EMBEDDED EIGENVALUES AND VIRTUAL POLES

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An embedded eigenvalue of simple multiplicity is perturbed by an operator of finite rank. The resulting spectral concentration is shown to be due to the pole of the continuation of a certain analytic function.

If λ_0 is point eigenvalue of the selfadjoint operator T_0, then it may happen that λ_0 "disappears" under perturbation in the sense that the perturbed operator $T_\varepsilon = T_0 + \varepsilon V$ has a purely continuous spectrum near λ_0 for arbitrarily small positive ε. As McLeod [10] remarks, there are two parts of the theory of this phenomenon. There is, first of all, the theory of *spectral concentration*, in which it is shown that as $\varepsilon \to 0+$ there is an asymptotic "concentration" of the spectral measure of T_ε near λ_0. Spectral concentration has been discussed for isolated eigenvalues by many authors (see [1], [10], and [8, Chap. VIII]), and for eigenvalues embedded in the continuous spectrum by Friedrichs [4] and the present author [5]. Secondly, there is the conjecture, verified in certain examples (see [2], [10]), that this concentration of the spectral measure is due to a *virtual pole* of the resolvent $R(z, \varepsilon) = (T_\varepsilon - z)^{-1}$. For example, if T_ε is a differential operator and $R(z, \varepsilon)$ an integral operator with Green's function kernel $g_\varepsilon(x, y; z)$, this means that g_ε can be continued meromorphicly from $\operatorname{Im} z > 0$ to a neighborhood of λ_0 (on which the continuation will *not* agree with the values of $g_\varepsilon(x, y; z)$ for $\operatorname{Im} z < 0$), and that this continuation will have, slightly below the axis, a pole $\zeta(\varepsilon)$ which tends to λ_0 as $\varepsilon \to 0+$, and which accounts for the concentration in much the same way that a pole *on* the axis might account for a point eigenvalue. This example defines the problem—in a more general situation, the first step toward a solution is to find a replacement for the Green's function which can be continued meromorphicly.

In [5], the author considered spectral concentration of embedded eigenvalues of simple multiplicity, under perturbations of finite rank. In the present paper, we shall discuss virtual poles in the context. The quantity chosen to replace the Green's function is the *inverse Weinstein-Aronszajn* matrix; that is to say, the inverse of the matrix of the restriction of $I + \varepsilon V(T_0 - z)^{-1}$ to the (finite dimensional) range of the perturbation V (cf. [8, Chap. IV, §6]).

The organization of the paper is as follows. In §1, we state the basic assumptions and note a formula (Lemma 1.1) which is fundamental to our approach. The formulation of the problem is similar to that of Kuroda [9]. The existence of a virtual pole $\zeta(\varepsilon)$ is then

proved for sufficiently small ε in § 2. The spectral theory of T_ε is discussed in § 4, where it is shown that T_ε has an eigenvalue near λ_0 if and only if $\zeta(\varepsilon)$ is real, in which case the eigenvalue is $\zeta(\varepsilon)$ itself. Section 3 contains a general result on continuations of Stieltjes transforms which is used in § 4. Next, we consider the asymptotic spectral theory of T_ε as $\varepsilon \to 0+$, in the case that $\zeta(\varepsilon)$ is *not* real. In § 5, we obtain an asymptotic formula on the range of V for the spectral density near λ_0, which corresponds to the formula of Wigner and Weiskopff mentioned by Friedrichs [4]. This formula is used in § 6 to prove a result on spectral concentration near λ_0. Finally, in § 7, we discuss, under certain conditions of nondegeneracy, the relationship between $\zeta(\varepsilon)$ and the formal perturbation series for the perturbed eigenvalue, obtaining in this case an asymptotic formula for the imaginary part of $\zeta(\varepsilon)$. Section 8 discusses three examples.

1. Assumptions and preliminary facts. Let T_0 be a self-adjoint operator on a Hilbert space $\mathscr{H}$ with domain $\mathscr{D}(T_0)$, and V a bounded self-adjoint operator of finite rank r. V may then be written in the form $V = \sum_{i=1}^{r} c_i(\,\cdot\,, \varphi_i)\varphi_i$ where $c_1, \cdots, c_r$ are nonvanishing real numbers and $\{\varphi_1, \cdots, \varphi_r\}$ is a finite orthonormal set spanning the range $\mathscr{R}(V)$ of V. If $\varepsilon > 0$, then the operator $T_\varepsilon = T_0 + \varepsilon V$ with $\mathscr{D}(T_\varepsilon) = \mathscr{D}(T_0)$ is also self-adjoint. Let

$$R_0(z) = (T_0 - z)^{-1} \quad \text{and} \quad R(z, \varepsilon) = (T_\varepsilon - z)^{-1}$$

be the resolvents of T_0 and T_ε, and $E_0(\lambda)$ and $E_\varepsilon(\lambda)$ the corresponding spectral resolutions. If S is a Borel set, we define

$$E_\varepsilon[S] = \int_S dE_\varepsilon(\lambda)$$

for $\varepsilon \geqq 0$.

The following assumption will be made throughout the paper, and is an implicit hypothesis of all theorems.

BASIC HYPOTHESIS. *Let λ_0 be an eigenvalue of T_0 of simple multiplicity and ψ_0 a corresponding normalized eigenvector. Let $P_0 = (\,\cdot\,, \psi_0)\psi_0$ and $P = I - P_0$. We shall assume that for $i, j = 1, \cdots, r$ the analytic function $(R_0(z)P\varphi_i, \varphi_j)$ has an analytic continuation from the upper half-plane into a neighborhood of λ_0 in the complex plane.*

In the sequel, we shall let $N = \{z \colon |z - \lambda_0| \leqq \delta_0\}$ be a fixed closed disc contained in the above neighborhoods for all $i, j = 1, \cdots, r$. Since $[R_0(z)]^* = R_0(\bar{z})$, it is a consequence of the above assumption that $(R_0(z)P\varphi_i, \varphi_j)$ has a corresponding continuation from the *lower* half-

plane to a neighborhood of N. These two continuations are in general *not identical on* N, since λ_0 need not be an isolated eigenvalue. In order to distinguish between them, we shall introduce the following notation: Let $F(z)$ be a function defined and analytic for $\text{Im } z \neq 0$. The continuation of $F(z)$ from $\text{Im } z > 0$ to N will be denoted by a subscript '$+$', while the continuation from $\text{Im } z < 0$ to N will be denoted by a subscript '$-$'. For example, $(R_0(z)\varphi_i, \varphi_j)_+$ denotes the continuation from $\text{Im } z > 0$. Similarly, we shall encounter such notations as $W_\pm(z, \varepsilon)$ and $[W_a(z, \varepsilon)]_\pm$.

The matrix whose elements are a_{ij}, $i, j = 1, \cdots, r$, will be denoted by mat $\{a_{ij}\}$.

