

INITIAL SEGMENTS OF ONE-ONE DEGREES

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Let 0 be the one-one degree consisting of the infinite recursive sets whose complements are also infinite. In this paper are studied initial segments of the one-one degrees ≥ 0 . A characterization is stated of the order types of those initial segments with greatest member which are at the same time initial segments of the many-one degrees. It is shown that any finite initial segment with greatest member is a lattice, and that any finite recursively enumerable initial segment with greatest member is a distributive lattice.

According to the usual definition of one-one reducibility a set of natural numbers A is one-one reducible to a set of natural numbers B written $A \leq_1 B$ if there is a one-one recursive function ρ such that $\rho^{-1}(B) = A$. The relation of one-one equivalence \equiv_1 defined by

$$A \equiv_1 B \Leftrightarrow [A \leq_1 B \text{ and } B \leq_1 A]$$

is an equivalence relation, and the equivalence classes into which it partitions the sets of natural numbers are called one-one degrees. The natural ordering of these degrees induced by \leq_1 is denoted by \leq . In this paper we shall be wholly concerned with the one-one degrees ≥ 0 . Notice that if A has one-one degree ≥ 0 then neither A nor its complement is immune. In this context it is convenient to adopt a new definition of one-one reducibility: we say that A is *one-one reducible* to a set of natural numbers B , again written $A \leq_1 B$, if there exists a one-one partial recursive (p.r.) function ρ such that $\text{dom } \rho$ (the domain of ρ) and $A \cup \text{dom } \rho$ are both recursive, and such that $\rho^{-1}(B) = A \cap \text{dom } \rho$. On sets whose one-one degrees in the old sense are ≥ 0 the new definition of \leq_1 is the same as the old. Also, for any set A there is a set B whose one-one degree in the old sense is ≥ 0 , and such that $A \equiv_1 B$ in the new sense. Thus adopting this new definition leaves the one-one degrees ≥ 0 unchanged, and suppresses the remaining one-one degrees.

We adopt the usual definition of many-one reducibility save that \emptyset, N (the set of all natural numbers) are both by convention in the zero many-one degree consisting of all recursive sets.

The only notations likely to be unfamiliar consist in writing $A \oplus B$ for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, A' for the complement of A , and $\text{rng } \rho$ for the range of ρ .

1. Common initial segments of the many-one and one-one degrees. In [3] we obtained a characterization of the isomorphism

types of the initial segments of the many-one degrees. We can sharpen the construction in § 2 of [3] to obtain:

THEOREM 1. *An upper semilattice L is isomorphic to a common initial segment of the many-one and one-one degrees with greatest member $\neq 0$ if and only if there are sequences $\langle D_i \rangle$ of finite distributive lattices of cardinality > 1 , and $\langle \chi_i \rangle$ of maps, such that for each i , χ_i is an upper semilattice homomorphism of D_i into D_{i+1} preserving 0 and 1, and such that L is isomorphic to the direct limit of the sequence*

$$D_0 \xrightarrow{\chi_0} D_1 \xrightarrow{\chi_1} \dots \xrightarrow{\chi_{i-1}} D_i \xrightarrow{\chi_i} \dots$$

(By saying that χ_i "preserves 0 and 1" we mean that χ_i maps the least element, greatest element respectively, of D_i into the least element, greatest element respectively, of D_{i+1} .)

Clearly if we consider only finite initial segments then the condition of the theorem reduces to: L is a finite distributive lattice of cardinality > 1 . By an argument not unlike Friedberg's maximal set construction one can establish:

THEOREM 2. *If L is a finite distributive lattice then there is a common recursively enumerable (r.e.) initial segment of the many-one and one-one degrees isomorphic to L .*

It follows from Theorem 1 that the elementary theory of the partial ordering of one-one degrees is undecidable (for details see [1], § 3). From Theorem 2 the same is true of the partial ordering of the r.e. one-one degrees, but for this one has to use the fact that the sets of first order sentences true of no distributive lattice, and true of some finite distributive lattice respectively, are recursively inseparable. (Unfortunately, we know no reference for this result.)

2. **Finite initial segments of the one-one degrees.** In this section we shall show that every finite initial segment of the one-one degrees is a lattice of a certain kind.

The *disjoint union* $a \oplus b$ of a pair of one-one degrees a, b is defined to be the one-one degree of $A \oplus B$ where A, B are any representatives of a, b respectively. The reader will easily verify that $a \oplus b$ depends only on a, b and that the operation \oplus is associative on one-one degrees. For each one-one degree a and natural number n , let na be 0 if $n = 0$, and be

$$\underbrace{a \oplus a \oplus \cdots \oplus a}_{n \text{ times}}$$

otherwise. A one-one degree a is called a *cylinder* just if $2a = a$; notice that a is a cylinder just if all its representatives are cylinders in the sense of Myhill [5].

A one-one degree a is called *decomposable* if there exist one-one degrees b, c both $< a$ such that $a = b \oplus c$; otherwise a is called *indecomposable*.

A finite sequence $\{a_1, a_2, \dots, a_n\}$ of one-one degrees is called a *canonical decomposition* of a one-one degree a if $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ where each a_i is indecomposable and where deleting any of a_1, a_2, \dots, a_n in the last equation destroys the equality. We shall first show that when canonical decompositions exist they are unique. To this end we prove:

LEMMA 1. *If*

$$a_0 \oplus a_1 \oplus \cdots \oplus a_{m-1} = b_0 \oplus b_1 \oplus \cdots \oplus b_{n-1}$$

with m, n both ≥ 1 then there exist one-one degrees $\{a_{i,j} \mid i < m, j < n\}$ such that

$$\begin{aligned} a_i &= a_{i,0} \oplus a_{i,1} \oplus \cdots \oplus a_{i,n-1}, \quad i < m; \\ b_j &= a_{0,j} \oplus a_{1,j} \oplus \cdots \oplus a_{m-1,j}, \quad j < n. \end{aligned}$$

