PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 3, 1969

# MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIAL AND A PRESCRIBED SUBMATRIX-II

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**Let $A = [a_{ij}]$ be an $n \times n$ complex matrix and $f(\lambda)$ be a polynomial with complex coefficients of degree $n + k$ and leading coefficient $(-1)^{n+k}$. In the present paper we solve the following problem: under what conditions does there exist an $(n + k) \times (n + k)$ complex matrix $B$ of which $A$ is the submatrix standing in the top left-hand corner and such that $f(\lambda)$ is its characteristic polynomial?**

In [3] we solved this problem for $k = 1$. It can be seen that from our Theorem 2 in [3] the solution of the general case $(k > 1)$ comes out very easily when $A$ is real symmetric (hermitian) and $B$ is required to be of the same kind. This last problem had actually already been solved by Ky Fan and G. Pall (see [1]). Now we will prove the following

THEOREM. *Let $A$ be an $n \times n$ complex matrix whose distinct characteristic roots are $w_i$ $(i = 1, \cdots, t)$. Let us suppose that in the Jordan normal form of $A$, $w_i$ appears in $r_i$ distinct diagonal blocks of orders $v_1^{(i)}, \cdots, v_{r_i}^{(i)}$ respectively. Let us assume that $v_1^{(i)} \leqq \cdots \leqq v_{r_i}^{(i)}$. Let $\theta_i = \sum_{j=1}^{r_i - k} v_j^{(i)}$, with $\theta_i = 0$ if $r_i - k < 1$. There exists an $(n + k) \times (n + k)$ complex matrix $B$ having $A$ in the top left-hand corner and with $f(\lambda)$ as characteristic polynomial if and only if $f(\lambda)$ is divisible by $\prod_{i=1}^{t}(w_i - \lambda)^{\theta_i}$.*

First we prove that the condition is necessary. Let $T$ be a nonsingular matrix that transforms $A$ into its Jordan normal form $J$: $TAT^{-1} = J$, with $J = \operatorname{diag}(J_1, \cdots, J_m)$. The block $J_i$ will be of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

and we will suppose that $J_i$ is of type $s_i \times s_i$. Let

$$B = \begin{bmatrix} A & X_1 \\ Y_1 & S_1 \end{bmatrix}$$

where $X_1$, $Y_1$, $S_1$ are blocks of type $n \times k$, $k \times n$, $k \times k$ respectively. Let us assume that $f(\lambda) = \det(B - \lambda E_{n+k})$ where $E_{n+k}$ denotes the identity matrix of order $n + k$. If

$$T_1 = \begin{bmatrix} T & 0 \\ 0 & E_k \end{bmatrix},$$

we will have

$$B_1 = T_1 B T_1^{-1} = \begin{bmatrix} J & X \\ Y & S \end{bmatrix}$$

with $X = TX_1$, $Y = Y_1 T^{-1}$ and $S = S_1$. As $i \neq j$ implies $w_i \neq w_j$ all we need to prove is that $\det(B_1 - \lambda E_{n+k})$ is divisible by $(w_i - \lambda)^{\theta_i}$ $(i = 1, \cdots, t)$. We will do it for $(w_1 - \lambda)^{\theta_1}$ as the proof is the same for the other cases. We can assume that $w_1$ appears in the first $u$ diagonal blocks of $J$ and that $s_1 \leqq s_2 \leqq \cdots \leqq s_u$. Let us expand $\det(B_1 - \lambda E_{n+k})$ by Laplace rule in terms of its first $\sum_{i=1}^{u} s_i$ rows. The necessity of the condition of the theorem will be proved if we show that all the nonzero minors contained in the first $\sum_{i=1}^{u} s_i$ rows have determinants which are divisible by $(w_1 - \lambda)^{\theta_1}$. These minors are $\operatorname{diag}(J_1 - \lambda E^{(1)}, \cdots, J_u - \lambda E^{(u)})$ ($E^{(i)}$ denotes the identity matrix of the same order as $J_i$) and all the minors obtained from this one by replacing no more than $k$ of its columns by the same number of columns taken from the matrix which remains after suppressing the last $\sum_{i=u+1}^{m} s_i$ rows of $X$. As $J_i$ $(i = 1, \cdots, u)$ are diagonal matrices with $w_1$ in the principal diagonal our assertion follows.