For $\text{Im } z \neq 0$, let $W(z, \varepsilon)$ be the restriction of $I + \varepsilon V R_0(z)$ to $\mathscr{R}(V)$. Since $\mathscr{R}(V)$ is invariant under $I + \varepsilon V R_0(z)$, $W(z, \varepsilon)$ is an operator on the finite dimensional space $\mathscr{R}(V)$, and is easily seen to have the matrix

$$(1.1) \qquad \text{mat } \{\delta_{ij} + \varepsilon c_j(R_0(z)\varphi_i, \varphi_j)\}$$

with respect to the basis $\varphi_1, \cdots, \varphi_r$. Moreover, $W^{-1}(z, \varepsilon)$ exists for $\text{Im } z \neq 0$ and every $\varepsilon \geqq 0$, and is the restriction to $\mathscr{R}(V)$ of the operator $I - \varepsilon V R(z, \varepsilon)$ (see [5, § 1] or [10]). We also define $W_a(z, \varepsilon)$ to be the restriction to $\mathscr{R}(V)$ of $I + \varepsilon V R_0(z)P$, so that

$$(1.2) \qquad W(z, \varepsilon) = W_a(z, \varepsilon) + (\lambda_0 - z)^{-1}VP_0 \ .$$

In analogy with (1.1), $W_a(z, \varepsilon)$ has the matrix

$$(1.3) \qquad \text{mat } \{\delta_{ij} + \varepsilon c_j(R_0(z)P\varphi_i, \varphi_j)\} \ .$$

The following formula for $W^{-1}(z, \varepsilon)$ is basic to our approach.

LEMMA 1.1. $W_a^{-1}(z, \varepsilon)$ *exists for* $\text{Im } z \neq 0$ *and*

$$(1.4) \qquad W^{-1}(z, \varepsilon) = W_a^{-1}(z, \varepsilon)[I - \varDelta(z, \varepsilon)(W_a^{-1}(z, \varepsilon)\cdot, \psi_0)V\psi_0]$$

where

$$(1.5) \qquad \varDelta(z, \varepsilon) = \varepsilon[\lambda_0 - z + \varepsilon(W_a^{-1}(z, \varepsilon)V\psi_0, \psi_0)]^{-1} \ .$$

For the proof, see [5, § 1, Lemma 1.1].

PROPOSITION. *Under the Basic Hypothesis, the following hold for all sufficiently small positive* ε:

(a) $W_a(z, \varepsilon)$ *and* $W_a^{-1}(z, \varepsilon)$ *have analytic continuations to a neighborhood of* N *from both half-planes. For a suitably small* $\varepsilon_0 > 0$, *these continuations,* $[W_a(z, \varepsilon)]_\pm$ *and* $[W_a^{-1}(z, \varepsilon)]_\pm$ *are bounded on* N, *uniformly for* $0 \leqq \varepsilon \leqq \varepsilon_0$.

(b) *If* $VP_0 \neq 0$, *then* $W(z, \varepsilon)$, $W^{-1}(z, \varepsilon)$, *and* $\Delta(z, \varepsilon)$ *have meromorphic continuations to a neighborhood of* N *from both half-planes.* $W_{\pm}(z, \varepsilon)$ *is analytic on* N *with the exception of a simple pole at* λ_0. *The poles of* $W_{\pm}^{-1}(z, \varepsilon)$ *coincide with those of* $\Delta_{\pm}(z, \varepsilon)$. *The residues at these poles have rank one.*

Proof. Existence and uniform boundedness of the continuations $[W_a(z, \varepsilon)]_{\pm}$ is immediate from the Basic Hypothesis and (1.3). The Neumann series expansion of $W_a^{-1}(z, \varepsilon)$ in powers of ε (cf. [5; §3]) converges uniformly on N, so that the rest of (a) follows. (b) follows easily from (a) and an inspection of formulae (1.2), (1.4) and (1.5).

2. **Virtual poles.** We are now able to prove the existence of a virtual pole of $W^{-1}(z, \varepsilon)$.

THEOREM 1. *Let* $VP_0 \neq 0$. *Then for all sufficiently small positive* ε, $W_+^{-1}(z, \varepsilon)$ *has a unique pole* $\zeta(\varepsilon)$ *in* N. $\zeta(\varepsilon)$ *is a simple pole with residue of rank one,* $\mathrm{Im}\, \zeta(\varepsilon) \leqq 0$ *and* $\zeta(\varepsilon) \to \lambda_0$ *as* $\varepsilon \to 0+$. *Moreover,* $\zeta(\varepsilon)$ *is a complex analytic function of* ε *at* $\varepsilon = 0$.

Proof. By the Proposition of §1, it suffices to consider solutions of the equation

$$(2.1) \qquad \varepsilon \Delta_+^{-1}(z, \varepsilon) = \lambda_0 - z + \varepsilon (W_a^{-1}(z, \varepsilon) V \psi_0, \psi_0)_+ = 0 \;.$$

Let $0 < \rho \leqq \rho_0$. Then since $(W_a^{-1}(z, \varepsilon) V \psi_0, \psi_0)_+$ is bounded on N uniformly in ε, $\varepsilon \Delta_+^{-1}(z, \varepsilon) - (\lambda_0 - z)$ can be made arbitrarily small uniformly on $|z - \lambda_0| = \rho$, by choosing ε sufficiently small. By Rouche's Theorem, $\varepsilon \Delta_+^{-1}(z, \varepsilon)$ and $(\lambda_0 - z)$ have the same number of zeros (counting multiplicities) inside $|z - \lambda_0| = \rho$. Choosing $\rho = \rho_0$ gives existence, uniqueness and simplicity of the pole $\zeta(\varepsilon)$ of $\Delta_+(z, \varepsilon)$ in N, while choosing ρ arbitrarily small shows that $\zeta(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0+$. Since the coefficient of $\Delta_+(z, \varepsilon)$ in (1.4) is of rank one, the residue of $W_+^{-1}(z, \varepsilon)$ has rank one. If $\mathrm{Im}\, z > 0$, $W_+^{-1}(z, \varepsilon)$ is the restriction of $I - \varepsilon V R(z, \varepsilon)$ to $\mathscr{R}(V)$, and therefore has no pole. Hence, $\mathrm{Im}\, \zeta(\varepsilon) \leqq 0$. Analyticity of $\zeta(\varepsilon)$ follows from a general result [3; p. 101, Th. 7.1] on analyticity of inverse functions.

COROLLARY. $W_-^{-1}(z, \varepsilon)$ *has, for sufficiently small positive* ε, *the unique pole* $\zeta_-(\varepsilon) = \overline{\zeta}(\varepsilon)$. *As a complex analytic function of* ε, $\zeta_-(\varepsilon) = \overline{\zeta}(\bar{\varepsilon})$.

Proof. We simply observe that for real ε, $\Delta_-(z, \varepsilon) = \overline{\Delta}_+(\bar{z}, \varepsilon)$ since $R_0(z) = R_0^*(\bar{z})$. The result follows easily.