Proof. For $i < m$ let A_i be a representative of a_i , for $j < n$ let B_j be a representative of b_j . Let

$$\begin{aligned} A &= \{mx + i \mid x \in A_i \text{ and } i < m\}, \\ B &= \{nx + j \mid x \in B_j \text{ and } j < n\} \end{aligned}$$

then A, B have one-one degrees

$$a_0 \oplus a_1 \cdots \oplus a_{m-1}, \quad b_0 \oplus b_1 \oplus \cdots \oplus b_{n-1}$$

respectively. By hypothesis these are equal and without loss of generality we may suppose that A, B are one-one equivalent in the old sense. By a result of Myhill ([4], §5) there is a recursive permutation ρ mapping A onto B . For $i < m, j < n$ define

$$A_{i,j} = \{x \mid \rho(mx + i) = j \bmod n \text{ and } x \in A_i\}.$$

Letting $a_{i,j}$ be the one-one degree of $A_{i,j}$ the lemma follows by inspection.

THEOREM 3. *If a one-one degree has a canonical decomposition*

then its canonical decomposition is unique to within a permutation.

Proof. Let $a_0 \oplus a_1 \oplus \dots \oplus a_{m-1}, b_0 \oplus b_1 \oplus \dots \oplus b_{n-1}$ be two canonical decompositions of the same one-one degree. Applying Lemma 1 and using the notation above, since a_i is indecomposable, there exists $\alpha(i)$ such that $a_i = a_{i, \alpha(i)}$. Similarly there exists $\beta(i)$ such that $b_i = a_{\beta(i), i}$. As i runs through the numbers $< m$, $\alpha(i)$ must run through all numbers $< n$; for suppose $\alpha(i)$ is never zero then

$$a_0 \oplus a_1 \oplus \dots \oplus a_{m-1} \leq b_1 \oplus b_2 \oplus \dots \oplus b_{n-1}$$

which contradicts the definition of canonical decomposition. Similarly $\beta(i)$ runs through all numbers $< m$. Hence $m = n$ and α, β are both permutations of the numbers $< m$. We claim that $a_i = b_{\alpha(i)}$ and $b_i = a_{\beta(i)}$ for each $i < m$. It is clear that $a_i \leq b_{\alpha(i)}$ and $b_i \leq a_{\beta(i)}$ for each $i < m$. For reductio ad absurdum suppose that our claim is false then the set

$$S = \{a_i \mid a_i < b_{\alpha(i)} \text{ and } i < m\} \cup \{b_i \mid b_i < a_{\beta(i)} \text{ and } i < m\}$$

is nonempty, and without loss of generality we may suppose that $a_k < b_{\alpha(k)}$ and that a_k is a maximal member of S . By choice of k we have

$$a_k < b_{\alpha(k)} = a_{\beta\alpha(k)} = b_{\alpha\beta\alpha(k)} = a_{\beta\alpha\beta\alpha(k)} = \dots .$$

But $\beta\alpha$ is a permutation and so one of the suffixes of a in the chain of equalities is equal to k . This contradiction completes the proof of the theorem.

Let a, b be one-one degrees then b/a is defined to be $\sup \{n \mid na \leq b\}$; b/a may take any of the values $0, 1, 2, \dots, \infty$. Notice that for any indecomposable $a, (b \oplus c)/a = (b/a) + (c/a)$ where it is understood that $x + \infty = \infty + x = \infty$. If a is not a cylinder then $a, 2a, 3a, \dots$ is a strictly increasing sequence of one-one degrees whence $a/a = 1$. If a is a cylinder then b/a is zero or ∞ , and if $a \leq b$ and a is a cylinder then $b \oplus a = b$.

Notice that if a one-one degree a has only finitely many predecessors then some sequence of the degrees $\leq a$ must constitute a canonical decomposition of a ; for example if a is indecomposable then it is its own canonical decomposition.

LEMMA 2. *Let a, b be one-one degrees such that a has only finitely many predecessors. Let $a_0 \oplus a_1 \oplus \dots \oplus a_m$ be the canonical decomposition of a . Then $a \leq b$ if and only if a/a_i for all $i \leq m$.*

Proof. The “only if” part is immediate. For the “if” part we

proceed by induction on m . Since the result is immediate for $m = 0$ we may suppose that $m > 0$. Let $\mathbf{a}^* = \mathbf{a}_0 \oplus \mathbf{a}_1 \oplus \dots \oplus \mathbf{a}_{m-1}$ and suppose that $\mathbf{a}/\mathbf{a}_i \leq \mathbf{b}/\mathbf{a}_i$ for all $i \leq m$. It follows from our induction hypothesis that $\mathbf{a}^* \leq \mathbf{b}$ since it is easy to see that $\mathbf{a}_0 \oplus \mathbf{a}_1 \oplus \dots \oplus \mathbf{a}_{m-1}$ is the canonical decomposition of \mathbf{a}^* . Without loss of generality we assume that \mathbf{a}_m is minimal among $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_m$.

Let B be a representative of \mathbf{b} . It is convenient to introduce a map Ψ from the r.e. sets onto the one-one degrees $\leq \mathbf{b}$. If W is finite let $\Psi(W) = \mathbf{0}$, if W is an infinite r.e. set let ω be any one-one recursive function whose range is W and let $\Psi(W)$ be the one-one degree of $\{x \mid \omega(x) \in B\}$. It is easy to see that $\Psi(W)$ does not depend upon the choice of w , that Ψ is order preserving, and that

$$\Psi(W) = w_0 \oplus w_1 \oplus \dots \oplus w_{e-1}$$

if and only if W is the union of disjoint r.e. sets W_0, W_1, \dots, W_{e-1} such that $\Psi(W_i) = w_i$ for each $i < e$. Since $\mathbf{a}^* \leq \mathbf{b}$ there exist disjoint r.e. sets A_0, A_1, \dots, A_{m-1} such that $\Psi(A_i) = \mathbf{a}_i$ for each $i < m$. There are two cases to consider:

Case 1. \mathbf{a}/\mathbf{a}_m is finite. Then \mathbf{a}_m is not a cylinder because of the remarks that precede this lemma. Let $\mathbf{a}^*/\mathbf{a}_m = k$ so that $\mathbf{b}/\mathbf{a}_m \geq \mathbf{a}/\mathbf{a}_m \geq k + 1$. It easily follows that there are disjoint r.e. sets C_0, C_1, \dots, C_k such that $\Psi(C_j) = \mathbf{a}_m$ for each $j \leq k$. Effectively enumerate the r.e. sets $A_0, A_1, \dots, A_{m-1}, C_0, C_1, \dots, C_k$ simultaneously stage by stage letting A_i^s and C_i^s be the finite sets enumerated up to stage s for each $i < m$ and $j \leq k$. We then define

$$F_i = \{x \mid (\exists s)[x \in A_i^s \text{ and } (\forall j)_{j \leq k} (\forall r)_{r < s} [x \notin C_j^r]]\}, \quad i < m ;$$

$$G_j = \{x \mid (\exists s)[x \in C_j^s \text{ and } (\forall i)_{i < m} (\forall r)_{r \leq s} [x \notin A_i^r]]\}, \quad j \leq k .$$

Also, for each $i < m$ let F_i^* be the r.e. set $A_i - F_i$, and for each $j \leq k$ let G_j^* be the r.e. set $C_j - G_j$. Observe that $F_i^* \subseteq G_0 \cup G_1 \cup \dots \cup G_k$ for all $i < m$ and $G_j^* \subseteq F_0 \cup F_1 \cup \dots \cup F_{m-1}$ for all $j \leq k$. Also, it can be seen that $F_0, F_1, \dots, F_{m-1}, G_0, G_1, \dots, G_k$ constitute a decomposition of $A_0 \cup A_1 \cup \dots \cup A_{m-1} \cup C_0 \cup C_1 \cup \dots \cup C_k$ into disjoint r.e. sets.

For each $i \leq m$, \mathbf{a}_i is indecomposable. This implies that either $\Psi(F_i) = \mathbf{a}_i$ or $\Psi(F_i^*) = \mathbf{a}_i$ for each $i < m$ and that either $\Psi(G_j) = \mathbf{a}_m$ or $\Psi(G_j^*) = \mathbf{a}_m$ for each $j \leq k$. Define P, Q by

$$P = \{i \mid i < m \text{ and } \Psi(F_i) = \mathbf{a}_i\},$$

$$Q = \{i \mid i < m \text{ and } i \notin P\} .$$

Now if $i \in Q$ then $\Psi(F_i^*) = \mathbf{a}_i$. But F_i^* is the union of the disjoint r.e. sets $F_i^* \cap G_0, \dots, F_i^* \cap G_k$ so that for each $i \in Q$ there exists $\alpha(i) \leq k$

such that $\Psi(F_i^* \cap G_{\alpha(i)}) = a_i$.

Consider the set

$$D = \alpha(Q) \cup \{j \mid j < k \text{ and } \Psi(G_j^*) = a_m\},$$

and let β be a map from $\alpha(Q)$ into Q such that $\alpha\beta(j) = j$ for all $j \in \alpha(Q)$. For each $j \in D$ we define

$$D_j = \begin{cases} G_j^* & \text{if } \Psi(G_j^*) = a_m, \\ F_{\beta(j)}^* \cap G_j & \text{otherwise.} \end{cases}$$

This is a sound definition because if $\Psi(G_j^*) \neq a_m$ then $j \in \alpha(Q)$ and so $\beta(j)$ is defined.

We first claim that $\Psi(D_j) = a_m$ for all $j \in D$. For, if $\Psi(G_j^*) \neq a_m$ then $\beta(j)$ is defined and $\Psi(G_j) = a_m$. Since Ψ is order preserving it follows that

$$a_{\beta(j)} = \Psi(F_{\beta(j)}^* \cap G_j) \leq \Psi(G_j) = a_m,$$

and because a_m is minimal we deduce that $a_{\beta(j)} = a_m$. It follows that $\Psi(F_{\beta(j)}^* \cap G_j) = \Psi(D_j) = a_m$. This proves our first claim.

Next we use this to deduce that $j \notin D$ for some $j \leq k$. For, otherwise $(k + 1)a_m = \Psi(D_0 \cup \dots \cup D_k) \leq \Psi(A_0 \cup \dots \cup A_{m-1}) = a^*$, because D_0, D_1, \dots, D_k are disjoint r.e. subset of $A_0 \cup \dots \cup A_{m-1}$. But $a^*/a_m = k$ and so this is impossible.

Now let j be a fixed number $\leq k$ which does not belong to D and for each $i \leq m$ define

$$E_i = \begin{cases} F_i & \text{if } i \in P \\ F_i^* \cap G_{\alpha(i)} & \text{if } i \in Q, \\ G_j & \text{if } i = m. \end{cases}$$

Then E_0, E_1, \dots, E_m are disjoint r.e. sets and $\Psi(E_i) = a_i$ for all $i \leq m$. Since $\Psi(E_0 \cup \dots \cup E_m) = a$ it follows that $a \leq b$ which completes this case.

Case 2. $a/a_m = \infty$. Then a_m is cylinder, otherwise a would have an infinite number of predecessors. Suppose $a_m \leq a^*$, then by the remarks preceding the lemma we have $a = a^* \oplus a_m = a^*$ which contradicts $a_0 \oplus a_1 \oplus \dots \oplus a_m$ being a canonical decomposition of a . Thus $a^*/a_m = 0$. Now, taking $k = 0$, proceed just as in the last case.