Let us now prove that the condition is sufficient. For this we need an auxiliary proposition.

LEMMA. *Let $A$ be an $n \times n$ complex matrix whose distinct characteristic roots are $w_1, \cdots, w_t$. Let us assume that in the Jordan normal form of $A$, $w_i$ $(i = 1, \cdots, t)$ appears in $r_i$ diagonal blocks of orders $v_1^{(i)} \leqq v_2^{(i)} \leqq \cdots \leqq v_{r_i}^{(i)}$. Then it is possible to construct a matrix $A_1$ of type $(n + 1) \times (n + 1)$ containing $A$ in its top left-hand corner and such that: $(\alpha)$ The characteristic polynomial of $A_1$ is $\prod_{i=1}^{t}(w_i - \lambda)^{\sigma_i}\varphi(\lambda)$, where $\sigma_i = \sum_{j=1}^{r_i-1} v_j^{(i)}$ and $\varphi(\lambda)$ is any polynomial in $\lambda$ of degree $\rho = n + 1 - \sum_{i=1}^{t}\sigma_i$, leading coefficient $(-1)^{\rho}$ and such that $\varphi(w_i) \neq 0$ $(i = 1, \cdots, t)$. $(\beta)$ In the Jordan normal form of $A_1$ the characteristic root $w_i$ appears in exactly $r_i - 1$ diagonal blocks of orders*

$$v_1^{(i)}, \cdots, v_{r_i-1}^{(i)} \qquad (i = 1, \cdots, t).$$

*Proof.* We can suppose, without loss of generality, that $A$ is in its Jordan normal form.

The matrix $A_1$, if it exists, will have the form

$$A_1 = \begin{bmatrix} J_1 & 0 & \cdots & 0 & X_1 \\ 0 & J_2 & \cdots & 0 & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_m & X_m \\ Y_1 & Y_2 & \cdots & Y_m & q \end{bmatrix}$$

with $X_i = [x_1^i \cdots x_{s_i}^i]^T$ and $Y_i = [y_1^i \cdots y_{s_i}^i]$. The $x_j^i$ and $y_j^i$ must satisfy

$$\sum_{j=1}^{h+1} (-1)^{s_i - h} y_j^i x_{j+s_i-1-h}^i = b_{ih} \qquad (h = 0, \cdots, s_i - 1)$$

where the $b_{ih}$ are calculated by a process we give in [3]. Moreover, we recall that for each $i$ we can give to the $x_j^i$ $(j = 1, \cdots, s_i)$ arbitrary nonzero values. Let us suppose that we have fixed all the matrices $X_1, \cdots, X_m$ with $x_j^i \neq 0$ $(i = 1, \cdots, m;\ j = 1, \cdots, s_i)$. We can assume that $w_i$ appears in the diagonal blocks $J_{u_{i-1}+1}, \cdots, J_{u_i-1}, J_{u_i}$ $(i = 1, \cdots, t;\ u_0 = 0,\ u_t = m)$ of orders $s_{u_{i-1}+1} \leqq \cdots \leqq s_{u_i-1} \leqq s_{u_i}$ respectively. Let us now choose $Y_{u_{i-1}+1} = 0, \cdots, Y_{u_i-1} = 0$ $(i = 1, \cdots, t)$. Let

$$A_2 = \begin{bmatrix} J_{u_1} & 0 & \cdots & 0 & X_{u_1} \\ 0 & J_{u_2} & \cdots & 0 & X_{u_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_{v_t} & X_{u_t} \\ Y_{u_1} & Y_{u_2} & \cdots & Y_{u_t} & q \end{bmatrix}.$$