3. Continuations of Stieltjes transforms. The following simple result on meromorphic continuations of Stieltjes transforms of matrix-valued measures will be used in the next section.

THEOREM 2. *Let M be a positive Hermitian matrix-valued, finite Borel measure on $(-\infty, +\infty)$, and*

$$F(z) = \int_{-\infty}^{+\infty}(t - z)^{-1}dM(t) \qquad \operatorname{Im} z \neq 0$$

its Stieltjes transform. If $F(z)$ has a meromorphic continuation $F_+(z)$ from $\operatorname{Im} z > 0$ to a disc D about the origin, then

(a) *$F(z)$ has a meromorphic continuation $F_-(z)$ from $\operatorname{Im} z < 0$ to D.*

(b) *The poles of $F_-(z)$ have nonnegative imaginary parts and are the reflections in the real axis of the poles of $F_+(z)$.*

(c) *If $F_+(z)$ has a simple pole at $z = 0$, then M has a point mass at the origin, equal to the residue of $F_+(z)$ at the origin.*

Proof. From $\operatorname{Im} z \neq 0$, we have $F(z) = F^*(\bar{z})$, and hence by continuation

(3.1) $$F_-(z) = F_+^*(\bar{z})$$

which proves (a) and (b). For (c), observe that if

(3.2) $$F_+(z) = (A/z) + O(1)$$

and

(3.3) $$F_-(z) = (B/z) + O(1)$$

on D, then by (3.1)

(3.4) $$B = A^* .$$

Now observe that for sufficiently small real λ, we have

$$M'(\lambda) = (1/2\pi i)[F_+(\lambda) - F_-(\lambda)]$$
$$= (A - A^*)/(2\pi i\lambda) + O(1)$$

where $M'(\lambda)$ is the spectral density for the absolutely continuous part of M. Since $M'(\lambda)$ is a positive Hermitian matrix both for $\lambda > 0$ and $\lambda < 0$, we must have

(3.5) $$A = A^* = B .$$

Now if $\delta > 0$ is sufficiently small, it is clear that M is absolutely continuous on $(-\delta, +\delta)$ with smooth derivative, except possibly at $z = 0$. It is therefore easy to see that

(3.6)
$$2\pi i M(-\delta, +\delta) = \int_{|z|=\delta} F(z)dz$$
$$= \int_{|z|=\delta} F_+(z)dz + \int_C [F_-(z) - F_+(z)]dz$$

where C is the lower half of the (positively oriented) contour $|z| = \delta$. But the first term of the right side of (3.6) is equal to $2\pi i A$, while the second is $O(\delta)$, since its integrand is bounded on C by (3.2), (3.3) and (3.4). Hence,

$$\lim_{\delta \to 0+} M(-\delta, +\delta) = A$$

which proves the result.

4. **Spectral theory.** If $T = \int \lambda dE(\lambda)$ is self-adjoint and S is a Borel subset of the reals, then the *part of T in S* is the operator $TE[S]$, considered as an operator on $E[S]\mathscr{H}$. T is *absolutely continuous on S* if and only if the part of T in S is absolutely continuous, or equivalently if and only if the measure $d(E(\lambda)E[S]x, x)$ is absolutely continuous with respect to Lebesgue measure for every x in $\mathscr{H}$.

If $\mathscr{M}$ is the smallest subspace of $\mathscr{H}$ which contains $\varphi_1, \cdots, \varphi_r$ and reduces T_0, then $\mathscr{M}$ also reduces T_ε, and $T_\varepsilon = T_0$ on the orthogonal complement $\mathscr{M}^\perp$ of $\mathscr{M}$. For our purposes, it therefore suffices to assume that $\mathscr{M} = \mathscr{H}$ (see [5, 7, 9]). Let $G = (\lambda_0 - \rho_0, \lambda_0 + \rho_0)$ be the intersection of the interior of N with the real axis.

THEOREM 3. *Assume that* $\mathscr{H} = \mathscr{M}$. *Then* $VP_0 \neq 0$ *and, for all sufficiently small positive* ε,

(a) T_ε *has an eigenvalue in* G *if and only if* $\operatorname{Im} \zeta(\varepsilon) = 0$, *in which case* $\zeta(\varepsilon)$ *itself is the unique eigenvalue of* T_ε *in* G, *and has simple multiplicity.*

(b) *The continuous part of* T_ε *in* G *is absolutely continuous and unitarily equivalent to the part of* T_0 *in* $G \sim \{\lambda_0\}$.

Proof. Since

(4.1) $$W^{-1}(z, \varepsilon) = I - \varepsilon \operatorname{mat} \{c_j(R(z, \varepsilon)\varphi_i, \varphi_j)\}$$

it follows that for $i, j = 1, \cdots, r$, $(R(z, \varepsilon)\varphi_i, \varphi_j)$ is continuous on $G \sim \{\zeta(\varepsilon)\}$, and hence that $d(E_\varepsilon(\lambda)\varphi_i, \varphi_i)$ is absolutely continuous on $G \sim \{\zeta(\varepsilon)\}$. Since $\mathscr{H} = \mathscr{M}$, it follows that T_ε is absolutely continuous on $G \sim \{\zeta(\varepsilon)\}$. (See [5], proof of Theorem 1). However, the absolutely continuous parts of T_0 and T_ε are unitarily equivalent by the Rosenblum-Kato Theorem [8; Th. 4.4, p. 540], so that part (b) follows. Since $G \sim \{\zeta(\varepsilon)\} = G$ when $\operatorname{Im} \zeta(\varepsilon) < 0$, it remains only to show that $\zeta(\varepsilon)$

is an eigenvalue of simple multiplicity whenever $\operatorname{Im} \zeta(\varepsilon) = 0$. Observe that the Stieltjes transform mat $\{(R(z, \varepsilon)\varphi_i, \varphi_j)\}$ of the matrixvalued measure $dM = \operatorname{mat} \{d(E_\varepsilon(\lambda)\varphi_i, \varphi_j)\}$ satisfies the hypotheses of Theorem 2 if $\zeta(\varepsilon)$ is taken as the origin. It follows that $\zeta(\varepsilon)$ is a point mass of M, and hence a point eigenvalue of T_ε.

