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A NOTE ON A THEOREM OF HILL

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Recently Hill has shown the existence of an abelian p-group with the property that each infinite subgroup can be embedded in a direct summand of the same cardinality but the group is not a direct sum of countable groups. Megibben has since observed that this phenomenon occurs in a larger class of abelian groups. In this note we show that such pathology is present in modules for a rather wide class of rings. In fact, the lack of such phenomena for a particular class of modules serves as a characterization for left perfect rings. Our results also yield some facts concerning pure injective modules.

All rings in this paper are associative with identity and all modules are unital.

2. **A characterization of left perfect rings.** Bass [1] calls a ring R left perfect if each left R-module has a projective cover (projective cover is the dual of injective envelope). Among several other characterizations of left perfect rings, Bass proves that R is left perfect if and only if R has the descending chain condition on principal right ideals. Hence, assuming that R is not left perfect, we can obtain a strictly decreasing sequence of principal right ideals of the form

$$a_1R \supset a_1a_2R \supset \cdots \supset a_1 \cdots a_nR \supset \cdots .$$

We set $P = \Pi_{n<\omega} Re_n$, where $Re_n \cong R$ for each n, and we denote by S the submodule of finitely nonzero sequences in P. We shall use the notation $\sum_{i=m}^{n} r_i e_i$, for $m \leqq n$, to denote a vector in P whose ith coordinate is zero for $i > n$ and $i < m$ and whose ith coordinate is $r_i e_i$ for $m \leqq i \leqq n$. We define elements

$$c^{(m)} = \sum_{i \geqq m} (a_m \cdots a_i)e_i \in P \quad \text{for} \quad m = 1, 2, \cdots .$$

Let A be the submodule of P generated by S and the elements $c^{(m)}$ for $m = 1, 2, \cdots$. With this notation established, we prove the following lemma.

LEMMA 2.1. *Let R be a ring that is not left perfect and let A and S be defined as above. Then A is free and S is not a direct summand of A.*

Proof. First we note that if $n < m$, then

$$(*) \qquad c^{(n)} = \sum_{i \geq n}^{m-1} (a_n \cdots a_i)e_i + (a_n \cdots a_{m-1})c^{(m)}$$

and in particular

$$c^{(n)} = a_n(e_n + c^{(n+1)}) \quad \text{for} \quad n = 1, 2, \cdots .$$

Now suppose that $A = S \oplus B$. Then $c^{(n)} = s_n + b_n$ where $s_n \in S$ and $b_n \neq 0 \in B$. From property $(*)$ above, we have that, for $n > 1$,

$$c^{(1)} = s_{1n} + (a_1 \cdots a_{n-1})c^{(n)} \quad \text{where} \quad s_{1n} \in S .$$

Therefore

$$\begin{aligned} s_1 + b_1 = c^{(1)} &= s_{1n} + (a_1 \cdots a_{n-1})c^{(n)} \\ &= s_{1n} + (a_1 \cdots a_{n-1})(s_n + b_n) \\ &= s_{1n} + (a_1 \cdots a_{n-1})s_n + (a_1 \cdots a_{n-1})b_n. \end{aligned}$$

Hence $s_1 = s_{1n} + (a_1 \cdots a_{n-1})s_n$ and $b_1 = (a_1 \cdots a_{n-1})b_n$ for each $n > 1$. Therefore $c^{(1)} = s_1 + (a_1 \cdots a_{n-1})b_n$ for $n = 2, 3, \cdots$. Since s_1 has only finitely many nonzero coordinates, it follows that there is a positive integer r such that $a_1 \cdots a_r = a_1 \cdots a_r a_{r+1} y$. But this implies that $a_1 \cdots a_r R = a_1 \cdots a_{r+1} R$ which is a contradiction. Thus S is not a summand of A.

To show that A is free, let $y_n = e_n + c^{(n+1)}$ for $n = 1, 2, \cdots$. Since $c^{(n)} = a_n y_n$ by property $(*)$ above, it follows that A is generated by $\{y_n\}_{n<\omega}$. Suppose that $r_1 y_1 + \cdots + r_n y_n = 0$ where $r_i \in R$. Then

$$r_1 c^{(2)} + r_2 c^{(3)} + \cdots + r_n c^{(n+1)} = -r_1 e_1 - r_2 e_2 - \cdots - r_n e_n .$$

Since the first coordinate of the left hand side is zero, it follows that $r_1 = 0$. A repetition of the preceding argument shows that $r_1 = r_2 = \cdots = r_n = 0$. This implies that A is free with $\{y_n\}_{n<\omega}$ for a basis.

We observe from [1] that a left R-module is torsionless if and only if it can be embedded as a submodule of a direct product of copies of R. We shall call a left R-module G $\aleph_1$-separable provided G is flat, torsionless and that each countably generated submodule of G is contained in a countably generated direct summand of G (this definition parallels the definition given by L. Fuchs [4] in the context of $\aleph_1$-free groups). We now prove the main result of this section. The proof is modeled after that of Hill's [5].

