PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

ON BOUNDARY BEHAVIOR OF THE BERGMAN KERNEL FUNCTION AND RELATED DOMAIN FUNCTIONALS

BRUCE L. CHALMERS

Bergman and Hörmander have studied the boundary behavior of the Bergman kernel function for domains in $\underset{\sim}{C}^n$, $2 \leqq n < \infty$, with strictly pseudo-convex boundaries. It is the purpose of this paper to extend these results to a larger and more general class of domains with so called strictly (p, q)-pseudo-convex boundaries.

Let B be a bounded domain in the space $\underset{\sim}{C}^n$ of n complex variables, say $z_k = x_k + iy_k$, $k = 1, 2, \cdots, n$, where $2 \leqq n < \infty$. (The euclidean norm of a point $z = (z_1, z_2, \cdots, z_n)$ will be denoted by

$$|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$$

and we let $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n)$ and $0 = (0, 0, \cdots, 0)$.) The space $H(B)$ of all square-integrable holomorphic functions on B forms a Hilbert space with the Bergman (reproducing) kernel function $K_B(z, \bar{t})$ with respect to the inner product $(f, g) = \int_B f\bar{g}\, d\omega$, where $f, g \in H(B)$ and $d\omega$ is the euclidean volume element of B. If $\{\varphi_n(z)\}_{n=1}^{\infty}$ is any complete orthonormal system for $H(B)$, then

$$(1) \qquad K_B(z, \bar{t}) = \sum_{n=1}^{\infty} \varphi_n(z)\overline{\varphi_n(t)}\ ,$$

which converges uniformly in the interior of $B \times B$. Then, if $z^* = z^*(z)$ is a biholomorphic map from B to $B^* \subset \underset{\sim}{C}^n$,

$$(2) \qquad K_B(z, \bar{t}) = K_{B*}(z^*(z), \overline{t^*(t)}) \frac{\partial(z^*)}{\partial(z)} \overline{\frac{\partial(t^*)}{\partial(t)}}\ ,$$

and we say $K_B(z, \bar{t})$ is a relative (pseudoconformal) invariant. Furthermore

$$(3) \qquad K_B(z, \bar{z}) = \sup\{|f(z)|^2 : f \in H(B), \|f\| \leqq 1\}$$

is a decreasing domain functional; i.e.,

$$(4) \qquad B \subset A \text{ implies } K_A(z, \bar{z}) \leqq K_B(z, \bar{z})\ .$$

In [1] and [10] respectively Bergman and Hörmander have investigated the behavior of $K(B; z) = K_B(z, \bar{z})$ near a point $Q \in \partial B$, the boundary of B, where B is strictly pseudoconvex at Q. In each case sufficient conditions were attained for $K(B; z)$ to become infinite of order $n + 1$ at $Q \in \partial B$ (i.e., $|z - Q|^{n+1}K(B; z)$ remains between two positive finite bounds as z nears Q) in a certain "nontangential" approach A. Both procedures use as "basic" domains the complex n-dimensional hypersphere $[|z| < 1]$ whose kernel function is known and becomes infinite of order $n + 1$ at the boundary.

We here introduce an extension of the notion of strictly pseudo-convex as follows:

DEFINITION. Let B be a domain in $\underset{\sim}{C}^n$. Suppose that there exists an analytic change of coordinates, one-to-one in a neighborhood η of $Q \in \partial B$, so that, with respect to the new (normal) coordinates, $Q = 0$ and, in a neighborhood of 0, $B = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn} + o(r^2_{1(q-1)} + r^{2/p}_{qn})]$, where $p > 0$, $2 \leqq q \leqq n$, and $r^2_{uv} = \sum_{k=u}^{v} |z_k|^2$. Then B is said to be strictly (p, q)-pseudo-convex at Q. If also $B \subset \eta$, then B is said to be globally representable (with respect to normal coordinates) at Q.

Note that if B is strictly pseudo-convex at Q then B is strictly $(1, n)$-pseudo-convex at Q.

As examples of domains in $\underset{\sim}{C}^2$ which are strictly $(p, 2)$-pseudo-convex at the points of the circle $[z_1 = 0, |z_2| = 1]$ consider the Reinhardt circular domains given by $[|z_2|^2 < R(|z_1|^{2/p})]$, where $|z_1| < R_0$, R is decreasing and $R'(0) \neq 0$.

