lemma 2, and since $z'_1 + \cdots + z'_l = z_1 + \cdots + z_l = y_1, \cdots, y_k$, the $l \geqq k$.

DEFINITION. For $z \in C_k^r(U)$, we define $R_r(z) = k$; i.e., $R_r\colon \wedge^r U \to J$ such that $R_r(z) = k$ if and only if $z \in C_k^r(U)$.

We will drop the index r when no confusion arises.

If $x \in U, z \in \wedge^r U$ such that $z = \sum p(\omega)x_\omega, \omega \in Q(r, 1, n)$, where $x_1, \cdots, x_n$ is a basis of U, then we write $x \wedge z$ for the vector

$$\sum p(\omega)x \wedge x_\omega,\ \omega \in Q(r, 1, n)\ .$$

If $z = x_1 \wedge \cdots \wedge x_r$ is a nonzero pure vector in $\wedge^r U$, then we shall denote the r-dimensional space $\langle x_1, \cdots, x_n \rangle$ by $U(z)$.

THEOREM 4. *Let* $y = y_1 + \cdots + y_k \in C_k^r(U), y_i \in C_1^r(U), 1 \leqq i \leqq k$.

(i) *Suppose* $x \wedge (y_1 + \cdots + y_k) = 0, x \in U$. *Then* $x \in U(y_i)$, $i = 1, \cdots, k$.

(ii) *Suppose* $x \in U, x \notin U(y_1) + \cdots + U(y_k)$. *Then* $x \wedge y \in C_k^{r+1}(U)$.

Proof. (i) Suppose on the contrary that $x \notin U(y_1)$. Then

$$x \wedge y_1 = x \wedge \left(-\sum_{i=2}^{k} y_i\right) \neq 0\ .$$

Thus, we can choose a basis $x_1, \cdots, x_n$ of U such that

$$x = x_1,\ y_1 = x_2 \wedge \cdots \wedge x_{r+1}\ .$$

Then

$$\left(-\sum_{i=2}^{k} y_i\right) = x_2 \wedge \cdots \wedge x_{r+1} + \sum p(1, \alpha)x_1 \wedge x_\alpha,\ (\alpha \in Q(r-1, 2, n))\ .$$

Hence $(-\sum_{i=2}^{k} y_i) = y_1 + x \wedge v$, where $v = \sum p(1, \alpha)x_\alpha \in \wedge^{r-1}U$. Taking

$s = 1$, $m = n$ in Lemma 2, it is easy to see that since $R(-\sum_{i=2}^{k}) = k - 1$, then $R(x \wedge v) \leqq k - 1$. But $x \wedge v = -(y_1 + \cdots + y_k)$ which implies $R(x \wedge v) = k$. We have a contradiction. Therefore $x \in U(y_1)$. Similarly, $x \in U(y_i)$, $i = 2, \cdots, k$.

(ii) Suppose that

$$x \wedge y = z_1 \cdots + z_l \in C_l^{r+1}(U), z_i \in C_1^{r+1}(U)\ , \qquad 1 \leqq i \leqq l\ .$$

Clearly $l \leqq k$.

To show $l \geqq k$, we choose a basis $x_1, \cdots, x_n$ of U such that $x = x_1$ and $x_2, \cdots, x_s$ is a basis of $U(y_1) + \cdots + U(y_k)$. Then

$$y = \sum p(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 2, n))\ .$$

Using (i) and the fact that $x \wedge (x \wedge y) = x_1 \wedge (z_1 + \cdots + z_l) = 0$, we can express each $z_j = x_1 \wedge (\sum p^{(j)}(\omega)\boldsymbol{x}_\omega;\ \omega \in Q(r, 2, n))$, $1 \leqq j \leqq l$.

Now $\sum_{j=1}^{l} p^{(j)}(\omega) = 0, (\omega = (i_1, \cdots, i_r))$, *unless*

$$\{i_1, \cdots, i_r\} \subseteq \{2, \cdots, s\}\ .$$

In the latter case, $\sum_{j=1}^{l} p^{(j)}(\omega) = p(\omega)$. Therefore, $z_1 + \cdots + z_l = z_1' + \cdots + z_l' = x \wedge y$ where

$$z_j' = \sum p^{(j)}(\omega)x_1 \wedge \boldsymbol{x}_\omega, (\omega \in Q(r, 2, s))\ .$$

Hence $y = z_1'' + \cdots + z_l''$, where $z_j'' = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 2, s))$, which implies $R(y) \leqq l$, i.e., $k \leqq l$.

THEOREM 5. *Let* $y_i \in C_1^r(U), z_i \in C_1^r(U)$, $(i = 1, \cdots, k)$ *such that* $y_1 + \cdots + y_k = z_1 + \cdots + z_k$.

Then $U(y_1) + \cdots + U(y_k) = U(z_1) + \cdots + U(z_k)$.