THEOREM 2.2. *A ring R is left perfect if and only if each $\aleph_1$-separable left R-module is a direct sum of countably generated modules.*

Proof. If R is left perfect, then by Theorem 3.2 [2] any flat left module is projective. Since an $\aleph_1$-separable left module is flat, it follows from Kaplansky's theorem [6] that each $\aleph_1$-separable left R-module is a direct sum of countably generated modules.

Now suppose that R is not left perfect. This implies by Theorem P [1] that R has a strictly decreasing sequence

$$a_1R \supset a_1a_2R \supset \cdots \supset a_1 \cdots a_nR \supset \cdots$$

of principal right ideals. Set $P^* = \Pi_{\alpha<\Omega}Re_\alpha$ where $Re_\alpha \cong R$ for each $\alpha < \Omega$ (Ω denotes the first uncountable ordinal). We construct a left submodule G of P^* such that $G = \bigcup_{\alpha<\Omega} G_\alpha$ where $\{G_\alpha\}_{\alpha<\Omega}$ is a monotone increasing chain defined as follows: $G_0 = 0$, $G_1 = Re_1$ and suppose that G_α has been defined for each $\alpha < \beta$ such that the following conditions hold:

(i) If α is a limit ordinal, $\alpha < \beta, G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$.

(ii) If $\alpha - 1$ and $\alpha - 2$ exist, $G_\alpha = G_{\alpha-1} \oplus Re_{\alpha-1}$.

(iii) If $\alpha - 1$ exists and is a limit, there is a monotone increasing sequence $\sigma_\alpha(n)$ of ordinals less than $\alpha - 1$ such that $\sigma_\alpha(n) - 2$ is defined for each n and such that $\sigma_\alpha(n)$ converges to $\alpha - 1$. Then $c_\alpha^{(m)} = \sum_{i\geqq m} (a_m \cdots a_i)e_{\sigma_\alpha(i)}$ for $m = 1, 2, \cdots$ and G_α is generated by $G_{\alpha-1}$ and $\{c_\alpha^{(n)}\}_{n<\omega}$.

(iv) If $\rho_{\gamma+1}$ denotes the natural projection of P^* onto $\Pi_{\lambda<\gamma}Re_\lambda$ and if $\gamma + 1 < \alpha < \beta$, then $\rho_{\gamma+1}(G_\alpha) = G_{\gamma+1}$.

(v) G_α is not a direct summand of $G_{\alpha+1}$ if α is a limit ordinal.

(vi) G_α is flat for $\alpha < \beta$.

If β is a limit ordinal we set $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ and if both $\beta - 1$ and $\beta - 2$ exist we set $G_\beta = G_{\beta-1} \oplus Re_{\beta-1}$. It is straightforward in either of the above two cases to show that (i)-(vi) hold for the collection $[G_\alpha]_{\alpha<\beta}$. Now suppose that $\beta - 1$ is a limit ordinal. Define $\sigma_\beta(n)$ and $c_\beta^{(m)}$ so that (iii) is satisfied and define G_β to be the submodule generated by $G_{\beta-1}$ and $\{c_\beta^{(c)}\}_{n<\omega}$. Suppose that $\gamma + 1 < \beta$ and consider $\rho_{\gamma+1}(G_\beta)$. To show that (iv) is satisfied, it clearly suffices to show that $\rho_{\gamma+1}(c_\beta^{(m)}) \in G_{\gamma+1}$. But this is a direct consequence of the fact that $c_\beta^{(m)} = \sum_{i\geqq m} (a_m \cdots a_i)e_{\sigma_\beta(i)}$ and that $\sigma_\beta(i) > \gamma + 1$ for all i larger than some integer i_0. To see that (v) holds, let Λ_β be the set of ordinals $\{\sigma_\beta(1), \sigma_\beta(2), \cdots\}$ and let I_β be the ordinals less than β that are not in Λ_β. Let $B = G_\beta \cap \Pi_{I_\beta}Re_\lambda$, $A = G_\beta \cap \Pi_{\Lambda_\beta}Re_\lambda$ and let S denote the finite sequences in $\Pi_{I_\beta} Re_\lambda$. It is routine to show that $G_\beta = B \oplus A$ and that $G_{\beta-1} = B \oplus S$. We observe that (up to isomorphism) our A and S here are the same as the A and S, respectively, in Lemma 2.1. It follows that $G_{\beta-1}$ is not a direct summand of G_β. We also see that $G_{\beta-1}$ is flat since B is necessarily flat and since A is free. Thus the collection $[G_\alpha]_{\alpha\leqq\beta}$ satisfies (i)-(vi) and hence we obtain $G = \bigcup_{\alpha<\Omega} G_\alpha$ where $\{G_\alpha\}_{\alpha<\Omega}$ satisfies (i)-(vi). Note that G is torsionless since G is a submodule of P^*. G is flat from (vi) since a direct limit of flat modules is flat. Property (v) implies that G is not a direct sum of countably generated modules. Finally, property (iv) guarantees that $\rho_{\gamma+1}$, when restricted to G, is a projection of G onto $G_{\gamma+1}$. Thus G is $\aleph_1$-separable.

