A GREEN'S FUNCTION APPROACH TO PERTURBATIONS OF PERIODIC SOLUTIONS

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Consider the nonlinear differential system

$$(1.1) \qquad \dot{y} = A(t)\cdot y + \varepsilon q(t, y, \varepsilon)$$

where y, q are column n-vectors, q is continuous in (t, y, ε) and has continuous second partial derivatives with respect to y, ε for all values of $t, 0 \leq y \leq R$ for some $R > 0$ and $0 \leq \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$. Further assume $A(t)$ is an $n \times n$ matrix such that $A \in C^1$, and both A and q are periodic in t with period T. Associated with system (1.1) are the general homogeneous and nonhomogeneous equations

$$(1.2) \qquad \dot{y} = A(t)\cdot y$$

$$(1.3) \qquad \dot{y} = A(t)\cdot y + f(t)$$

where $f(t)$ is an arbitrary n-vector function periodic in t of period T. In this paper we consider the classical problem of proving the existence of T-periodic solutions $y = y(t)$ of (1.1) when the homogeneous system (1.2) has nontrivial T-periodic solutions.

In his book "Oscillations in Nonlinear Systems" [5], J. K. Hale treats this problem in Chapter 11 as an extension of the general method developed in Chapter 6 for the case when $A(t)$ is a constant matrix A. Among other treatments of this problem is the work of D. C. Lewis [6] utilizing a generalized Green's matrix. It is the aim of the present paper to incorporate certain ideas from [6] into the general method employed in [5] in Chapter 6. We feel that the approach presented here brings into sharper focus the generality of the methods employed in [5] and also has certain computational advantages which will be mentioned in the body of the paper. The present paper has many points of contact with a recent paper of S. Bancroft, J. K. Hale, and D. Sweet [2]. See also [1], [7] and [8].

2. We must first consider the general nonhomogeneous equation (1.3). We will state a necessary and sufficient condition for the existence of T-periodic solutions of (1.3) when (1.2) has nontrivial T-periodic solutions. Following [6] we then construct a generalized Green's function in order to find an explicit periodic solution of (1.3).

Let $S = \{f : f(t) = f(t + T); f \in C;$ and $f(t)$ is a real valued n-vector for all $t\}$. Then S is a Banach space if we define the norm $V(f)$ to

be $V(f) = \sup \{\|f(t)\|; 0 \leqq t \leqq T\}$. Further let $X(t)$ be any real funda-
mental matrix solution of (1.2), and let $n-k$ be the rank of the $n \times n$
matrix $L = X(0) - X(T)$. Then there exist k linearly independent
column vectors $b_1, b_2, \cdots, b_k$ such that $L \cdot b_i = 0$ for $i = 1, 2, \cdots, k$.
Let B denote the $n \times k$ matrix $(b_1, b_2, \cdots, b_k)$. Then the columns
$\varphi_1(t), \varphi_2(t), \cdots, \varphi_k(t)$ of the matrix $\Phi(t)$ defined by $\Phi(t) = X(t) \cdot B$ form
a set of k linearly independent solutions of (1.2) which are T-periodic.
By assumption $k > 0$, and of course the columns of $\Phi(t)$ span the
space S_1 of all T-periodic solutions of (1.2). Explicitly we have

$$S_1 = \{f \in S: f(t) = \Phi(t) \cdot a = \sum a_i \cdot \varphi_i(t) \text{ for all } a = (a_1, a_2, \cdots, a_k) \in R^k\} .$$

Let A' denote the transpose of an $r \times s$ matrix A. Let $N(A)$ and
$R(A)$ denote the nullity and range of A, respectively. Consider the
$n \times n$ matrix $L' = X'(0) - X'(T)$. Since $\dim N(L') = \dim N(L)$, we
can find k linearly independent column vectors $u_1, u_2, \cdots, u_k$ such that
$L' \cdot u_i = u_i' \cdot L = 0$ for $i = 1, 2, \cdots, k$. Let U denote the $k \times n$ matrix
with rows $u_1', u_2', \cdots, u_k'$. We define an $n \times k$ matrix $\Psi(t)$ to be

$$\Psi(t) = [U \cdot X(T) \cdot X^{-1}(t)]' .$$

Then $\Psi'(t) = U \cdot X(T) \cdot X^{-1}(t)$. Clearly, $\Psi(t)$ has rank k and each of its
columns is a solution of the differential equation

(2.1) $$\dot{Z} = -A'(t) \cdot Z$$

which is adjoint to (1.2).