Let Q_0 be the orthogonal projection on the eigenspace of $\zeta(\varepsilon)$ and Q the orthogonal projection on $\mathscr{R}(V)$. Then the point mass of M at $\zeta(\varepsilon)$ is the matrix of the operator QQ_0Q on $\mathscr{R}(V)$, and since this is the residue of mat $\{(R(z, \varepsilon)\varphi_i, \varphi_j)\}$, it has rank one. It follows that if ψ_1 and ψ_2 are in the range of Q_0, there exists a linear combination $\psi = a_1\psi_1 + a_2\psi_2$ such that $QQ_0Q\psi = 0$. But then

$$0 = (Q_0Q\psi, Q\psi) = \| Q_0Q\psi \|^2$$

so that $Q_0Q\psi = 0$ and

$$0 = (Q\psi, Q_0\psi) = (Q\psi, \psi) = \| Q\psi \|^2 .$$

Thus, if $\mathscr{M}_0$ is the orthogonal complement of the subspace spanned by ψ, then $\mathscr{M}_0$ reduces T_ε (since ψ is an eigenvector) and contains $\mathscr{R}(V)$ (since $Q\psi=0$). Hence, $\mathscr{M}_0$ also reduces T_0, so that $\mathscr{M}_0=\mathscr{H}$ and $\psi = 0$. Therefore ψ_1 and ψ_2 are linearly dependent, and $\zeta(\varepsilon)$ has simple multiplicity.

REMARK. Note that by analyticity of $\zeta(\varepsilon)$, either (a) $\operatorname{Im} \zeta(\varepsilon) \neq 0$ for *all* sufficiently small positive ε or (b) $\operatorname{Im} \zeta(\varepsilon)=0$ for *all* sufficiently small positive ε.

5. **Asymptotic spectral theory.** In this section, an asymptotic formula for the spectral density matrix defined by

$$E'(\lambda, \varepsilon) = \operatorname{mat} \{d(E_\varepsilon(\lambda)\varphi_i, \varphi_j)/d\lambda\}$$

is obtained. The result is valid near $\operatorname{Re} \zeta(\varepsilon)$ as $\varepsilon \to 0+$ in the case $\operatorname{Im} \zeta(\varepsilon)\neq 0$, and corresponds to the formula of Wigner and Weisskopf referred to by Friedrichs [4].

LEMMA 5.1. *If we define*

$$R_+(z, \varepsilon) = \operatorname{mat} \{(R(z, \varepsilon)\varphi_i, \varphi_j)_+\}$$

then $(z - \zeta(\varepsilon))R_+(z, \varepsilon)$ *is analytic in z and ε on $N \times \{\varepsilon: |\varepsilon| < \varepsilon_1\}$ for some sufficiently small positive ε_1.*

Proof. Observe that $\varepsilon\Delta_+^{-1}(z, \varepsilon) = \lambda_0 - z + O(\varepsilon)$ uniformly on N. If $|\lambda_0 - z| = \rho_0$, then $|\varepsilon\Delta_+^{-1}| \geqq (\rho_0/2)$ for ε small, and hence $|\Delta_+| \leqq (2\varepsilon/\rho_0)$ uniformly on $|\lambda_0 - z| = \rho_0$. From (4.1) and Theorem 1, $R_+(z, \varepsilon)$ is

analytic in z for each ε, and by the above is analytic in ε on $|z - \lambda_0| = \rho_0$. Thus, we have by Cauchy's formula

$$(z - \zeta(\varepsilon))\boldsymbol{R}_+(z, \varepsilon) = (1/2\pi i) \int_{\partial N} (\xi - z)^{-1}(\xi - \zeta(\varepsilon))\boldsymbol{R}_+(\xi, \varepsilon)d\xi$$

where the contour ∂N has the positive orientation. By differentiation under the integral sign, we have analyticity in ε in a neighborhood of $\varepsilon = 0$.

Let $\zeta(\varepsilon) = \lambda_\varepsilon - i\Gamma_\varepsilon$, so that $\Gamma_\varepsilon \geqq 0$, and let $|J|$ be the Lebesgue measure of J.

THEOREM 4. *Assume that* $\operatorname{Im} \zeta(\varepsilon) \neq 0$, *and let* $\{J_\varepsilon: \varepsilon > 0\}$ *be an arbitrary family of open intervals containing* λ_ε *and such that* $|J_\varepsilon| = O(\Gamma_\varepsilon^{1/2})$ *as* $\varepsilon \to 0+$. *Then for* $\lambda \in J_\varepsilon$

$$\boldsymbol{E}'(\lambda, \varepsilon) = (\Gamma_\varepsilon/\pi)[(\lambda - \lambda_\varepsilon)^2 + \Gamma_\varepsilon^2]^{-1}B(\varepsilon) + O(1)$$

as $\varepsilon \to 0+$, *where* $B(\varepsilon)$ *is the real part of the residue of* $\boldsymbol{R}_+(z, \varepsilon)$ *at the pole* $\zeta(\varepsilon)$. $B(\varepsilon)$ *is analytic in* ε *and* $B(0) = \operatorname{mat}\{(P_0\varphi_i, \varphi_j)\}$.

Proof. By Lemma 5.1, we have

(5.1) $\boldsymbol{R}_+(z, \varepsilon) = (z - \zeta(\varepsilon))^{-1}A(\varepsilon) + O(1)$

as $\varepsilon \to 0+$, uniformly on N. Since

$$A(\varepsilon) = (1/2\pi i) \int_{\partial N} \boldsymbol{R}_+(z, \varepsilon)dz$$

differentiation under the integral sign shows that $A(\varepsilon)$ is analytic near $\varepsilon = 0$, and hence that $B(\varepsilon)$ is real analytic for real ε. Let

$$A(\varepsilon) = B(\varepsilon) + iC(\varepsilon) .$$

Then by (5.1) we have for $\lambda \in J_\varepsilon$

(5.2) $\boldsymbol{E}'(\lambda, \varepsilon) = (1/\pi) \operatorname{Im} \boldsymbol{R}_+(\lambda, \varepsilon)$

$= (\Gamma_\varepsilon/\pi)[(\lambda - \lambda_\varepsilon)^2 + \Gamma_\varepsilon^2]^{-1}B(\varepsilon)$

$+ (1/\pi)(\lambda - \lambda_\varepsilon)[(\lambda - \lambda_\varepsilon)^2 + \Gamma_\varepsilon^2]^{-1}C(\varepsilon) + O(1)$

as $\varepsilon \to 0+$. Observe now that $C(\varepsilon) = O(\Gamma_\varepsilon^{1/2})$; for if we note that $\boldsymbol{E}'(\lambda, \varepsilon) \geqq 0$ and set $\lambda = \lambda_\varepsilon \pm \Gamma_\varepsilon^{1/2}$, then (5.2) implies that

(5.3) $\pm\Gamma_\varepsilon^{1/2}[\Gamma_\varepsilon + \Gamma_\varepsilon^2]^{-1}(C(\varepsilon)x, x) + O(1) \geqq 0$

for every $x \in \mathscr{R}(V)$. But if $\Gamma_\varepsilon^{-1/2}(C(\varepsilon)x, x)$ were unbounded as $\varepsilon \to 0+$, then since *either sign* of Γ_ε can be taken, (5.3) could not hold. It now follows easily that the second term of (5.2) is $O(1)$ as $\varepsilon \to 0+$,

which completes the proof.