From the above proof, we obtain the following corollary.

COROLLARY 2.3. *A ring R is left perfect if and only if each $\aleph_1$-separable left R-module is projective.*

3. Some remarks on pure injective modules over artinian rings. An interesting consequence of our Lemma 2.1 is that the direct sum of $\aleph_0$ copies of a ring R (as a left R-module) is not a direct summand of the corresponding direct product of $\aleph_0$ copies of R if R is not left perfect. In this section we wish to consider in part the question of when the direct sum of infinitely many copies of R (as a left R-module) is a direct summand of the corresponding direct product of copies of R. More generally, we consider the problem of determining when projective modules are pure injective modules in the sense of Warfield [7]. For commutative Noetherian rings we obtain a complete answer to both of the above questions. A submodule A of a left R-module B is called a pure submodule provided, for any right module M, the natural homomorphism $M \otimes A \rightarrow M \otimes B$ is injective. A module Q is called pure injective, if for every module B and pure submodule A, each homomorphism of A into Q extends to a homomorphism of B into Q. Hence, if a pure injective module Q is a pure submodule of a module B, then Q is a direct summand of B. Our main theorems of this section follow the next lemma.

LEMMA 3.1. *If R is a left artinian ring, then any pure submodule of a left projective R-module is a direct summand.*

Proof. Suppose that A is a pure submodule of a left projective module P and suppose that M is an arbitrary right R-module. From the exact sequence

$$0 = \mathrm{Tor}_1^R(M, P) \rightarrow \mathrm{Tor}_1^R(M, P/A) \rightarrow M \otimes A \rightarrow M \otimes P ,$$

we obtain that $\mathrm{Tor}_1^R(M, P/A) = 0$ since the homomorphism $M \otimes A \rightarrow M \otimes P$ is injective. Hence P/A is a flat left R-module. By Theorem P [1], P/A is projective and thus A is a direct summand of P.

In what follows, $\sum A_i$ will denote the finitely nonzero vectors in the direct product ΠA_i.

THEOREM 3.2. *If R is a commutative artinian ring, then each projective R-module is pure injective. Moreover, if R is a commutative Noetherian ring and if each projective R-module is pure injective, then R is artinian.*

Proof. First suppose that R is a commutative artinian ring. It suffices to show that each free R-module is pure projective. By Proposition 9 [7], R is pure injective as a module over itself. Let $F = \sum_\alpha R$ be an arbitrary free R-module and let P denote the direct product $P = \Pi_\alpha R$ containing F. It is elementary to see that F is a pure submodule of P and that P is pure injective since R is pure injective. By Theorem 3.4 [2], P is also a projective R-module. Hence, by Lemma 3.1, F is a direct summand of P and therefore is pure injective.

Now suppose that R is a commutative Noetherian ring for which each projective module is pure injective. Let S and A be as in Lemma 2.1. Note that $S = \sum_{\aleph_0} R$ and that $S \subseteqq A \subseteqq \Pi_{\aleph_0} R$. Therefore S is pure in A and is therefore a direct summand of A. Hence Lemma 2.1 yields that R is a perfect ring. Since R is also Noetherian, we have that R is artinian.

COROLLARY 3.3. *If R is a commutative artinian ring, then the direct sum $\sum_\alpha R$ is a direct summand of the direct product $\Pi_\alpha R$ for each cardinal number α. Moreover, if R is a commutative Noetherian ring and if $\sum_{\aleph_0} R$ is a direct summand of $\Pi_{\aleph_0} R$, then R is artinian.*

We conclude our consideration of pure injective modules with an answer to the converse problem answered in Theorem 3.2, that is, we classify those rings for which every pure injective R-module is projective. Our solution here needs no initial assumptions on the ring.

THEOREM 3.4. *A ring R has the property that each pure injective left R-module is projective if and only if R is semi-simple and artinian.*

Proof. The sufficiency is clear. Hence suppose that R has the property that each pure injective left R-module is projective. Since each injective left module is pure injective, it follows that each injective left R-module is also projective. By Theorem 5.3 [3] of Faith and Walker, we have that R is quasi-Frobenius. Since each left R-module can be embedded as a pure submodule of a pure injective left R-module by Corollary 6 [7], we have that any left R-module is isomorphic to a pure submodule of a projective module. Since a quasi-Frobenius ring is left artinian, it follows by Lemma 3.1 that each left R-module is projective. It is well-known that such a ring is a semi-simple artinian ring.

REFERENCES

1. H. Bass, *Finitistic homological dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. 95 (1960), 466–488.
2. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.
3. C. Faith and E. A. Walker, *Direct sum representations of injective modules*, J. of Algebra **5** (1967), 203–221.
4. L. Fuchs, *Abelian groups*, Hungarian Academy of Sciences, Budapest, 1958.
5. P. Hill, *On the decomposition of groups* (to appear).
6. I. Kaplansky, *Projective modules*, Ann. of Math. **68** (1958), 372–377.
7. R. B. Warfield, *Purity and algebraic compactness for modules* (to appear).

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