Proof. Suppose on the contrary that there exists a vector $x \in U(y_1)$ such that $x \notin U(z_1) + \cdots + U(z_k)$. Since $x \wedge (y_1 + \cdots + y_k) = x \wedge (z_1 + \cdots + z_k)$, then

$$R(x \wedge (y_1 + \cdots + y_k)) = R(x \wedge (z_1 + \cdots + z_k)) \leqq k - 1\ .$$

But, by Theorem 4 (ii), $R(x \wedge (z_1 + \cdots + z_k)) = k$, which is a contradiction.

DEFINITION. Let

$$z = z_1 + \cdots + z_k \in C_k^r(U), z_i \in C_1^r(U), \qquad i = 1, \cdots, k\ .$$

Then we define $U(z)$ to be the subspace $U(z_1) + \cdots + U(z_k)$.

THEOREM 6. *Let* $z_i \in C_1^r(U)$, $i = 1, \cdots, k$, *and let*

$$\dim [U(z_1) + \cdots + U(z_k)] = rk .$$

Then $R(z_1 + \cdots + z_k) = k$.

Proof. Suppose the Theorem is false. Let k be the smallest integer for which it fails. Clearly $k \geqq 2$. Let

$$z_1 + \cdots + z_k = y_1 + \cdots + y_l \in C_l^r(U),\ y_i \in C_1^r(U) .$$

Let $x \in U(z_1)$. Then $x \notin U(z_2) + \cdots + U(z_k)$. By the choice of

$$k,\ z_2 + \cdots + z_k \in C_{k-1}^r(U) .$$

Hence, by Theorem 4 (ii),

$$x \wedge (z_2 + \cdots + z_k) = x \wedge (z_1 + \cdots + z_k) = x \wedge (y_1 + \cdots + y_l) ,$$

and $l \geqq k - 1$. But we assumed $l < k$. Therefore $l = k - 1$.

By Theorem 5,

$$U(x \wedge z_2) + \cdots + U(x \wedge z_k) = U(x \wedge y_1) + \cdots + U(x \wedge y_{k-1}) .$$

Hence $\langle x \rangle + U(z_2) + \cdots + U(z_k) = \langle x \rangle + U(y_1) + \cdots + U(y_{k-1})$.

Now let $x' \in U(z_1)$, independent of x. Then again

$$\langle x' \rangle + U(z_2) + \cdots + U(z_k) = \langle x' \rangle + U(y_1) + \cdots + U(y_{k-1}) .$$

Taking intersections, we obtain

$$U(z_2) + \cdots + U(z_k) = U(y_1) + \cdots + U(y_{k-1}) .$$

By a similar argument,

$$\begin{aligned} V_i &= U(z_1) + \cdots + U(z_{i-1}) + U(z_{i+1}) + \cdots + U(z_k) \\ &= U(y_1) + \cdots + U(y_{k-1}) . \end{aligned}$$

Hence $U(y_1) + \cdots + U(y_{k-1}) = \bigcap_{i=1}^k V_i = \{0\}$, which is impossible. The result follows.

THEOREM 7. $\sum_{i=1}^s x_i \wedge y_1 \in C_s^2(U)$ *if and only if* $(\{x_1, y_1, \cdots x_s, y_s\}$ *is independent.*

Proof. If $\{x_1, y_1, \cdots, x_s, y_s\}$ is dependent, it is easy to show that $R(\sum_{i=1}^s x_i \wedge y_i) \leqq s - 1$. It follows that the condition is necessary.

The converse follows easily from Theorem 6.

COROLLARY 8. *Let* $f = \sum_{i=1}^s x_i \wedge y_i$, *and* $\dim \langle x_1, y_1, \cdots, x_s, y_s \rangle \langle 2k, k \leqq s$. *Then* $R(f) \leqq k - 1$.

We shall now direct our attention to the rank 2 subspaces of $\wedge^2 U$.

DEFINITION. A *rank* 2 *subspace* H in $\wedge^2 U$ is a subspace whose nonzero members are in $C_2^2(U)$.

In this paper, we shall restrict our considerations to the case $\dim U = 4$. It is clear from Theorem 7 that $C_2^2(U)$ is empty when $\dim U < 4$.

LEMMA 9. *Let $f \in C_2^2(U)$ and let $\{y_1, \cdots, y_4\}$ be any basis of $U(f)$. Then f has a representation $f = y_1 \wedge u + v \wedge w$, where $\langle u, v, w\rangle = \langle y_2, y_3, y_4\rangle$.*

Proof. Since $f \in \wedge^2 \langle y_1, \cdots, y_4\rangle$, then

$$\begin{aligned} f &= \sum p(\omega)\boldsymbol{y}_\omega,\ (\omega \in Q(2, 1, 4)),\ p(\omega) \in F\ , \\ &= y_1 \wedge (\textstyle\sum_{j=2}^4 p(1, j)y_j) + \sum p(\alpha)\boldsymbol{y}_\alpha\ ; \qquad (\alpha \in Q(2, 2, 4))\ , \end{aligned}$$

which is of the form $y_1 \wedge u + v \wedge w$. It follows from Theorem 7 and its corollary, and the fact that $R(f) = 2$ that

$$\langle u, v, w\rangle = \langle y_2, y_3, y_4\rangle\ .$$

THEOREM 10. *Let* $\dim U = 4$ *and let H be a rank* 2 *subspace in $\wedge^2 U$. Then* $\dim H = 1$, *provided F is algebraically closed.*