6. **Spectral concentration.** Let $\{T_\varepsilon : \varepsilon \geqq 0\}$ be a family of self-adjoint operators. If $E_\varepsilon(\lambda)$ is the spectral resolution of T_ε, we define $E_\varepsilon[S] = \int_S dE_\varepsilon(\lambda)$ for every Borel set S. Following [1], we say that *the spectrum of* T_ε *is concentrated at* λ_0 *to order* p *as* $\varepsilon \to 0+$ if and only if there exists a family $\{J_\varepsilon : \varepsilon > 0\}$ of intervals such that $E_\varepsilon[J_\varepsilon] \to E_0[\{\lambda_0\}]$ strongly and $|J_\varepsilon| = O(\varepsilon^p)$ as $\varepsilon \to 0+$, where $|J_\varepsilon|$ is the Lebesgue measure of J_ε.

THEOREM 5. *Let $\{J_\varepsilon : \varepsilon > 0\}$ be an arbitrary family of intervals, symmetric about λ_ε, and such that $|J_\varepsilon| = O(\Gamma_\varepsilon^{1/2})$ as $\varepsilon \to 0+$. Assume that* $\operatorname{Im} \zeta(\varepsilon) \neq 0$. *Then*

(a) *If $\beta = \lim |J_\varepsilon|/2\Gamma_\varepsilon$ exists as*

$$\varepsilon \to 0+, 0 \leqq \beta \leqq \infty,$$

then $E_\varepsilon[J_\varepsilon] \to (2/\pi) \arctan(\beta) P_0$ *weakly as* $\varepsilon \to 0+$

(b) $E_\varepsilon[J_\varepsilon] \to P_0$ *strongly if and only if* $\lim |J_\varepsilon|/\Gamma_\varepsilon = \infty$.

COROLLARY 6.1. *Let $\Gamma_\varepsilon = O(\varepsilon^q)$ as $\varepsilon \to 0+$. Then the spectrum of T_ε is concentrated at λ_0 to order p for $0 \leqq p < q$, but not for $p \geqq q$.*

Proof of Theorem 5. If we recall that $(B(\varepsilon)x, y) \to (P_0 x, y)$ as $\varepsilon \to 0+$ for every $x, y \in \mathscr{R}(V)$, the proof is essentially the same as the proof of Theorem 4 of [5]. Only a few simple changes need to be made to compensate for the fact that in [5], $\Gamma_\varepsilon = O(\varepsilon^2)$.

REMARK. In defining concentration of order p, the authors of [1] permit $\{J_\varepsilon\}$ to be Borel sets. The distinction is relevant for eigenvalues λ_0 of multiplicity greater than one, but not for simple multiplicity—at least for isolated eigenvalues. We have therefore chosen the simpler definition.

7. **Perturbation series.** One of the classical problems of perturbation theory (discussed, for example, by Frierdrichs [4] and Kato [8, Chap. III]) is the interpretation of the formal perturbation series in cases where it exists (at least to a certain number of terms), but there is no corresponding perturbed eigenvalue. This is usually done in terms of spectral concentration (see [1, 8, 10] where further references are given). In the preceding section, we have related spectral concentration to the position of the pole $\zeta(\varepsilon)$, and we shall now discuss the connection between $\zeta(\varepsilon)$ and the formal series. Under some rea-

sonable nondegeneracy conditions, we obtain rather instructive results, which involve the order of vanishing of the unperturbed spectral density at λ_0. To be precise, we shall prove the following theorem:

THEOREM 6. *Let ν be the order of vanishing of the matrixvalued function* mat $\{d(E_0(\lambda)P\varphi_i, \varphi_j)/d\lambda\}$ *at λ_0, and assume that*

$$d(E_0(\lambda)PV\psi_0, \psi_0)/d\lambda$$

vanishes at λ_0 of order exactly ν. Let

$$(7.1) \qquad\qquad \zeta(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$$

be the series expansion of $\zeta(\varepsilon)$, and let

$$(7.2) \qquad\qquad \zeta_0(\varepsilon) = \lambda_0 + \lambda_p \varepsilon^p + O(\varepsilon^{p+1})$$

where $\lambda_p \neq 0$, $p \geq 1$ and $\zeta_0(\varepsilon)$ is the quantity defined in Lemma 7.2 below. Then

(a) *The first nonreal coefficient in* (7.1) *is $a_{2+\nu p}$ and we have*

$$(7.3) \qquad\qquad \Gamma_\varepsilon = +\varepsilon^{2+\nu p} \lambda_p^\nu \gamma(\lambda_0) + O(\varepsilon^{\nu p+3})$$

where

$$\gamma(\lambda) = \pi(\lambda - \lambda_0)^{-\nu} d(E_0(\lambda)PV\psi_0, V\psi_0)/d\lambda \ .$$

(b) *The formal perturbation series exists up through terms of order $\nu p + 1$ and coincides with $\sum_{n=0}^{\nu p+1} a_n \varepsilon^n$.*

For the proof, we shall require two lemmas.

LEMMA 7.1. *Let C be the contour consisting of the real axis outside of $G = (\lambda_0 - \rho_0, \lambda_0 + \rho_0)$, together with the lower boundary of N, all positively oriented. Then for any $n \geq 0$ and $\mathrm{Im}\, z > 0$, we have*

$$(7.4) \qquad W_a(z, \varepsilon) = I + \varepsilon Q_n(z) + \varepsilon(z - \lambda_0)^n D_n(z)$$

where $Q_n(z)$ is the matrix polynomial $\sum_{k=0}^{n-1} Q^{(k)}(z-\lambda_0)^k$, with coefficients

$$(7.5) \qquad Q^{(k)} = \mathrm{mat}\left\{ c_j \int_C (\lambda - \lambda_0)^{-k-1} d(E_0(\lambda)P\varphi_i, \varphi_j) \right\}$$

and

$$(7.6) \qquad D_n(z) = \mathrm{mat}\left\{ c_j \int_C (\lambda - \lambda_0)^{-n} (\lambda - z)^{-1} d(E_0(\lambda)P\varphi_i, \varphi_j) \right\} .$$

REMARKS. (1) The meaning of the rather cryptic integrals on the right sides of (7.5) and (7.6) is the following. The integral has its usual meaning over $|\lambda - \lambda_0| \geqq \rho_0$, while on the lower part of ∂N, $d(E_0(\lambda)P\varphi_i, \varphi_j)$ is to be replaced by

$$(7.7) \qquad (1/2\pi i)[(R_0(\lambda)P\varphi_i, \varphi_j)_+ - (R_0(\lambda)P\varphi_i, \varphi_j)_-]d\lambda \ .$$

The coefficient of $d\lambda$ in (7.7) is simply the analytic continuation of $d(E_0(\lambda)\varphi_i, \varphi_i)/d\lambda$ from G to N.