We have

$$\det(A_1 - \lambda E_1) = \prod_{i=1}^{t} (w_i - \lambda)^{\sigma_i} \det(A_2 - \lambda E_2)$$

where $\sigma_i = \sum_{j=u_{i-i}+1}^{u_i-1} s_j$ and $E_j$ is the identity matrix of the same order as $A_j$ $(j = 1, 2)$. The matrix $\operatorname{diag}(J_{u_1}, J_{u_2}, \cdots, J_{u_t})$ is obviously a nonderogatory matrix and so according to the corollary to Theorem 1 in [3] we can choose $Y_{u_1}, \cdots, Y_{u_t}$ and $q$ such that

$$\det(A_2 - \lambda E_2) = \varphi(\lambda) .$$

With this choice $A_1$ has the required characteristic polynomial. Let us find the diagonal blocks of the Jordan normal form of $A_1$ corresponding to $w_i$ $(i = 1, \cdots, t)$. This amounts to finding all the elementary divisors of $A$ of the form $(\lambda - w_i)^\gamma$ $(i = 1, \cdots, t)$. Let us consider, for example, the case $i = 1$ as the other cases can be treated in the same fashion. $A_1$ can be written in the form

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $A_{11} = \operatorname{diag}(J_1, \cdots, J_{u-1})$ and the matrix $A_{22}$ has not the characteristic root $w_1$. Therefore (see [2], p. 85) the elementary divisors of $A_1$ of the form $(\lambda - w_1)^{\gamma}$ are exactly

$$(\lambda - w_1)^{s_1}, (\lambda - w_1)^{s_2}, \cdots, (\lambda - w_1)^{s_{u-1}}$$

and the proof of the lemma is concluded.

Let us now complete the proof of the theorem.

Let

$$\theta_{ih} = \sum_{j=1}^{r_i-h} v_j^{(i)} \qquad (h = 1, \cdots, k-1;\ \theta_{ih} = 0 \text{ if } r_i - h < 1).$$

Let

$$f_j(\lambda) = \prod_{i=1}^{t}(w_i - \lambda)^{\theta_{ij}}\varphi_j(\lambda) \qquad (j = 1, \cdots, k-1),$$

where the $\varphi_j(\lambda)$ are polynomials in $\lambda$ chosen arbitrarily but with the following properties:

($\alpha$) The leading coefficient and the degree of $\varphi_j(\lambda)$ $(j = 1, \cdots, k-1)$ are such that $f_j(\lambda)$ has degree $n + j$ and leading coefficient $(-1)^{n+j}$

($\beta$) For $j = 1, \cdots, k-1$ the roots of $\varphi_j(\lambda)$ are distinct, $\varphi_j(w_i) \neq 0$ $(i = 1, \cdots, t)$ and if $\varphi_j(\xi) = 0$ then $\varphi_{j+1}(\xi) \neq 0$.

Obviously there are infinitely many possibilities of choice for the $\varphi_j(\lambda)$ $(j = 1, \cdots, k-1)$.

Because of the lemma we can border $A$ with a row (below) and a column (on the right hand side) to obtain a matrix $A_1$ with characteristic polynomial $f_1(\lambda)$ and such that in its Jordan normal form $w_i$ $(i = 1, \cdots, t)$ appears in exactly $r_i - 1$ diagonal blocks whose orders are $v_1^{(i)}, \cdots, v_{r_i-1}^{(i)}$. Now we can border $A_1$ with another row (below) and a column (on the right hand side) in such a way that we obtain a matrix $A_2$ with $f_2(\lambda)$ as characteristic polynomial and such that in the Jordan normal form of $A_2$ the characteristic root $w_i$ $(i = 1, \cdots, t)$ appears in exactly $r_i - 2$ diagonal blocks of orders $v_1^{(i)}, \cdots, v_{r_i-2}^{(i)}$. We can continue in this fashion up to the matrix $A_{k-1}$. Using now Theorem 1 of [3] with $A_{k-1}$, the proof is complete.

In an $(n + k) \times (n + k)$ matrix any principal minor of type $n \times n$ can be brought to the top left-hand corner by a permutation of rows and the same permutation of columns. This remark combined with the Theorem above solves the following problem: under what conditions does there exist an $(n + k) \times (n + k)$ complex matrix $B$ of which $A$ is the principal minor contained in the rows of orders $i_1, \cdots, i_n$

$(1 \leqq i_1 < \cdots < i_n \leqq n + k)$ and such that $f(\lambda)$ is its characteristic polynomial?

## References

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Received May 10, 1968.

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