In §2 and §3 we prove the following extensions of theorems by Hörmander and Bergman respectively:

THEOREM A. *In $\underset{\sim}{C}^n$ let B be a bounded strictly (p, q)-pseudo-convex domain at Q. Suppose further that B is a domain of holomorphy. Then $K(B; z)$ becomes infinite of order $p(n - q + 1) + q$ in a "nontangential"* (see §1) *approach to Q.*

THEOREM B. *In $\underset{\sim}{C}^n$ let B be a bounded strictly (p, q)-pseudo-convex globally representable domain at Q. Suppose further that passing through Q is a "normal"* (see §3) *analytic hypersurface lying entirely outside B. Then $K(B; z)$ becomes infinite of order $p(n - q + 1) + q$ in a "nontangential" approach to Q.*

Since p is an arbitrary positive number, these theorems provide sufficient conditions for the kernel function to become infinite of any order $r > 2$. Setting $p = 1$ and $q = n$ in Theorem A yields Hörmander's result, while taking $p = 1$ and $q = n = 2$ in Theorem B gives Bergman's result.

The proofs of Theorems A and B will use as "basic" domains the nonhomogeneous (if $p \neq 1$) domains $R_p(n, s) = [r^{2/p}_{1(s-1)} + r^2_{sn} < 1]$ which we introduce in §1. There we will determine the kernel function of these domains and show that it becomes infinite of order $p(n - q + 1) + q$, $q = n - s + 2$, at the points on the boundary given by $r_{1(s-1)} = 0$.

Under the hypotheses of Theorem B, we will determine in §4 the boundary behavior of other domain functionals related to the kernel function K, such as the Bergman metric given by $ds^2 = T^{\nu\bar{\mu}} dz_\nu d\bar{z}_\mu$, where $T^{\nu\bar{\mu}} = \partial^2 \log K/\partial z_\nu \partial \bar{z}_\mu$, and the scalar (pseudoconformal) invariant $I = K/\det(T^{\nu\bar{\mu}})$.

1. Determination of the Bergman kernel function and its boundary behavior for the domains $R_p(n, s)$.

Notation. If $f(x) \in C^k$, then denote the k-th partial derivative of $f(x)$ with respect to x by $[f(x)]_{(x,k)}$.

THEOREM 1.1. *For the kernel function of $R = R_p(n, s)$ we have*

$$K(R; z) = \frac{[[t^{n'}(1-t^p)^{-1}]_{(t,n-s')}]_{t^p,s')}}{\pi^n(1-r_{sn}^2)^{n'+1}}, \tag{1}$$

where $t = r_1^{2/p}(s-1)(1-r_{sn}^2)^{-1}$, $s' = s-1$, $p' = p-1$, and $n' = n + p's'$.

Proof. The monomials $z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$, $m = (m_1, m_2, \cdots, m_n) \geqq 0$ form a complete set of orthogonal functions in $H(R)$. Thus by (0.1) $K(z) = K(R; z) = \sum_{m \geqq 0} c_m |z^m|^2$, where $c_m^{-1} = \int_R |z^m|^2 d\omega$. It is clear that R admits the group G_1 of unitary transformations of the first $s-1$ variables $z_1, \cdots, z_{s-1}$ and the group G_2 of unitary transformations of the last $n-s+1$ variables $z_s, \cdots, z_n$. The Jacobian of a unitary transformation has modulus one and so $K(z) = K(r_{1(s-1)}, 0, \cdots, 0, r_{sn}, 0, \cdots, 0) = \sum_{m_1=0}^{\infty} \sum_{m_s=0}^{\infty} c_{m_1 0 \cdots 0 m_s 0 \cdots 0}\, r_{1(s-1)}^{2m_1} r_{sn}^{2m_s}$ and our problem has been reduced to only a double sum. After a straightforward calculation we find

$$c^{-1}_{m_1 0 \cdots 0 m_s 0 \cdots 0} = \frac{\pi^n m_1!\, m_s!}{(s+m_1-1)!\,[p(s+m_1-1)+1] \cdots [p(s+m_1-1)+m_s+n-s+1]}.$$