Let $\psi_1(t), \psi_2(t), \cdots, \psi_k(t)$ denote the k linearly independent columns
of $\Psi(t)$. These functions span the space S_2 of all T-periodic solutions
of the adjoint equation (2.1). In fact, since $\Psi(0) = \Psi(T)$ and $\Psi(t)$ is
a matrix solution of (2.1) we see that $\Psi(s) = \Psi(s + T)$ for $0 \leqq s \leqq T$.
If (2.1) had l linearly independent periodic solutions with $l > k$ then
the original system (1.2), which is adjoint to (2.1), would also have
at least l linearly independent T-periodic solutions, contrary to as-
sumption. Explicitly, we have

$$S_2 = \{f \in S: f(t) = \Psi(t) \cdot a = \sum a_i \cdot \psi_i(t) \text{ for all } a = (a_1, a_2, \cdots, a_k) \in R^k\} .$$

It is easy to see that S becomes a real Euclidean space when an
inner product (f, g) is defined for $f, g \in S$ by

(2.2) $$(f, g) = \frac{1}{T} \cdot \int_0^T \langle f(t), g(t) \rangle dt = \sum_{i=1}^n \frac{1}{T} \cdot \int_0^T f_i(t) \cdot g_i(t) dt$$

where $\langle a, b \rangle = \sum_{i=1}^n a_i \cdot b_i$ denotes the usual inner product for $a, b \in R^n$.
Without loss of generality we may assume that both $\{\varphi_i(t)\}$ and $\{\psi_i(t)\}$
are orthonormal bases for their respective subspaces S_1 and S_2; i.e.,

$(\varphi_i, \varphi_j) = (\psi_i, \psi_j) = \delta ij$ for $1 \leqq i, j \leqq k$.

By P and Q we denote the orthogonal projections of S onto S_1 and S_2, respectively.   Explicitly we have

$$(2.3) \qquad P: S \to S_1 , \qquad Pf(t) = \sum_{i=1}^{n} (f, \varphi_i) \cdot \varphi_i(t)$$

$$(2.4) \qquad Q: S \to S_2 , \qquad Qf(t) = \sum_{i=1}^{n} (f, \psi_i) \cdot \psi_i(t) .$$

For convenience we summarize certain well-known general properties of projections in Euclidean space.

(i) $f \in S_1 \Rightarrow Pf = f$ and $f \in S_2 \Rightarrow Qf = f$.

(ii) $P^2 f = Pf$ and $Q^2 f = Qf$ for all $f \in S$.

(iii) $Pf = 0 \Leftrightarrow (f, \varphi_i) = 0$ for $i = 1, 2, \cdots k$.

(iv) $Qf = 0 \Leftrightarrow (f, \psi_i) = 0$ for $i = 1, 2, \cdots k$.

Let us consider the condition $Qf = 0$.   By (iv) we have $Qf = 0 \Leftrightarrow$

$$0 = (f, \psi_i) = (\psi_i, f) = \frac{1}{T} \int_0^T \langle \psi_i(t), f(t) \rangle dt$$

for $i = 1, 2, \cdots, k$.   In matrix notation these k equations can be written as

$$(2.5) \qquad 0 = \int_0^T \Psi'(t) \cdot f(t) dt = \int_0^T U \cdot X(T) \cdot X^{-1}(t) \cdot f(t) dt .$$

Again using matrix notation we note explicitly that for $f \in S$ we have

$$(2.6) \qquad Qf(t) = \Psi(t) \cdot \frac{1}{T} \cdot \int_0^T \Psi'(s) \cdot f(s) ds .$$

LEMMA 1. *The differential system* (1.3) *with* $f \in S$ *has a T-periodic solution if and only if* $Qf = 0$.

Proof. It is well known that the unique solution $y(t)$ of (1.3) satisfying $y(0) = y_0$ is given by

$$(2.7) \qquad y(t) = X(t) \cdot X^{-1}(0) \cdot y_0 + \int_0^t X(t) \cdot X^{-1}(s) \cdot f(s) ds .$$

Let $\beta = X^{-1}(0) \cdot y_0$. The solution $y(t)$ is T-periodic if and only if $y(0) = y(T)$, and from (2.7) this is equivalent to the statement that the matrix equation

$$(2.8) \qquad L \cdot \beta = \int_0^T X(T) \cdot X^{-1}(s) \cdot f(s) ds$$

can be solved for β. Suppose first that such a vector β has been

found. The matrix U was defined in such a way that $U \cdot L = 0$. Multiplying (2.8) by U and making use of (2.5) we see that $Qf = 0$.