Before turning to the next theorem, we note that any finite upper semilattice is in fact a lattice as long as it has a least element. For, if least upper bounds exist then greatest lower bounds exist by the definition:

$$b \cap c = \sup \{x \mid x \leq b \text{ and } x \leq c\} .$$

THEOREM 4. *Let a be a one-one degree which has only a finite number of predecessors then the predecessors of a form a lattice.*

Proof. In view of the observation that precedes the theorem, we have only to show that the predecessors of a form an upper semilattice. Let b, c be any one-one degrees $\leq a$ and suppose that a has only a finite number of predecessors. Let a_1, a_2, \dots, a_p be an enumeration of all the indecomposable one-one degrees $\leq a$ arranged so that for any i, j

$$p \geq i > j \geq 1 \text{ implies } a_i \not\leq a_j .$$

To show that $b \cup c$ exists we shall define an increasing sequence e_1, e_2, \dots, e_p of one-one degrees $\leq a$ and simultaneously we shall show that for all $i, 1 \leq i \leq p$, and all $d \leq a$

$$(1) \quad b \leq d \text{ and } c \leq d \rightarrow e_i \leq d$$

and

$$(2) \quad e_i/a_i \geq \max (b/a_i, c/a_i) .$$

Define

$$e_1 = \begin{cases} a_1 & \text{if one of } b/a_1, c/a_1 \text{ is } \infty \\ \max (b/a_1, c/a_1)a_1 & \text{otherwise} \end{cases}$$

then $e_1 \leq a$ and (1), (2) are certainly satisfied for $i=1$. Suppose that e_i has been defined for $1 \leq i \leq k$ where $k < p$ so that (1), (2) are satisfied for $1 \leq i \leq k$. If

$$e_k/a_{k+1} \geq \max (b/a_{k+1}, c/a_{k+1}) ,$$

then set $e_{k+1} = e_k$, in which case (1), (2) are clearly satisfied for $i = k+1$. Otherwise let r, s be the numbers on the left, right respectively of the last inequality with the proviso that if a_{k+1} is a cylinder then $s = 1$. Set $e_{k+1} = e_k \oplus (s - r)a_{k+1}$. Any degree appearing in the canonical decomposition of e_{k+1} is clearly one of a_1, a_2, \dots, a_{k+1} . Hence by Lemma 2, $e_{k+1} \leq a$. Suppose that $b \leq d$ and $c \leq d$ then from (1) for $i = k$ we know that $e_k \leq d$ which implies that $d/a_i \geq e_k/a_i$ for $1 \leq i \leq k$. But since we are assuming $e_{k+1} \neq e_k$ we can deduce that a_{k+1} is different from all of a_1, a_2, \dots, a_k and does not exceed any of them. Thus

$$d/a_i \geq e_k/a_i = e_{k+1}/a_i , \quad 1 \leq i \leq k .$$

Also

$$d/a_{k+1} \geq s = e_{k+1}/a_{k+1}.$$

Any degree which appears in the canonical decomposition of e_{k+1} is one of a_1, a_2, \dots, a_{k+1} . Hence $e_{k+1} \leq d$ by Lemma 2. It is immediate that (2) is satisfied for $i = k + 1$. This completes the induction step. It is clear that e_p is the least upper bound of b, c . For $b \leq e_p$ and $c \leq e_p$ by (2) and Lemma 2, and e_p is the least among the common upper bounds of b, c from (1). This completes the proof of the theorem.

The next theorem gives a necessary condition for a finite lattice to be an initial segment of one-one degrees.

THEOREM 5. *Let a be a one-one degree with only a finite number of predecessors then there exist finite separated distributive lattices L_1, L_2 and an order preserving map Ψ of L_2 onto the one-one degrees $\leq a$ satisfying the following four conditions in which \subseteq, \emptyset denote the partial order and least element of L_2 respectively.*

(i) L_1 is an initial segment of L_2 containing at least \emptyset and $\Psi(L_1)$ is a distributive lattice,

(ii) if $x \cap y = \emptyset$ then $\Psi(x \cup y) = \Psi(x)$ if and only if $y \in L_1$ and $\Psi(y) \leq \Psi(x)$,

(iii) if $x_1 \cap x_2 = y_1 \cap y_2 = \emptyset$ and $\Psi(x_i) = \Psi(y_i)$ for $i = 1, 2$ then $\Psi(x_1 \cup x_2) = \Psi(y_1 \cup y_2)$,

(iv) for any $x \in L_2$ and $y \leq \Psi(x)$ there exists $y \subseteq x$ such that $\Psi(y) = y$.

Proof. We proceed by induction on the number of predecessors of a . When $a = \mathbf{0}$ the theorem is obvious. Suppose that $b > \mathbf{0}$ has only a finite number of predecessors and assume the result for $a < b$. Assume in addition that for any $a < b$ we can choose L_1, L_2, Ψ such that $\Psi(L_1)$ consists of the cylinders $\leq a$, and if $x_1 \cap x_2 = \emptyset$ then $\Psi(x_1 \cup x_2) = \Psi(x_1) \oplus \Psi(x_2)$.

Case 1. Suppose that b has just one maximal predecessor b_1 . Let $L_1^{(1)}, L_2^{(1)}, \Psi^{(1)}$ be the finite separated distributive lattices and map whose existence is predicted by the theorem, and satisfying the additional conditions, when $a = b_1$. Let $L_2 = L_2^{(1)} \cup \{b\}$ where $b \in L_2^{(1)}$, and let

$$L_1 = \begin{cases} L_1^{(1)} \cup \{b\} & \text{if } b \text{ is a cylinder} \\ L_1^{(1)} & \text{otherwise.} \end{cases}$$

Let $x \subseteq y$ in L_2 if and only if $y = b$ or x, y are both in $L_2^{(1)}$ and $x \subseteq y$ in $L_2^{(1)}$. Let $\Psi(b) = b$ and otherwise let Ψ agree with $\Psi^{(1)}$. It is easy

to check (i) – (iv) and the additional conditions for $\mathbf{a} = \mathbf{b}$.