Now

$$\sum_{m_s=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+m_s-1)}{m_s!} (r_{sn}^2)^{m_s} = \frac{1}{(1-r_{sn}^2)^\alpha}, \quad \text{where}$$
$$\alpha = p(s+m_1-1)+n-s+2\,.$$

Thus,

$$K(z) = \frac{1}{\pi^n(1-r_{sn}^2)^{n'+1}} \sum_{m=0}^{\infty} (m+1)\cdots(m+s')[p(m+s')+1]\cdots \tag{2}$$
$$\cdots [p(m+s')+n-s'](t^p)^m\,.$$

Now if Q is a boundary point of $R_p(n, s)$ with $r_{1(s-1)} = 0$, then by a unitary transformation in G_2 followed by a translation, we can effect an analytic change of coordinates with respect to which $Q = 0$ and the domain can be written as

$$(3) \qquad R'_p(n, s) = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn}] , \qquad q = n - s + 2 ,$$

with kernel function (obtained from formulas (2) and (0.2)):

$$(4) \quad K(R'_p(n, s); z) = \frac{1}{\pi^n(2x_1 - r^2_{1(q-1)})^{p(s-1)+q}} \sum_{m=0}^{\infty} a_m \left[\frac{r^2_{qn}}{(2x_1 - r^2_{1(q-1)})^p}\right]^m ,$$

where the a_m are given in (2).

Notation. We shall write $\lim^A_{z\to 0}$, or sometimes simply $\lim^A$, to indicate a limit is being taken as $z \to 0$ in the set

$$(5) \qquad \left[0 < \beta < \frac{x_1}{|z|}\right] \text{ if } p < 2, \text{ and } \left[0 < \beta < \frac{x_1^p}{|z|}\right] \text{ if } p \geqq 2 ,$$

and say $z \to 0$ via an A-approach. (If for a strictly (p, q)-pseudoconvex domain the z-coordinates are normal, such an approach to 0 is nontangential in the usual sense.)

Thus from (4) we have

THEOREM 1.2. *$K(R'_p(n, s); z)$ becomes infinite of order $p(n-q+1)+q$ at 0 in a "nontangential" A-approach; in fact*

$$(6) \quad \lim_{z\to 0}{}^A (2x_1)^{ps'+q} K(R'_p(n, s); z) = \pi^{-n} s'! \, [ps' + 1] \cdots [ps' + q'] = L .$$

2. Poof of Theorem A. The following lemma is due to Hörmander (see [10], p. 146):

LEMMA 2.1. (*Hörmander*) *Let B be a bounded domain of holomorphy. Let $Q \in \partial B$ and assume that for some neighborhood U of Q there is an analytic function u_0 in $B' = B \cap U$ such that $|u_0| \leqq 1$ in B', $|u_0(z)| \to 1$ when $z \to Q$, and $|u_0(z)|$ has an upper bound < 1 in $B' \cap {}^{\sim}U_0$ for some neighborhood U_0 of Q with compact closure contained in U. Then $K(B; z)/K(B'; z) \to 1$, $z \to Q$.*

LEMMA 2.2. *$R_p(n, s)$ is a domain of holomorphy.*

Proof. It follows easily from (1.1) that $K(R_p(n, s); z)$ becomes infinite at every boundary point of $R_p(n, s)$. But it is well known that a domain whose kernel function becomes infinite at every boundary point is a domain of holomorphy (see e.g., [7], p. 357).

LEMMA 2.3. *If B is strictly (p, q)-pseudo-convex at Q, then, with respect to local normal coordinates*

$$\overline{\lim}^A (2x_1)^{ps'+q} K(B; z) \leqq L . \tag{1}$$

Proof. B can be written locally in a neighborhood of $Q = 0$ as $B = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn} + o(r^2_{1(q-1)} + r^{2/p}_{qn})]$. For $\varepsilon > 0$ let

$$R_\varepsilon = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn} + \varepsilon(r^2_{1(q-1)} + r^{2/p}_{qn})] .$$