Conversely suppose $Qf = 0$. It follows from (2.5) that for each column u_i of the matrix U' we have

$$\left\langle u_i, \int_0^T X(T) \cdot X^{-1}(s) \cdot f(s)ds \right\rangle = 0 .$$

Thus for each vector $x \in N(L')$ we have

$$\left\langle x, \int_0^T X(T) \cdot X^{-1}(s) \cdot f(s)ds \right\rangle = 0 .$$

From the compatibility theorem of linear algebra this guarantees the existence of some solution of (2.8).

For an arbitrary linear equation $L \cdot \beta = \gamma$, it is known that one can find a nonsingular $n \times n$ matrix M, called a pseudoinverse of L, such that $L \cdot M \cdot L = L$ and for each $\gamma \in R(L)$ we have $L \cdot (M \cdot \gamma) = \gamma$. Pseudo inverses are constructed from the given matrix L by means of elementary row operations (see [3], [4]). The following corollary is an immediate consequence of this observation.

COROLLARY. *If $f \in S$ and $Qf = 0$, then (2.8) has a solution of the form $\beta = \int_0^T P(s) \cdot f(s)ds$, where $P(s) = M \cdot X(T) \cdot X^{-1}(s)$ and M is a nonsingular pseudo inverse of L.*

LEMMA 2. *Consider (1.3) and suppose $f \in S$ and $Qf = 0$. Then there exists an $n \times n$ matrix $G(t, s)$ which is independent of the function f and continuous in t and s except at $t = s$ where it has a finite jump, and such that*

(i) *The function $y_0(t) = \int_0^T G(t, s) \cdot f(s)ds$ is a T-periodic solution of (1.3) and moreover is the only T-periodic solution satisfying $Py_0 = 0$.*

(ii) *The columns of $\Psi(s)$ are orthogonal to the rows of $G(t, s)$ for each t; i.e.,*

(2.9) $$\int_0^T G(t, s)^0 \cdot \Psi(s)ds = 0 .$$

Proof. Since $Qf = 0$, (2.8) has a solution β of the form indicated in the preceding corollary. Substituting β into (2.7) we get the equation

$$y(t) = \int_0^t X(t) \cdot [P(s) + X^{-1}(s)] \cdot f(s)ds + \int_t^T X(t) \cdot P(s) \cdot f(s)ds ,$$

or

$$y(t) = \int_0^T K(t, s) \cdot f(s) ds$$

(2.10)
$$K(t, s) = X(t) \cdot [P(s) + X^{-1}(s)] \qquad 0 \leqq s \leqq t$$
$$= X(t) \cdot P(s) \qquad\qquad\qquad t \leqq s \leqq T$$

for the unique T-periodic solution of (1.3) satisfying $y(0) = y_0$. Thus any T-periodic solution of (1.3) will be in the form

(2.11)
$$y(t) = \Phi(t) \cdot c + \int_0^T K(t, s) \cdot f(s) ds$$

for a suitable choice of $c \in R^k$.

It is easy to see that at most one T-periodic solution y_0 of (1.3) can satisfy $Py_0 = 0$. For if $y^* \in S$ is another solution with $Py^* = 0$, then $y = y_0 - y^* \in S$ and solves (1.2). Thus $y \in S_1$ and so $Py = y$. On the other hand we have $Py = Py_0 - Py^* = 0$, which implies $y_0 = y^*$.

To verify (i), we first observe that the condition $Py = 0$ is equivalent to the equation

(2.12)
$$\int_0^T \Phi'(t) \cdot y(t) dt = 0 \; .$$

We want to find $c \in R^k$ such that both (2.11) and (2.12) are satisfied. For this it is necessary and sufficient that

$$0 = \frac{1}{T} \int_0^T \Phi'(t) \cdot \Phi(t) \cdot c\, dt + \frac{1}{T} \int_0^T \Phi'(t) \cdot \left[\int_0^T K(t, s) \cdot f(s) ds \right] dt \; .$$