Case 2. Suppose that \mathbf{b} has at least two maximal predecessors $\mathbf{b}_1, \mathbf{b}_2$. Using the technique of Lemma 2 it is easy to show that there exist $\mathbf{c}_1 \leq \mathbf{b}_1$ and $\mathbf{c}_2 \leq \mathbf{b}_2$ such that $\mathbf{b} = \mathbf{c}_1 \oplus \mathbf{c}_2$. For $i = 1, 2$ let $L_1^{(i)}, L_2^{(i)}, \Psi^{(i)}$ be the finite separated distributive lattices and the map predicted by the theorem, and satisfying the additional conditions, when $\mathbf{a} = \mathbf{c}_i$. For $j = 1, 2$ let L_j be the direct union of $L_j^{(1)}, L_j^{(2)}$. Then for $j = 1, 2, L_j$ is a finite separated distributive lattice, and L_1 is an initial segment of L_2 . For any pair (x, y) in L_2 define

$$\Psi(x, y) = \Psi^{(1)}(x) \oplus \Psi^{(2)}(y) .$$

It is again easy to check (i)–(iv) and the additional conditions for $\mathbf{a} = \mathbf{b}$. This completes the proof of the theorem.

We conjecture that the converse of Theorem 5 is true in the following sense. Let P be any partially ordered set such that there exist finite separated distributive lattice L_1, L_2 and an order preserving map Ψ of L_2 onto P such that (i)–(iv) are satisfied. We conjecture that there exists a one-one degree \mathbf{p} such that the lattice of one-one degree $\leq \mathbf{p}$ is isomorphic to P , and that under the isomorphism $\Psi(L_1)$ maps into the set of cylinders $\leq \mathbf{p}$, and that for any x_1, x_2 disjoint in L_2 the image of $\Psi(x_1 \cup x_2)$ under the isomorphism is the disjoint union of the images of $\Psi(x_1), \Psi(x_2)$. In the next section we shall prove a particular case of this conjecture. Although it is by no means obvious this conjecture is consistent with the earlier results of this section, e.g., it follows from the conditions imposed upon P that P is a lattice. This can be shown by a suitable modification of the proof of the earlier results.

There are a number of observations which are almost immediate from what has been proved above. First notice that if $\mathbf{a}_1, \mathbf{a}_2$ both have only a finite number of predecessors then $\mathbf{a}_1 \oplus \mathbf{a}_2$ has only a finite number of predecessors and so from Theorem 4, $\mathbf{a}_1 \cup \mathbf{a}_2$ and $\mathbf{a}_1 \cap \mathbf{a}_2$ exist in the partial ordering of one-one degrees. Next, if $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are distinct one-one degrees minimal $> \mathbf{0}$, then

$$\mathbf{a}_1 \cup \mathbf{a}_2 \cup \dots \cup \mathbf{a}_n = \mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \dots \oplus \mathbf{a}_n$$

and the one-one degrees $\leq \mathbf{a}_1 \oplus \mathbf{a}_2 \oplus \dots \oplus \mathbf{a}_n$ form a Boolean algebra of order 2^n . This shows that neither of the two nondistributive lattices of order five can be isomorphic to initial segments of the one-one degrees. One can also show rather easily that many other non-distributive lattices of small order cannot be isomorphic to initial segments of the one-one degrees, e.g., the lattices depicted in the usual way:

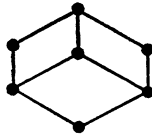


Fig. 1

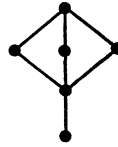


Fig. 2



Fig. 3

As an example take the second of these and suppose for reductio and absurdum that the initial segment whose greatest member is a is isomorphic to this lattice. Let L_1, L_2, Ψ be the separated distributive lattices and the map given by Theorem 5 and let the additional conditions of the proof of Theorem 5 be satisfied. Let a_1, a_2, a_3 be maximal below a , and let b be minimal above 0 . Let a_1, a_2 be elements of L_2 such that $\Psi(a_i) = a_i$ for $i = 1, 2$. Since L_2 is separated there exist b_1, b_2 such that $b_i \subseteq a_i$ for $i = 1, 2$ and such that $b_1 \cup b_2 = a_1 \cup a_2$ while $b_1 \cap b_2 = \emptyset$. Clearly, $\Psi(b_i) \leq a_i$ for $i = 1, 2$. Also,

$$\Psi(a_1 \cup a_2) = \Psi(b_1 \cup b_2)$$

is \geq both a_1 and a_2 since Ψ is order preserving. Thus $\Psi(b_1 \cup b_2) = a$. From (iv) there exists $a_3 \subseteq b_1 \cup b_2$ such that $\Psi(a_3) = a_3$. Now for $i = 1, 2$ $\Psi(a_3 \cap b_i)$ is \leq both a_3 and $\Psi(b_i)$ and hence is either 0 or b . But $\Psi(a_3) = \Psi(a_3 \cap b_1) \oplus \Psi(a_3 \cap b_2)$, whence $a_3 = b \oplus b$. Similarly, $a_1 = b \oplus b$, contradiction.

3. Not every initial segment is distributive. In this section we verify one particular consequence of our conjecture and in so doing we construct an initial segment of one-one degrees which is isomorphic to the lattice:

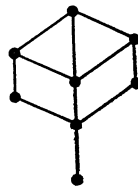


Fig. 4

Let M be a maximal r.e. set. Let $\mathcal{K}, \mathcal{K}(M), \mathcal{L}(M)$ denote the classes of all r.e. sets, all r.e. subsets of M , all recursive subsets of M respectively. For our construction we shall require the following:

LEMMA 3. *There exists a class \mathcal{M} of r.e. sets, $\mathcal{L}(M) \subseteq \mathcal{M} \subseteq \mathcal{K}(M)$, such that if W is a finite union of members of \mathcal{M} then $W \in \mathcal{M}$ and $M - W$ is infinite, and such that if $W \in \mathcal{K}(M) - \mathcal{M}$ then some member of \mathcal{M} complements W relative to M .*