Then $R^\delta_\varepsilon = R_\varepsilon \cap [|z| < \delta]$ is included in B if δ is sufficiently small. Since R_ε is obtained from $R'_p(n, s)$ by the change of coordinates $z_k \leftarrow (1+\varepsilon)^{-1} z_k$, $k = 1, \cdots, q-1$, $z_l \leftarrow (1+\varepsilon)^{-p} z_l$, $l = q, \cdots, n$, R_ε is also a domain of holomorphy. Thus

$$\begin{aligned}\overline{\lim}^A (2x_1)^{ps'+q} K(B; z) &\leqq \overline{\lim}^A (2x_1)^{ps'+q} K(R^\delta_\varepsilon; z) \\ &= \overline{\lim}^A (2x_1)^{ps'+q} K(R_\varepsilon; z) = (1+\varepsilon)^{ps'+q-2} L\end{aligned}$$

by (0.4), Lemma 2.1 (taking $u_0(z) = e^{-z_1}$ and $U = [|z| < \delta]$), and (0.2) together with Theorem 1.2, respectively.

Conclusion of proof of Theorem A. Let $0 < \varepsilon < 1$. Then, analogously as in the proof of Lemma 2.3., $B^\delta = B \cap [|z| < \delta] \subset R_{-\varepsilon}$ if δ is sufficiently small. Now since B is a domain of holomorphy we can apply Lemma 2.1 with $u_0(z) = e^{-z_1}$ and $U = [|z| < \delta]$ to obtain

$$\begin{aligned}\underline{\lim} (2x_1)^{ps'+q} K(B; z) &= \underline{\lim} (2x_1)^{ps'+q} K(B^\delta; z) \geqq \underline{\lim} (2x_1)^{ps'+q} K(R_{-\varepsilon}; z) \\ &= (1-\varepsilon)^{ps'+q-2} L ,\end{aligned}$$

using also (0.4) and (0.2) together with Theorem 1.2. Combining this result with that of Lemma 2.3, we conclude that under the hypotheses of Theorem A,

$$\lim^A (2x_1)^{p(n-q+1)+q} K(B; z) = L . \tag{2}$$

3. Proof of Theorem B.

DEFINITION 3.1. Suppose that B is a strictly (p, q)-pseudo-convex globally representable domain at Q. Suppose h is an analytic hypersurface through Q which can be represented as $h = [z_1 = 0]$ with respect to normal coordinates for B and $h \cap \bar{B} = Q$. Then we say that through Q passes the "normal" analytic hypersurface h lying entirely outside B.

LEMMA 3.2. *Let B be a bounded domain whose closure intersects the hyperplane $[z_1 = 0]$ only at the origin. Then an arbitrary neigh-*

borhood of the origin includes the section $B \cap [z_1 = \gamma]$ if γ is sufficiently small.

Proof. Let $P = \sup_{z \in B} r_{2n}$. Since the set of points $[z_1 = 0, b \leqq r_{2n} \leqq P]$ is closed, for each $b > 0$ there exists a number $\tau(b)$, $0 < \tau(b) \leqq b$, such that $[|z_1| \leqq \tau(b), b \leqq r_{2n} \leqq P]$ contains no point of B. Now if U is any neighborhood of the origin, there is a $\nu > 0$ such that $[|z_1| < \nu, r_{2n} < \nu] \subset U$. Then if $z \in B$ and $b \leqq \nu$, we have that $|z_1| < \tau(b)$ implies $r_{2n} < b$, and so the section $B \cap [z_1 = \gamma] \subset U$ for $|\gamma| < \tau(b)$.

LEMMA 3.3. *Let B be a domain satisfying the hypotheses of Theorem B. Then there exists a transformation*

$$W\colon z_1 \leftarrow \frac{z_1}{1 - \alpha z_1}, z_k \leftarrow z_k \frac{1 + (\beta - \alpha) z_1}{1 - \alpha z_1}, k = 2, \cdots, n$$

such that for α and β positive and sufficiently large, the domain $R_{-\varepsilon}^{\alpha\beta} = W(R_{-\varepsilon})$ includes B in its interior, where $0 < \varepsilon < 1$.

Proof.

$$B = [2x_1 > r_{1(q-1)}^2 + r_{qn}^{2/p} + o(r_{1(q-1)}^2 + r_{qn}^{2/p})] \text{ and}$$

$R_{-\varepsilon} = [2x_1 > (1-\varepsilon)(r_{1(q-1)}^2 + r_{qn}^{2/p})]$. Thus $\exists$ a neighborhood η of 0 such that $B \cap \eta \subset R_{-\varepsilon} \cap \eta$. Hence for $|\gamma| < \delta$, $B \cap [z_1 = \gamma] \subset R_{-\varepsilon} \cap [z_1 = \gamma]$ if δ is sufficiently small, by Lemma 3.2.