(2) Note that since $D_n(z)$ is well defined and analytic on the interior of N, and $Q_n(z)$ is a polynomial, (7.4) holds on the interior of N, provided that $W_a(z, \varepsilon)$ is replaced by $[W_a(z, \varepsilon)]_+$ on the left. A similar formula can be obtained for $[W_a(z, \varepsilon)]_-$ simply by making C traverse the (negatively oriented) upper half of the boundary of N rather than the lower half.

Proof of Lemma 7.1. Fix z with $\text{Im } z > 0$ and note the finite binomial expansion

$$(7.8) \qquad \begin{aligned} (\lambda - z)^{-1} &= (\lambda - \lambda_0)^{-1} \sum_{k=0}^{n-1} \{(z - \lambda_0)^k(\lambda - \lambda_0)^{-k} \\ &\quad + (z - \lambda_0)^n(\lambda - \lambda_0)^{-n}(\lambda - z)^{-1}\} \ . \end{aligned}$$

Integrate (7.8) over C with respect to the matrix measure

$$\text{mat } \{\varepsilon c_j d(E_0(\lambda)P\varphi_i, \varphi_j)\}$$

(see Remark (1) above). On the right, we obtain

$$\varepsilon Q_n(z) + \varepsilon(z - \lambda_0)^n D_n(z)$$

and on the left

$$(7.9) \qquad \text{mat}\left\{\varepsilon c_j \int_C (\lambda - z)^{-1}d(E_0(\lambda)P\varphi_i, \varphi_j)\right\} \ .$$

For $\text{Im } z > 0$, the integrand of (7.9) is analytic between C and the real axis, and we may therefore deform the contour C in (7.9) to the real axis. When this is done, we find that (7.9) is equal to $W_a(z, \varepsilon) - I$, which yields the result.

LEMMA 7.2. (a) *For sufficiently small* ε, *there exists in* N *a unique solution* $z = \zeta_0(\varepsilon)$ *of the equation*

$$(7.10) \qquad z = \lambda_0 + \varepsilon([I + \varepsilon Q_\nu(z)]^{-1}V\psi_0, \psi_0)$$

$\zeta_0(\varepsilon)$ *is analytic in* ε *at* $\varepsilon = 0$, *and* $\zeta_0(0) = \lambda_0$.

Moreover, (b) $\zeta_0(\varepsilon)$ *is real for real* ε *and*

(c) *coincides with the formal perturbation series up to and including terms of order* $\varepsilon^{\nu p+1}$.

Proof of parts (a) *and* (b). The equation (7.10) is sufficiently like (2.1) that part (a) can be proved just as Theorem 1. For part (b), observe that since mat $\{d(E_0(\lambda)P\varphi_i, \varphi_j)/d\lambda\}$ vanishes at λ_0 of order ν, the contour of integration in (7.5) can be deformed back to the real axis to obtain

$$Q^{(k)} = \text{mat}\left\{c_j \int_{-\infty}^{+\infty} (\lambda - \lambda_0)^{-k-1} d(E_0(\lambda)P\varphi_i, \varphi_j)\right\}$$

$$= \text{mat}\{c_j(S^{k+1}\varphi_i, \varphi_j)\}$$

where S is the *reduced resolvent* (cf. [1] and [8]), defined by

$$S = (T_0 P - \lambda_0 P)^{-1} \quad \text{on} \quad P\mathscr{H}$$
$$= 0 \quad\quad\quad\quad\quad \text{on} \quad P_0\mathscr{H}.$$

Now S is self-adjoint, and hence the matrix polynomial

$$S(z) = \sum_{k=0}^{n-1} \text{mat}\{(S^{k+1}\varphi_i, \varphi_j)\}z^k$$

is Hermitian for real z. It follows by a simple calculation that the matrix

$$[I + \varepsilon Q_\nu(z)]^{-1} V = [I + \varepsilon V S(z)]^{-1} V$$

is Hermitian, and hence that the right side of (7.10) is real for real z in N. However, it now follows by the Schwartz reflection principle that the complex conjugate of any solution of (7.10) is also a solution. Since $\zeta_0(\varepsilon)$ is unique, it is therefore real.

For the proof of (c), we shall require the following.

PROPOSITION. *Let* $F(z, \varepsilon)$ *and* $F_0(z, \varepsilon)$ *be analytic in* z *on the unit disc, for each fixed* ε, $0 \leqq \varepsilon \leqq \varepsilon_2$. *Let* dF/dz *be bounded on the unit disc uniformly in* ε, *and suppose that for some* $m \geqq 0$

$$F(z, \varepsilon) - F_0(z, \varepsilon) = O(\varepsilon z^m)$$

uniformly in ε. *Then if there exist unique solutions* $z(\varepsilon)$ *and* $z_0(\varepsilon)$ *of the equations* $z = \varepsilon F(z, \varepsilon)$ *and* $z_0 = (z_0, \varepsilon)$ *in the unit disc, we have*

(7.11) $$z(\varepsilon) - z_0(\varepsilon) = O(\varepsilon^2 z_0^m)$$

as $\varepsilon \to 0+$.

Proof. We have, uniformly in ε,

$$z - z_0 = \varepsilon F(z, \varepsilon) - \varepsilon F_0(z_0, \varepsilon)$$
$$= \varepsilon F(z, \varepsilon) - \varepsilon F(z_0, \varepsilon) + \mathrm{O}(\varepsilon^2 z_0^m)$$
$$= \varepsilon \mathrm{O}(z - z_0) + \mathrm{O}(\varepsilon^2 z_0^m)$$

where the final equality uses uniform boundedness of dF/dz. Hence

$$(z - z_0)(1 + \mathrm{O}(\varepsilon)) = \mathrm{O}(\varepsilon^2 z_0^m)$$

which implies (7.11) as $\varepsilon \to 0+$.

Proof of part (c). Consider first the case where λ_0 is an isolated eigenvalue of T_0, (see [8, Chap. VII]) so that $dE_0(\lambda)P/d\lambda = 0$ on a neighborhood of G. In this case, there is a perturbed eigenvalue $\zeta(\varepsilon)$ which satisfies the equation

(7.12) $$z = \lambda_0 + \varepsilon([I + \varepsilon K(z)]^{-1} V\psi_0, \psi_0)$$

where $K(z)$ is the matrix mat $\{c_j(R_0(z)P\varphi_i, \varphi_j)\}$. Now as an operator on $\mathscr{H}$, $R_0(z)P$ is analytic on a neighborhood of N, so that

$$VR_0(z)P = V \sum_{k=0}^{\infty} [R_0(z)P]^{k+1}(z - \lambda_0)^k$$

for z in N, from which it is immediate that

$$K(z) = Q_n(z) + \mathrm{O}((z - \lambda_0)^n)$$

for *every* $n \geq 0$. By the above proposition, taking λ_0 as the origin, we may compare the solutions of (7.10) and (7.12) for $n = \nu$ to obtain that

(7.13) $$\zeta(\varepsilon) - \zeta_0(\varepsilon) = \mathrm{O}(\varepsilon^{p\nu+2})$$

where ζ_0 corresponds to Q_ν, and p is given by (7.2).