Since this is easy we omit its proof; notice that the property we require of \mathcal{M} is simply that $\mathcal{H}(M) - \mathcal{M}$ should be the intersection with $\mathcal{H}(M)$ of a nonprincipal ultrafilter over M which contains no member of $\mathcal{L}(M)$. It is clear that there exists an increasing sequence M_0, M_1, \dots of members of \mathcal{M} such that every member of \mathcal{M} is $\subseteq M_i$ for some i . Choose a strictly increasing sequence m_0, m_1, \dots of members of M such that for all i , $m_i \in M - M_i$. Since $M - M_i$ is infinite for each i , the sequence m_0, m_1, \dots can be found. Let $M^* = \{m_i \mid i \geq 0\}$, then $M^* \cap M_i$ is finite for each i . For $j = 0, 1$ let $R_j = \{2x + j \mid x \geq 0\}$ and $S_j = \{2x + j \mid x \in M\}$. Let P_0, P_1 be infinite disjoint sets both $\subseteq M'$. Let a be the one-one degree of the set

$$A = \{2x + j \mid x \in M^* \text{ and } j \leq 1\} \cup \{2x + j \mid x \in P_j \text{ and } j \leq 1\}.$$

We shall show that the degrees $\leq a$ with their natural ordering do form a lattice of order eight isomorphic to the one depicted above. We shall again use the map Ψ introduced in Lemma 2. If W is finite then $\Psi(W) = 0$, while if W is an infinite r.e. set we let ω be any one-one recursive function whose range is W and $\Psi(W)$ is defined to be the one-one degree of $\{x \mid \omega(x) \in A\}$. We shall use the properties of Ψ which were stated above without proof. Let a_j be $\Psi(R_j)$ for $j = 0, 1$ and let $m = \Psi(S_0) = \Psi(S_1)$; it is apparent that $\Psi(S_0) = \Psi(S_1)$ when one considers the map: $2x$ goes to $2x+1$, restricted to S_0 . It is clear that $a = a_0 \oplus a_1$, that $m \leq a_j$ for $j = 0, 1$. Thus all of $0, m, m \oplus m, a_0, a_1, a_0 \oplus m, a_1 \oplus m, a$ are $\leq a$. It now suffices to show that these are the only one-one degrees $\leq a$ and that they are all different. Let b be any one-one degree $\leq a$ then there exists some r.e. set B such that $\Psi(B) = b$. Now

$$\Psi(B) = \Psi(B \cap R_0) \oplus \Psi(B \cap R_1),$$

thus to prove that the degrees listed are the only ones $\leq a$ it is sufficient for $j = 0, 1$ to show that $\Psi(B \cap R_j)$ is one of $a_j, m, 0$. Now Ψ does not distinguish between r.e. sets whose symmetric difference is finite, hence because M is maximal we may suppose that $B \cap R_j \subseteq S_j$ or $(B \cap R_j) \cup S_j = R_j$. We now consider three cases which exhaust all the possibilities. Let $\mathcal{M}_j = \{\{2x + j \mid x \in X\} \mid X \in \mathcal{M}\}$.

Case 1. $B \cap R_j \in \mathcal{M}_j$. Then $B \cap R_j \cap A$ is finite, and so

$$\Psi(B \cap R_j) = 0.$$

Case 2. $B \cap R_j \in \mathcal{H}(S_j) - \mathcal{M}_j$. Then there exists W in \mathcal{M}_j such that $(B \cap R_j) \cup W = S_j$. Since \mathcal{H} is separated there exist disjoint r.e. sets T, U included in $B \cap R_j, W$ respectively such that $T \cup U = S_j$. Now $\Psi(S_j) = \Psi(T) \oplus \Psi(U)$. But $\Psi(W) = 0$, whence $\Psi(U) = 0$. Thus

$\Psi(T) = \Psi(S_j) = m$. Since $T \subseteq B \cap R_j \subseteq S_j$, we also have $\Psi(B \cap R_j) = m$.

Case 3. $(B \cap R_j) \cup S_j = R_j$. Then since \mathcal{K} is separated there exist recursive sets T, U included in $B \cap R_j, S_j$ respectively such that $T \cup U = R_j$. Since $\mathcal{L}(M) \subseteq \mathcal{M}$, we have $U \in \mathcal{M}_j$ whence $\Psi(U) = \mathbf{0}$. Thus $\Psi(T) = \Psi(R_j) = a_j$. Therefore $\Psi(B \cap R_j) = a_j$.

This completes the proof that each one-one degree $\leq a$ is one of the eight listed. We now turn to the task of showing that all eight are distinct. It is easy to show that no cylinder $> \mathbf{0}$ can be represented by an immune or coimmune set. But $A \cap S_j$ is immune since otherwise $A \cap S_j$ would have an infinite recursive subset which is impossible. Hence $m = \Psi(S_j)$ is represented by an immune set, whence $m \oplus m \neq m$. Suppose for reductio ad absurdum that $a_0 \leq a_1$ then there exists a one-one partial recursive function ρ such that $\text{dom } \rho \subseteq R_0$, $\text{rng } \rho \subseteq R_1$, such that $\text{dom } \rho$ and $R_0 \cap (A \cup \text{dom } \rho)$ are both recursive, and such that for all x in $\text{dom } \rho$

$$(3) \quad x \in A \Leftrightarrow \rho(x) \in A.$$

It is easy to see that $R_0 \cap (\text{dom } \rho)' \cap A$ is finite since A is immune. Therefore $(\text{dom } \rho)'$ cannot contain all but a finite number of members of $R_0 - S_0$, therefore $\text{dom } \rho$ does contain all but a finite number of members of $R_0 - S_0$. Now $\rho(R_0 - S_0)$ is clearly cohesive and so either $S_1 \cap \rho(R_0 - S_0)$ is finite or $(R_1 - S_1) \cap \rho(R_0 - S_0)$ is finite.