Since $\{\varphi_i(t)\}$ forms an orthonormal basis,

$$\frac{1}{T} \int_0^T \Phi'(t) \cdot \Phi(t) dt = I(k \times k)$$

and from $\Phi(t) = X(t) \cdot B$ we find

(2.13)
$$c = -\frac{1}{T} \int_0^T B' \cdot \left[\int_0^T X'(T) \cdot K(t, s) dt \right] \cdot f(s) ds \; .$$

Thus the vector c is completely determined by an equation of the form

(2.14)
$$c = \int_0^T \mathscr{L}(s) \cdot f(s) ds$$

where $\mathscr{L}(s)$ for a given $K(t, s)$ is a continuous $k \times n$ matrix function of s, independent of f, given explicitly by (2.13). Substituting (2.14) into (2.11) and introducing the function

(2.15)
$$G^*(t, s) = \Phi(t) \cdot \mathscr{L}(s) + K(t, s)$$

we find the T-periodic solution of (1.3) satisfying $Py_0 = 0$ to be

(2.16) $$y_0(t) = \int_0^T G^*(t, s) \cdot f(s) ds .$$

From (2.5), however, we see that for any $n \times k$ matrix function $R(t)$ the solution $y_0(t)$ in (2.16) can also be written as

(2.17) $$y_0(t) = \int_0^T [G^*(t, s) - R(t) \cdot \Psi'(s)] \cdot f(s) ds .$$

Choosing in particular the matrix

$$R(t) = \frac{1}{T} \int_0^T G^*(t, s) \cdot \Psi(s) ds$$

and recalling that $\{\psi_i(t)\}$ forms an orthonormal basis, we see that (2.9) is satisfied for

(2.18) $$G(t, s) = G^*(t, s) - R(t) \cdot \Psi'(s) .$$

Finally, it is clear that $G(t, s)$ has the specified smoothness properties.

COROLLARY. *Let $f \in S$ and $Qf = 0$. The unique solution $y_0(t)$ of* (1.3) *with $Py_0 = 0$ satisfies the inequality*

(2.18) $$V(y_0) \leqq K \cdot V(f)$$

for $K = T \cdot \sup \{\|G(t, s)\| : 0 \leqq s, t \leqq T\}$.

LEMMA 3. *The matrix $G(t, s)$ given above is uniquely determined by its properties specified in the statement of Lemma 2.*

Proof. Let $g(s)$ be any element of S. Set $f(s) = g(s) - Qg(s)$. Then $Qf = Qg - Q^2g = 0$ and so we may apply Lemma 1. From (2.6) and (2.9), however, we find that

(2.19) $$\int_0^T G(t, s) \cdot f(s) ds = \int_0^T G(t, s) \cdot g(s) ds$$

and so the unique solution $y_0(t)$ satisfying $Py_0 = 0$ is given by

$$y_0(t) = \int_0^T G(t, s) \cdot g(s) ds .$$

If $G_1(t, s)$ were a second matrix with the properties given in Lemma 2, then the same argument shows that

$$y_0(t) = \int_0^T G_1(t, s) \cdot g(s) ds$$

as well. From

$$\int_0^T [G(t, s) - G_1(t, s)] \cdot g(s) ds = 0$$

it follows immediately that $G_1(t, s) = G(t, s)$.

REMARKS. The subspaces S_1 and S_2 were introduced by Hale in [5]. The proofs of Lemma 1, 2 and 3 are modeled on the treatment in [6]. A discussion of these ideas employing the Moore-Penrose notion of generalized inverse can be found in [7]. It turns out that the estimate (2.18) is the essential information required for the iteration scheme which we construct in the next section. We note, however, that the existence of the function $G(t, s)$ has been established by constructive methods which in principle make it possible to write out explicitly every T-periodic solution of (1.3) for arbitrary $f \in S$.

3. In this section we indicate how the above analysis can be applied to the question of the existence of periodic solutions of (1.1) for $\varepsilon \neq 0$. Our treatment is a modification of the analysis in Chapters 6 and 11 of [5], and the reader is referred there for details and further results.