Now let $R_{-\varepsilon}^{\alpha} = W^{\alpha}(R_{-\varepsilon})$, where $W^{\alpha}\colon z_1 \leftarrow z_1/(1 - \alpha z_1)$, $z_k \leftarrow z_k$, $k = 2, \cdots, n$. Then consider $R_{-\varepsilon}^{\alpha} \cap [r_{2n} = 0] = \sim c_{\alpha}$, where c_{α} is the circle

$$\left[\left| z_1 + \frac{1}{2\alpha - (1 - \varepsilon)} \right| \leqq \frac{1}{2\alpha - (1 - \varepsilon)}\right].$$

We can assume α is so large that $c_{\alpha} \cap \bar{B} \cap [r_{2n} = 0] = 0$. Then the closed set $M = \bar{B} \cap [r_{2n} = 0] \cap [|z_1| \geqq \delta]$ lies entirely in the interior of the section $R_{-\varepsilon}^{\alpha} \cap [r_{2n} = 0]$. Now the sections $R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma]$, $\gamma \in M$, include hyperspheres with center $(\gamma, 0, \cdots, 0)$ and the radii of these hyperspheres have a positive lower bound r. If one applies to $R_{-\varepsilon}^{\alpha}$ the transformation $W_{\beta}\colon z_1 \leftarrow z_1$, $z_k \leftarrow z_k(1 + \beta z_1)$, $k = 2, \cdots, n$, to obtain the domain $R_{-\varepsilon}^{\alpha\beta} = W_{\beta}(R_{-\varepsilon}^{\alpha})$, then the section $R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma]$ goes into a domain including a hypersphere with center $(\gamma, 0, \cdots, 0)$ and radius $\geqq r|1 + \beta\gamma|$. β can be chosen so large that $r|1 + \beta\gamma| > P$ for $|\gamma| \geqq \delta$, where $P = \sup_{z \in B} r_{2n}$. Thus $B \cap [z_1 = \gamma] \subset R_{-\varepsilon}^{\alpha\beta} \cap [z_1 = \gamma]$ if $|\gamma| \geqq \delta$.

On the other hand consider $\gamma \in R_{-\varepsilon}^{\alpha} \cap [r_{2n} = 0] = \sim c_{\alpha}$. Thus if $\beta > 2\alpha - (1 - \varepsilon)$, then $|\gamma + 1/\beta| > 1/\beta$ and so $|1 + \beta\gamma| > 1$. Hence $R_{-\varepsilon}^{\alpha\beta} \cap [z_1 = \gamma] \supset R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma]$. But for $|\gamma| < \delta$ we know $R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma] \supset B \cap [z_1 = \gamma]$. Thus $R_{-\varepsilon}^{\alpha\beta} \cap [z_1 = \gamma] \supset B \cap [z_1 = \gamma]$ for all γ.

Conclusion of proof of Theorem B. Since $R^{\alpha\beta}_{-\varepsilon} \supset B$, we have $\underline{\lim}^A (2x_1)^{ps'+q}K(B; z) \geqq \underline{\lim}^A (2x_1)^{ps'+q}K(R^{\alpha\beta}_{-\varepsilon}; z) = (1-\varepsilon)^{ps'+q-2}L$. This result together with Lemma 2.3 shows that (2.2) also holds for B under the hypotheses of Theorem B.

4. Boundary behavior of other functionals on the domains of Theorem B.

THEOREM 4.1. *Under the hypotheses of Theorem B and using normal coordinates, we have further, denoting* $\lim^A (2x_1)^\lambda F(B; z)$ *by* (λ, F) *and letting* $m = p(q-1)$ *and* $\alpha = m + s$,

(1) $(\alpha + 1, K_{z_1}) = -L\alpha; (\alpha + 1/2, K_{z_k}) = 0, 2 \leqq k \leqq q-1;$
$(\alpha + p/2, K_{z_k}) = 0, q \leqq k \leqq n$,