Case 1. $S_1 \cap \rho(R_0 - S_0)$ is finite. Define the one-one partial recursive function σ by

$$\sigma(x) = y \Leftrightarrow \rho(2x) = 2y + 1.$$

Then $\sigma(M') \cap M$ is finite. Write $x \sim y$ if either $x = y$, or x can be obtained by iterating σ on y , or y can be obtained by iterating σ on x . Since σ is one-one, \sim is an equivalence relation on N . Consider the equivalence classes intersecting M' ; none of them can have infinite intersection with M' because this would mean that M' had an infinite r.e. subset. Thus there are an infinite number of equivalence classes intersecting M' . Since $\sigma(M') \cap M$ is finite and M is maximal, $M' - \sigma(M')$ is also finite. Hence only a finite number of equivalence classes intersect both M and M' . Since P_0, P_1 are disjoint it follows from (3) that $\sigma(x) \neq x$ for infinitely many x in M' . Hence there exist infinitely many equivalence classes $\subseteq M'$ of cardinality > 1 . We now have a contradiction, because we can construct two disjoint r.e. sets each of which intersects all the finite equivalence classes of cardinality > 1 , and hence each of which has infinite intersection with M' .

Case 2. $\rho(R_0 - S_0) \cap (R_1 - S_1)$ is finite. By Theorem 3 of [2] it follows that $R_1 - \rho(R_0 - S_0)$ is r.e. and hence there exists a recursive set $T \supseteq \rho(R_0 - S_0)$ such that $T - S_1$ is finite. Now $T \cap S_1$ being recursive is in \mathcal{M}_1 and so has only a finite intersection with A . Hence $\rho(R_0 - S_0)$ has only a finite intersection with A . This contradicts (3), because ρ is one-one and $R_0 - S_0$ has infinite intersection with A .

We have now shown that $\mathbf{a}_0 \not\leq \mathbf{a}_1$; similarly $\mathbf{a}_1 \not\leq \mathbf{a}_0$. It only remains to show that $\mathbf{a}_j/m = 1$ for $j = 0, 1$. Suppose for reductio ad absurdum that $\mathbf{a}_j/m \neq 1$, then $\mathbf{a}_j = m \oplus m$, because we have shown above that the only one-one degrees $< \mathbf{a}_j$ are $\mathbf{0}$ and m . Since $\mathbf{a}_j = \Psi(R_j) = m \oplus m$ there exist disjoint recursive sets T, U whose union is R_j such that $\Psi(T) = \Psi(U) = m$. One of T, U has only finite intersection with $R_j - S_j$. Thus we may suppose $T \subseteq S_j$, whence $T \in \mathcal{M}_j$, which means that $T \cap A$ is finite. Therefore $\Psi(T) = \mathbf{0}$, again a contradiction. Using the results of §2 it now follows easily that the initial segment of one-one degrees $\leq \mathbf{a}$ is isomorphic to the nondistributive lattice depicted above.

The use of Theorem 3 of [2] may be eliminated if we construct $A \cap (R_0 - S_0), A \cap (R_1 - S_1)$ with the immediate intention that $\mathbf{a}_0, \mathbf{a}_1$ should be incomparable and both $> m$.

4. Recursively enumerable initial segments. In this section we prove a theorem about r.e. one-one degrees and deduce that all r.e. finite initial segments are distributive. The theorem we shall prove is of the same genre as Theorem 3 of [2] quoted above.

THEOREM 6. *Let \mathbf{a}, \mathbf{b} be indecomposable r.e. one-one degrees such that $\mathbf{b} < \mathbf{a}$, then $\mathbf{a} = \mathbf{a} \oplus \mathbf{b}$.*

Proof. Let A be a representative of \mathbf{a} . Let Ψ be the map of the r.e. sets onto the one-one degrees $\leq \mathbf{a}$ defined in the last section. It is clear that $\Psi(W) = \Psi(W \cup A)$ for every r.e. set W . Let $\{\varphi_i\}$ be a recursive enumeration of all p.r. functions of one argument. Let $\{\varphi_{x,y}\}$ be an s.r.e. double sequence of finite functions such that for each $x, \varphi_{x,0} \subseteq \varphi_{x,1} \subseteq \dots$ and $\lim_y \varphi_{x,y} = \varphi_x$. Let B be an infinite r.e. set such that $B \supseteq A$ and $\Psi(B) = \mathbf{b}$. Let β be a one-one recursive function enumerating B . We shall effectively enumerate in steps $0, 1, 2, \dots$ disjoint r.e. sets B^0, B^1 whose union is B . For $j = 0, 1$ let B_s^j be the finite set of numbers enumerated in B_j before step s . While enumerating B^0, B^1 we simultaneously enumerate A . Let A_s be the finite set of numbers enumerated in A before step s . Let P be the set of all pairs (e, j) of natural numbers with $j \leq 1$. We will order P according to

$$(e, j) < (f, k) \Rightarrow [e < f \cdot \mathbf{V} \cdot e = f \text{ and } j < k] .$$

In order to describe step s we need two auxiliary functions:

$$\gamma_s^j(e) = \sup \{x \mid x = e \mathbf{V} (y)_{y < x} [y \in B_s^j \rightarrow \varphi_{e,s}(y)]\}$$

is defined $\cdot \& \cdot \varphi_{e,s}(y) \in B_s^{1-j} \cdot \& \cdot y \in A_s \Rightarrow \varphi_{e,s}(y) \in A_s$],

$$\delta_s^j(e) = \sup [\gamma_x^k(y) \mid x \leq s \ \& \ (y, k) \leq (e, j) \ \& \ (y, k) \in P]$$

each defined only for (e, j) in P . The construction can now be stated very easily. In step s let (e, j) be the least pair in P such that $\beta(s) < \delta_s^j(e)$. Enumerate $\beta(s)$ in B^j .