Now the formal perturbation series may be defined as the series whose terms are formally the same as those of $\zeta(\varepsilon)$ in the isolated case, leaving aside all questions of convergence and existence of terms. Moreover, the coefficients of $\zeta_0(\varepsilon)$ obtained from (7.10) are formally the same in *all* cases, provided only that the terms of $Q_\nu(z)$ exist. *Hence, even in the present case, the equation* (7.13) *must hold in the formal sense*, which is all that is required. This rather devious proof avoids a direct calculation of the formal series, a task which sceptics are invited to perform.

Proof of Theorem 6. Applying the formula

$$(I + a + b)^{-1} = (I + a)^{-1} - (I + a)^{-1}b(I + a)^{-1} + \mathrm{O}(b^2)$$

to (7.4) with $n = \nu$, we obtain

$$
\begin{aligned}
W_a^{-1}(z, \varepsilon) &= [I + \varepsilon Q_\nu(z)]^{-1} \\
&\quad - [I + \varepsilon Q_\nu(z)]^{-1}\varepsilon(z - \lambda_0)^\nu D_\nu(z)[I + \varepsilon Q_\nu(z)]^{-1} \\
&\quad + O(\varepsilon^2(z - \lambda_0)^{2\nu}) \\
&= [I + \varepsilon Q_\nu(z)]^{-1} - \varepsilon(z - \lambda_0)^\nu D_\nu(z) + O(\varepsilon^2(z - \lambda_0)^\nu) \;.
\end{aligned}
$$

(7.14)

Define $\zeta_1(\varepsilon) = \zeta(\varepsilon) - \zeta_0(\varepsilon)$. Since $\zeta(\varepsilon)$ satisfies (2.1), we obtain from (7.14)

(7.15)
$$
\begin{aligned}
(\zeta_0(\varepsilon) - \lambda_0) + \zeta_1(\varepsilon) &= \varepsilon([I + \varepsilon Q_\nu(\zeta(\varepsilon))]^{-1} V\psi_0, \psi_0) \\
&\quad - \varepsilon^2(\zeta(\varepsilon) - \lambda_0)^\nu (D_\nu(\zeta(\varepsilon)) V\psi_0, \psi_0) + O(\varepsilon^3(\zeta(\varepsilon) - \lambda_0)^\nu) \;.
\end{aligned}
$$

Now $\zeta(\varepsilon)$ can be replaced by $\zeta_0(\varepsilon)$ in the first term on the right side of (7.15), to within a term which is $O(\varepsilon\zeta_1(\varepsilon))$. By definition of $\zeta_0(\varepsilon)$, the resulting term will exactly cancel $\zeta_0(\varepsilon) - \lambda_0$ on the left side, yielding

(7.16)
$$
\begin{aligned}
\zeta_1(\varepsilon) + O(\varepsilon\zeta_1(\varepsilon)) &= -\varepsilon^2(\zeta(\varepsilon) - \lambda_0)^\nu (D_\nu(\zeta(\varepsilon)) V\psi_0, \psi_0) \\
&\quad + O(\varepsilon^3(\zeta(\varepsilon) - \lambda_0)^\nu) \;.
\end{aligned}
$$

The second term on the left side may clearly be dropped as $\varepsilon \to 0+$. Observe now that if $\nu > 0$, then

$$
(\zeta(\varepsilon) - \lambda_0)^\nu = (\zeta_0(\varepsilon) - \lambda_0)^\nu + (\zeta_0(\varepsilon) - \lambda_0)^{\nu-1}O(\zeta_1(\varepsilon)) \;.
$$

If this expression is inserted into the two terms on the right side of (7.16), it is seen in a similar way that of the four resulting terms, the two involving $\zeta_1(\varepsilon)$ may both be dropped. We therefore obtain

(7.17) $\zeta_1(\varepsilon) = -\varepsilon^2(\zeta_0(\varepsilon) - \lambda_0)^\nu (D_\nu(\zeta(\varepsilon)) V\psi_0, \psi_0) + O(\varepsilon^3(\zeta_0(\varepsilon) - \lambda_0)^\nu)$

for $\nu > 0$; however, it is trivial from (7.16) that (7.17) also holds for $\nu = 0$.

Now, since $\zeta_0(\varepsilon)$ is real for real ε,

$$
\Gamma_\varepsilon = -\operatorname{Im}\zeta(\varepsilon) = -\operatorname{Im}\zeta_1(\varepsilon) \;.
$$

Taking the imaginary part of (7.17) and using (7.2), we have that

(7.18) $\Gamma_\varepsilon = \varepsilon^{\nu p+2}\lambda_p^\nu \operatorname{Im}(D_\nu(\zeta(\varepsilon)) V\psi_0, \psi_0) + O(\varepsilon^{\nu p+3})$.

Since $D_\nu(z)$ is analytic, $\zeta(\varepsilon)$ may be replaced by λ_0 on the right side of (7.18), with the remainder being absorbed in the second term. However, by (7.6), we have

$$
\operatorname{Im}(D_\nu(\lambda_0) V\psi_0, \psi_0) = \lim_{z \to \lambda_0 + i0} \int_{-\infty}^{+\infty} [\operatorname{Im}(\lambda - z)^{-1}](\lambda - \lambda_0)^{-\nu} d(E_0(\lambda) P V\psi_0, V\psi_0)
$$

which is just a Poisson integral, convergent to $\gamma(\lambda_0)$. (7.3) now follows from (7.18), and the remainder of part (a) follows from (7.3). For part (b), observe that by (7.17)

$$\zeta(\varepsilon) - \zeta_0(\varepsilon) = O(\varepsilon^{\nu p + 2})$$

and that, by Lemma 7.2, $\zeta_0(\varepsilon)$ differs from the formal series by terms which are $O(\varepsilon^{\nu p + 2})$.

8. **Examples.** We shall first consider the classical example of Friedrichs [4, §'s 6-8 and 10]. Let $\mathscr{J}$ be a closed interval and $\mathscr{H} = L_2(\mathscr{J}) \oplus C$ where C is the one-dimensional space of complex numbers. Let

$$T_0[u(\lambda), \xi] = [\lambda u(\lambda), \lambda_0 \xi]$$

where λ_0 is in the interior of $\mathscr{J}$. T_0 is self-adjoint with spectrum $\mathscr{J}$ and has absolutely continuous spectral measure, except for an eigenvalue λ_0 of simple multiplicity with eigenvector $\psi_0 = [0, 1]$. Let

$$V = (\cdot, f)\psi_0 + (\cdot, \psi_0)f$$

where $f = [f(\lambda), 0]$. We shall assume that (a) $f(\lambda)$ is Holder continuous on $(-\infty, +\infty)$ and has its support in $\mathscr{J}$, (b) $f(\lambda)$ does not vanish on the interior of $\mathscr{J}$ and (c)

$$\int_{\mathscr{J}} |f(\lambda)|^2 \, d\lambda = 1 \, .$$

V is self-adjoint of rank two, and $\mathscr{R}(V)$ has the orthonormal basis $\{f, \psi_0\}$. We shall compute matrices on $\mathscr{R}(V)$ with respect to this basis; the fact that V is not diagonal is of no importance.