(2) $(2, ds^2) = \alpha \,|\, dz_1 |^2$ *if* $0 < p < 2$;

$$(2, ds^2) = \alpha \,|\, dz_1 |^2 + c \sum_{k=s}^{n} |\, dz_k |^2 \text{ if } p = 2 ,$$

$$(p, ds^2) = c \sum_{k=s}^{n} |dz_k|^2 \text{ if } p > 2, \text{ where}$$

$$c = q(p + m + 1) \cdots (p + \alpha - 1)/[(m + 1) \cdots (\alpha - 1)] ,$$

(3) (o, I)

$$= \pi^{-n}(q-1)!\, \alpha^{-n-1}[(m+1) \cdots \alpha]^q [q(p+m+1) \cdots (p+\alpha-1]^{1-q} .$$

Proof (Sketch). One can show analogously as in Lemma 3.3 that B includes in its interior the domain $R^{\alpha'\beta'}_\varepsilon = W(R_\varepsilon)$ where

$$W: z_1 \leftarrow \frac{z_1}{1+\alpha' z_1} , \; z_k \leftarrow \frac{z_k(1 + \alpha' z_1)}{1 + (\alpha' + \beta')z_1} , \; k = 2, \cdots, n ,$$

for sufficiently large α' and β'. Then if $F(B; z)$ is any decreasing functional of the domain B, *e.g.*, $K(B; z)$, we have

$$F(R^{\alpha'\beta'}_\varepsilon; z) \geqq F(B; z) \geqq F(R^{\alpha\beta}_{-\varepsilon}; z) .$$

It is well known that the quantities ds^2 and I are rational functions of decreasing domain functionals F_i which can be determined from the kernel function and its derivatives. But from our results in §1, and using (0.2), we can calculate the kernel function and its derivatives for the domains $R^{\alpha'\beta'}_\varepsilon$ and $R^{\alpha\beta}_{-\varepsilon}$. Thus we can calculate the decreasing domain functionals F_i and show that there exist λ_i such that $\lim_{\varepsilon\to 0} (\lambda_i, F_i(R^{\alpha'\beta'}_\varepsilon; z)) = \lim_{\varepsilon\to 0} (\lambda_i, F_i(R^{\alpha\beta}_{-\varepsilon}; z)) = l_i$, whence $(\lambda_i, F_i(B; z)) = l_i$. From these results (2) and (3) follow immediately. Although the derivatives K_{z_k} are not themselves rational functions of monotone domain functionals, they can be approximated above and below by such functions and the same method yields (1).

The author wishes to express his thanks to Professor S. Bergman for his guidance and to Professor C. Loewner and M. Skwarczynski for their pertinent criticism in the preparation of this paper.

BIBLIOGRAPHY

1. S. Bergman, *Uber die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math. **169** (1933), 1-42; and **172** (1934), 89-128.
2. ———, *The behaviour of the kernel function at boundary points of the second order*, Amer. J. Math. **65** (1943), 679-700.
3. ———, *Sur les functions orthogonales de plusieurs variables complexes*, Mém. des Sciences Math. **106** (1947).
4. ———, *Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations psuedo-conformes*, Mém. des Sciences Math. **108** (1948).
5. ———, *The kernel function and conformal mapping*, Math. Surveys 5, Amer. Math. Soc., New York, 1950.
6. ———, *Zur Theorie von Pseudokonformen Abbildungen*, Recueil Math. **1** (1936), 79-96.
7. H. J. Bremermann, "Holomorphic continuation of the kernel function and the Bergman metric in several complex variables,"in *Lectures on functions of a complex variable*, ed. by W. Kaplan, Univ. of Michigan press, 1955.
8. K. Diederich, *Das Rand verhalten der Bergmanschen Kernfunktion und Metrik in streng pseudokonvexen Gebieten* (to appear)
9. B. A. Fuks, *Special chapters in the theory of analytic functions of several complex variables*, (Russian) Moscow, 1963 (Engl. Transl., A.M.S. 1965).
10. L. Hörmander, *Existence theorem for the $\bar{\partial}$-operator by L^2 methods*, Acta Math. **113** (1965), 89-152.
11. H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, Berlin, 1962.

Received June 24, 1968, and in revised form August 19, 1968. This paper includes the essential contents of the author's doctoral thesis at Stanford University.

UNIVERSITY OF CALIFORNIA, RIVERSIDE