Let $a^*(t)$ be any T-periodic solution of (1.2). For some $a \in R^k$ we have $Pa^*(t) = a^*(t) = \Phi(t) \cdot a$. For given constants b, d with $0 < b < d < R$ and $a^*(t) \in S_1$ we define $S_0 \subset S$ as

$$S_0 = \{f \in S: Pf(t) = a^*(t); V(f) \leq d\} \ .$$

THEOREM 1. *Given a constant* $b, 0 < b \leq R$, *there exists an* $\varepsilon_1 > 0$ *such that corresponding to each* k-*vector function* $a^*(t) \in S_1$ *with* $a^*(t) = \Phi(t) \cdot a$ *and* $V(a^*) \leq b$ *and to each* ε *with* $|\varepsilon| \leq \varepsilon_1$, *there is a unique function* $y^*(t) = y^*(t, a, \varepsilon) \in S_0$ *which has a continuous first derivative with respect to* t, *satisfies* $y^*(t, a, 0) = a^*(t)$ *and*

$$(3.1) \qquad \dot{y}^* = A(t) \cdot y^* + \varepsilon \cdot [q(t, y^*, \varepsilon) - Qq(t, y^*, \varepsilon)] \ .$$

Finally, the function $y^*(t)$ *may be obtained by the iteration procedure in* (3.4) *below.*

Proof. Since the function $G(t, s)$ constructed in Lemma 2 is independent of $f \in S$, it makes sense to define an operator $\mathscr{F}$ for $y = y(t, a, \varepsilon)$ in S_0 as

$$(3.2) \qquad w = \mathscr{F}y = a^*(t) + \varepsilon \cdot \int_0^T G(t, s) \cdot q(s, y(s, a, \varepsilon), \varepsilon) ds \ .$$

As in the proof of Lemma 3 we see that the function $Z(t)$ defined by

$$Z(t) = \varepsilon \cdot \int_0^T G(t, s) \cdot q(s, y(s, a, \varepsilon), \varepsilon) ds$$

satisfies $\dot{Z} = A(t) \cdot Z + \varepsilon \cdot [q(t, y, \varepsilon) - Qq(t, y, \varepsilon)]$ and also $PZ = 0$. Thus we get $Pw(t) = Pa^*(t) + PZ(t) = a^*(t)$.

Without difficulty one can show that $\mathscr{F}$ is a contraction map in S_0 for $0 \leqq |\varepsilon| \leqq \varepsilon_1$ and ε_1 sufficiently small. Thus $\mathscr{F}$ has a unique fixed point $y^*(t) = y^*(t, a, \varepsilon)$ given by

$$(3.3) \qquad y^*(t) = a^*(t) + \varepsilon \cdot \int_0^T G(t, s) \cdot q(s, y^*(s, a, \varepsilon), \varepsilon) ds \,.$$

Again from (2.6) and (2.9) we see that $y^*(t)$ is a solution of (3.1). The contraction mapping principle states that the solution $y^*(t)$ can be obtained as the limit of a sequence $\{y^k(t)\}$ defined by

$$y_0(t) = a^*(t) = \Phi(t) \cdot a$$

$$(3.4) \qquad y^{k+1}(t) = \mathscr{F}y^k(t) = a^*(t) + \varepsilon \cdot \int_0^T G(t, s) \cdot q(s, y^k(s), \varepsilon) ds \,.$$

From (3.4) we conclude $y(t, a, 0) = a^*(t)$. To prove uniqueness, let $y_1 = y_1(t, \varepsilon)$ be another solution of (3.1) in S_0 for $|\varepsilon| \leqq \varepsilon_1$. Let us define $w(t) = y_1(t) - a^*(t)$. Then w is a T-periodic solution of

$$\dot{w} = A(t) \cdot w + \varepsilon \cdot [q(t, y_1, \varepsilon) - Qq(t, y_1, \varepsilon)]$$

satisfying $Pw = 0$. It follows now from (2.6) and Lemma 2 that y_1 is a fixed point of $\mathscr{F}$, which can only mean that $y_1 = y^*$.

REMARK. Consider the special case when $A(t)$ is a constant matrix of the form diag $(0, B)$, where B is an $(n - k) \times (n - k)$ matrix such that $\dot{X} = B \cdot X$ has no T-periodic solution $X(t)$. Thus (1.2) has precisely k linearly independent constant solutions, and so we may choose an orthonormal basis of S_1 to be the first k unit n-vectors in R^n. Then $\Phi(t)$ is the $n \times k$ matrix whose only nonzero terms are k 1's along the diagonal. Clearly we may choose the same basis for S_2, and so $\Psi(t) = \Phi(t)$. Thus $Pf = Qf$ for all $f \in S$ and the spaces S_1 and S_2 coincide. This is the case treated in Chapter 6 of [5].