To prove the theorem notice that

$$\mathbf{b} = \Psi(B) = \Psi(B^0) \oplus \Psi(B^1) .$$

Since \mathbf{b} is indecomposable, for either $j = 0$ or $j = 1$ we have $\Psi(B^j) \leq \Psi(B^{1-j})$. Thus there exists a partial recursive one-one function φ_e such that $B^j - \text{dom } \varphi_e, B^j - (A \cup \text{dom } \varphi_e)$ are both r.e., such that $\text{dom } \varphi_e \subseteq B^j, \text{rng } \varphi_e \subseteq B^{1-j}$, and

$$(4) \quad y \in A \Leftrightarrow \varphi_e(y) \in A$$

for all y in $\text{dom } \varphi_e$. Now $A \cap B^{1-j}$ is obviously infinite and r.e., thus we may suppose that $B^j - \text{dom } \varphi_e \subseteq A'$. Further, if $B^{1-j} - A$ has an infinite recursive subset, we may suppose that $B^j = \text{dom } \varphi_e$. However, if $B^{1-j} - A$ is immune, we may suppose that B was chosen so that $B - A$ is since $\Psi(B^{1-j}) = \mathbf{b}$. Thus in any case we may suppose that $B^j - \text{dom } \varphi_e$ is finite. By a suitable adjustment of φ_e we can make $\text{dom } \varphi_e = B^j$ unless $\Psi(B^j) = \Psi(B^{1-j})$. However, if $\Psi(B^j) = \Psi(B^{1-j}), \mathbf{b}$ is a cylinder and the theorem is immediate. Thus we shall suppose that $\text{dom } \varphi_e = B^j$. Clearly, $\lim_x \gamma_x(e, j) = \infty$ whence $\lim_x \delta_x(e, j) = \infty$. Let (e, j) now be the least pair in P such that $\lim_x \delta_x(e, j) = \infty$, then $\lim_x \gamma_x(e, j) = \infty$ and $\Psi(B^j) \leq \Psi(B^{1-j}) = \mathbf{b}$. For each pair (f, k) in P which is $< (e, j)$, $\lim_x \delta_x(f, k)$ is finite because $\delta_s(f, k)$ is increasing with s . Let

$$m = \sup \{\delta_x(f, k) \mid x \geq 0 \ \& \ (f, k) \in P \ \& \ (f, k) < (e, j)\} .$$

At any step s where $\beta(s) \geq m$, we can have $\beta(s)$ enumerated in B^{1-j} only if $\delta_s(e, j) \leq \beta(s)$. But $\delta_s(e, j)$ increases with s and $\lim_x \delta_x(e, j) = \infty$, therefore B^{1-j} is recursive. Now

$$\mathbf{a} = \Psi(N) = \Psi(B^{1-j}) \oplus \psi((B^{1-j})') .$$

But \mathbf{a} is indecomposable and $\Psi(B^{1-j}) = \mathbf{b} < \mathbf{a}$, whence $\Psi((B^{1-j})') = \mathbf{a}$. The last equation is now $\mathbf{a} = \mathbf{b} \oplus \mathbf{a}$ which proves the theorem.

THEOREM 7. *Let a be an r.e. one-one degree which has only a finite number of predecessors then the one-one degrees $\leq a$ form a distributive lattice.*

Proof. Let b, c, a^* be one-one degrees $\leq a$ such that one of $b/a^*, c/a^*$ is finite. Then $\min(b/a^*, c/a^*)a^*$ is \leq both b and c . Hence

$$(5) \quad b \cap c/a^* = \min(b/a^*, c/a^*) .$$

It is clear that we can relax the condition that one of $b/a^*, c/a^*$ be finite. Now let a^* be a maximal indecomposable degree $\leq a$. Then in the proof of Theorem 4 we may choose $a_1 = a^*$ and so from the construction of $b \cup c$ in that proof

$$(6) \quad b \cup c/a^* = \max(b/a^*, c/a^*) .$$

It is clear that (6) also holds when a^* is a cylinder.

Now let x, y, z be any r.e. one-one degrees $\leq a$, to prove the theorem it is sufficient to prove

$$(x \cup y) \cap z \leq (x \cap z) \cup (y \cap z) .$$

From Lemma 2 it is therefore sufficient to show that

$$\{(x \cup y) \cap z\}/a^* \leq \{(x \cap z) \cup (y \cap z)\}/a^*$$

for every indecomposable degree $a^* \leq a$. From Theorem 6 any such a^* is either a maximal indecomposable degree $\leq a$ or is a cylinder, thus (5) and (6) are applicable. Using (5) and (6) this reduces to showing that

$$\min(\max(x/a^*, y/a^*), z/a^*) \leq \max(\min(x/a^*, z/a^*),$$

$$\min(y/a^*, z/a^*)) .$$

However, it is easy to verify that equality obtains in the last line using only the definitions of max and min. This completes the proof of the theorem.

5. Conclusion. Above we have hardly touched upon the general question of characterizing arbitrary initial segments of the one-one degrees. A certain amount of information can be obtained by studying the map Ψ introduced in the proof of Lemma 2. However, there is a serious obstacle to a complete solution of the problem. Suppose that we are considering the one-one degrees $\leq a$ where a is a cylinder. If there is a noncylinder $b \leq a$ then from the work of Young in [7] there exists a collection of one-one degrees $\leq a$, and in the same many-one degree as b , which has the order type of the rationals. Thus it

seems that the approach to initial segments via finite substructures which has been successful for the many-one degrees in [3], is doomed to failure in the case of one-one degrees.

Our final comment concerns the relation between the notion of one-one degree used in this paper and the usual notion. Let \mathbf{a} be the one-one degree of a simple set such that the order types of the initial segments of one-one degrees $\leq \mathbf{a}$, under our definition and the usual definition, are θ and θ^* respectively. From Theorem 1 of [6] it is easy to see that

$$\theta^* = \omega \oplus \{(\theta - \{1\}) \circ (\omega^* \oplus \omega)\} \oplus \omega^* ,$$

where $\theta - \{1\}$ is the order type obtained from θ by deleting its greatest member, where \oplus and \circ denote ordinal sum, and product respectively, where ω is the order type of the natural numbers, and where ω^* is the converse of ω .

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