Let

$$T_\varepsilon = T_0 + \varepsilon V \, .$$

For the matrix of $I + \varepsilon V R_0(z)$ on $\mathscr{R}(V)$ with respect to the basis $\{f, \psi_0\}$ we obtain

(8.1)
$$W(z, \varepsilon) = \begin{vmatrix} 1 & \varepsilon(\lambda_0 - z)^{-1} \\ \varepsilon F(z) & 1 \end{vmatrix}$$

where

$$F(z) = \int_{\mathscr{J}} |f(\lambda)|^2 (\lambda - z)^{-1} d\lambda \, .$$

The inverse matrix is

(8.2)
$$W^{-1}(z, \varepsilon) = [\lambda_0 - z - \varepsilon^2 F(z)]^{-1} \begin{vmatrix} (\lambda_0 - z) & -\varepsilon \\ -\varepsilon(\lambda_0 - z)F(z) & (\lambda_0 - z) \end{vmatrix} .$$

Since $f(\lambda)$ is Holder continuous, $F(z)$ is continuous up to the real axis and

(8.3)
$$\mathrm{Im}\, F(\lambda + i0) = \pi |f(\lambda)|^2 \, .$$

By assumption (b), this quantity does not vanish on the interior of $\mathscr{I}$. Hence, T_ε is absolutely continuous on the interior of $\mathscr{I}$, and the eigenvalue λ_0 disappears under the perturbation εV.

Now if $|f(\lambda)|^2$ is analytic near λ_0, then by Theorem 2 and (8.3), $F(z)$ will have a continuation $F_+(z)$. It is clear from (8.2) that the virtual poles are solutions of

$$(8.4) \qquad \lambda_0 - z - \varepsilon^2 F_+(z) = 0 \ .$$

In the following three examples, $\mathscr{I} = (-\infty, +\infty)$ and $\lambda_0 = 0$.

EXAMPLE 8.1. Let

$$(8.5) \qquad |f(\lambda)|^2 = (1/\pi)(1 + \lambda^2)^{-1} \ .$$

Then for $\mathrm{Im}\, z > 0$

$$F(z) = -(z + i)^{-1}$$

so that by continuation

$$F_+(z) = -(z + i)^{-1}$$

for all z. (8.4) therefore becomes

$$-z + \varepsilon^2(z + i)^{-1} = 0$$

which, by the quadratic formula, has one solution $-i + O(\varepsilon^2)$ and another

$$(8.6) \qquad z(\varepsilon) = -i\varepsilon^2 + O(\varepsilon^4) \ .$$

In order to apply (7.3), note that $\nu = 0$ and hence by (8.3) that

$$(8.7) \qquad \gamma(\lambda) = \pi d(E_0(\lambda)PV\psi_0, V\psi_0)/d\lambda = \pi\,|f(\lambda)|^2 \ .$$

By (8.5), this gives

$$\Gamma(\varepsilon) = \varepsilon^2 + O(\varepsilon^3)$$

which agrees with the actual value

$$\Gamma(\varepsilon) = \varepsilon^2 + O(\varepsilon^4)$$

obtained from (8.6).

EXAMPLE 8.2. If, instead of (8.5), we choose

$$(8.8) \qquad |f(\lambda)|^2 = (2/\pi)\lambda^2(1 + \lambda^2)^{-2} \ .$$

Then

$$F_+(z) = -z(z + i)^{-2}$$

and (8.4) becomes

$$-z + \varepsilon^2 z(z + i)^{-2} = 0$$

which has the exact solutions

$$z = 0 \quad \text{and} \quad z = -i \pm \varepsilon^2 .$$

Therefore, this perturbation does not remove the eigenvalue $\lambda_0 = 0$. Theorem 6 does not apply; for although the condition involving ν is satisfied, a computation shows that, in the basis $\{f, \psi_0\}$,

$$Q_\nu(z) = Q_2(z) = \begin{vmatrix} z & 0 \\ 0 & 0 \end{vmatrix}$$

from which it follows that $\zeta_0(\varepsilon) \equiv 0$.

EXAMPLE 8.3. In order to illustrate (7.3) when $\nu \neq 0$, we consider a slightly different perturbation. Let

$$V_1[u(\lambda), \xi] = [0, \lambda_1 \xi]$$

where λ_1 is real and nonzero, and define

$$T_\varepsilon = T_0 + \varepsilon V + \varepsilon V_1 .$$

Then $\mathscr{R}(V + V_1) = \mathscr{R}(V)$, and (8.1) acquires an additional term to become

$$(8.9) \qquad W(z, \varepsilon) = \begin{vmatrix} 1 & \varepsilon(\lambda_0 - z)^{-1} \\ \varepsilon F(z) & 1 + \varepsilon\lambda_1(\lambda_0 - z)^{-1} \end{vmatrix} .$$

The inverse is

$$W^{-1}(z, \varepsilon) = [\lambda_0 + \varepsilon\lambda_1 - z - \varepsilon^2 F(z)]^{-1} \begin{vmatrix} (\lambda_0 + \varepsilon\lambda_1 - z) & -\varepsilon \\ -\varepsilon(\lambda_0 - z)F(z) & (\lambda_0 - z) \end{vmatrix}$$

and the equation for virtual poles becomes

$$\lambda_0 + \varepsilon\lambda_1 - z + \varepsilon^2 F(z) = 0 .$$

If $|f(\lambda)|^2$ is again given by (8.8), this equation becomes

$$(-z + \varepsilon\lambda_1)(z + i)^2 + \varepsilon^2 z = 0 .$$

By successive approximations, the relevant root is

$$z(\varepsilon) = \varepsilon\lambda_1 - \varepsilon^3\lambda_1 - 2\varepsilon^4\lambda_1^2 i + O(\varepsilon^5)$$

so that

(8.10) $\Gamma(\varepsilon) = 2\lambda_1^2\varepsilon^4 + O(\varepsilon^5)$

by actual computation. To apply (7.3), note that $\nu = 2$ and $\zeta_0(\varepsilon) = \varepsilon\lambda_1 + O(\varepsilon^2)$ so that $p = 1$. Hence,

(8.11) $\Gamma(\varepsilon) = \varepsilon^4\lambda_1^2\gamma(0) + O(\varepsilon^5)$

where, by (8.7)

$$\gamma(\lambda) = \lambda^{-2}\pi \, |f(\lambda)|^2 = 2(1 + \lambda^2)^{-2} \, .$$

Thus $\gamma(0) = 2$, and (8.11) agrees with (8.10).

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