THEOREM 2. *Let $y^*(t) = y^*(t, a, \varepsilon)$ be the solution of (3.1) defined for all $a \in R^k$ and ε such that $0 \leqq |\varepsilon| \leqq \varepsilon_1$ and $V(a^*) = V(\Phi(t) \cdot a) \leqq b$. If there exists an $\varepsilon_2 \leqq \varepsilon_1$ and a continuous function $a = a(\varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2$ such that*

$$(3.5) \qquad\qquad Qq(t, y^*(t, a(\varepsilon), \varepsilon), \varepsilon) = 0$$

then $y^*(t, a(\varepsilon), \varepsilon)$ is a T-periodic solution of (1.1) for $|\varepsilon| \leqq \varepsilon_2$. Conversely, suppose (1.1) has a T-periodic solution $\bar{y} = \bar{y}(t, \varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2 \leqq \varepsilon_1$, continuous in ε with $V(P\bar{y}) < R$. Then for some continuous function $a(\varepsilon)$ satisfying (3.5) we have $\bar{y}(t, \varepsilon) = y^*(t, a(\varepsilon), \varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2$

In this sense we say that the existence of a continuous function $a(\varepsilon)$ satisfying (3.5) is a necessary and sufficient condition for the existence of a periodic solution of (1.1).

Proof. The sufficiency of the condition is obvious. Conversely, given $\bar{y} = \bar{y}(t, \varepsilon)$ define $a(\varepsilon)' = ((\bar{y}, \varphi_1), (\bar{y}, \varphi_2), \cdots, (\bar{y}, \varphi_k))$. Then for $a^*(t, \varepsilon) = \varPhi(t) \cdot a(\varepsilon)$ we have $P\bar{y}(t, \varepsilon) = a^*(t, \varepsilon)$. Clearly $a(\varepsilon)$ is continuous and by assumption $V(P\bar{y}) = VP(\bar{y}(\cdot, \varepsilon)) = V(a^*(\cdot, \varepsilon)) < R$. Thus for some b, d with $0 < b < d < R$ we have $V(a^*(\cdot, \varepsilon)) \leqq b < R$ for $0 \leqq |\varepsilon| \leqq \varepsilon_2$. Since $\bar{y}$ is a solution of (1.1) we obviously have

$$(I - Q)[\dot{\bar{y}} - A(t) \cdot \bar{y} - \varepsilon \cdot q(t, \bar{y}, \varepsilon)] = 0$$
$$Q[\dot{\bar{y}} - A(t) \cdot \bar{y} - \varepsilon \cdot q(t, \bar{y}, \varepsilon)] = 0 \ .$$

Since $\bar{y}$ is assumed known we can regard it as a solution of an equation of the form (1.3). From Lemma 1 this implies $Qq(t, \bar{y}(t, \varepsilon), \varepsilon) = 0$. Thus the above equations reduce to

$$(3.6) \qquad \dot{\bar{y}} = A(t) \cdot \bar{y} + \varepsilon \cdot [q(t, \bar{y}, \varepsilon) - Qq(t, \bar{y}, \varepsilon)]$$

$$(3.7) \qquad Qq(t, \bar{y}, \varepsilon) = 0$$

for $0 \leqq |\varepsilon| \leqq \varepsilon_2$. From Theorem 1, however, the function $y^*(t, a(\varepsilon), \varepsilon)$ is the only T-periodic solution of (3.6) satisfying

$$P(y^*(t, a(\varepsilon), \varepsilon)) = a^*(t, \varepsilon)$$

for $|\varepsilon| \leqq \varepsilon_1$. Since $\varepsilon_2 \leqq \varepsilon_1$, from (3.6) we conclude $\bar{y}(t, \varepsilon) = y^*(t, a(\varepsilon), \varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2$ and from (3.7) we see that $a(\varepsilon)$ satisfies (3.5).

REMARK. Condition (3.5) is most easily verified, of course, when the implicit function theorem can be applied in the manner of Chapter 6 of [5]. If this fails, then one can use (3.4) to find approximate determining equations for (1.1) involving the successive approximations $y_0, y^1, \cdots y^k$, as done in Chapter 7 of [5]. One advantage of the present methods is that these higher order equations can be explicitly written out, due to the constructive method used to define the kernel $G(t, s)$ of the integral operator $\mathscr{T}$.

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