Proof. Let f be a nonzero member of H. Then f has a representation $f = x_1 \wedge x_2 + x_3 \wedge x_4$ in $C_2^2(U)$. By Theorem 7,

$$U = U(f) = \langle x_1, \cdots, x_4\rangle\ .$$

If f' is any other nonzero member of H, then $U(f') = \langle x_1, \cdots, x_4\rangle$. By Lemma 9, $f' = x_1 \wedge u + v \wedge w, \langle u, v, w\rangle = \langle x_2, x_3, x_4\rangle$. Hence $\dim \langle v, w\rangle \cap \langle x_3, x_4\rangle \leqq 1$. Without loss of generality, we shall assume $x_3 \in \langle v, w\rangle \cap \langle x_3, x_4\rangle$. Hence

$$f' = x_1 \wedge u + x_3 \wedge w', \langle u, w'\rangle \subset \langle x_2, x_3, x_4\rangle\ .$$

Let $u = \sum_{i=2}^4 b_i x_i$; $w' = \sum_{i=2,4} d_i x_i$; $b_i, d_i \in F$. Then for

$$\begin{aligned} \lambda \in F, z = \lambda f + f' &= x_1 \wedge (\lambda x_2 + b_2 x_2 + b_3 x_3 + b_4 x_4) \\ &\quad + x_3 \wedge (\lambda x_4 + d_2 x_2 + d_4 x_4)\ . \end{aligned}$$

The condition that the vectors

$$x_1, (\lambda x_2 + b_2 x_2 + b_3 x_3 + b_4 x_4), x_3, (\lambda x_4 + d_2 x_2 + d_4 x_4)$$

be independent; i.e., $R(z) = 2$, is equivalent to the condition that the determinant

$$\Gamma(\lambda, f_1, f_2) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda + b_2 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \\ 0 & d_2 & 0 & \lambda + d_4 \end{vmatrix} \quad be$$

nonzero. Now

$$\Gamma(\lambda, f_1, f_2) = \lambda^2 + \lambda(d_4 + b_2) + (b_2d_4 - d_2b_4) = g(\lambda) .$$

Since u, w' are independent, $(b_2d_4 - d_2b_4) \neq 0$. Hence $g(\lambda)$ is a nontrivial polynomial in λ, and hence, for some nonzero λ in F, $g(\lambda) = 0$; i.e., $\Gamma(\lambda, f_1, f_2) = 0$. For such a λ, $R(z) \leqq 1$. It follows that dim $H = 1$.

The above theorem is false when F is *nonalgebraically closed.* For example, the vectors

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$

and

$$f_2 = x_1 \wedge (x_3 + x_4) + (x_3 - x_2) \wedge x_4$$

in $C_2^2(U)$, where $U = \langle x_1, \cdots, x_4\rangle$, dim $U = 4$, $F \equiv$ Reals, generate a 2-dimensional rank 2 subspace in $\wedge^2 U$.

It is interesting to note that if F (nonalgebraically closed) has an *irreducible quadratic* polynomial $h(\lambda)$, and dim $U = 4$, then we can construct 2 independent vectors f_1, f_2 in $C_2^2(U)$, which will generate a 2-dimensional rank 2 subspace in $\wedge^2 U$, and such that $\Gamma(\lambda, f_1, f_2) = h(\lambda)$ (see Theorem 10). The construction is as follows:

Let dim $U = 4$, $U = \langle x_1, \cdots, x_4\rangle$. Let $h(\lambda) = \lambda^2 + a_1\lambda + a_0$ be irreducible in F. The companion matrix of $h(\lambda)$ is

$$B = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}; \qquad \lambda I - B = \begin{bmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{bmatrix}.$$

Now

$$\det(\lambda I - B) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & a_0 & 0 & \lambda + a_1 \end{vmatrix} = h(\lambda) \neq 0 .$$

Taking this determinant to be $\Gamma(\lambda, f_1, f_2)$ corresponding to $z = \lambda f_1 + f_2$, where $f_1, f_2 \in C_2^2(U)$, $\lambda \in F$, we have

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$
$$f_2 = x_1 \wedge (-x_4) + x_3 \wedge (a_0x_2 + a_1x_4) .$$

The construction is complete. Thus, for example, if $F \equiv$ Rationals and $h(\lambda) = \lambda^2 - 2$, then

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$

and

$$f_2 = x_1 \wedge (-x_4) + (-2)x_3 \wedge x_2 \ ,$$

and f_1, f_2 generate a 2-dimensional rank 2 subspace in $\wedge^2 U$.

For the work in this paper, I am greatly indebted to Dr. R. Westwick for his generous assistance.

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Received May 10, 1968.

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