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THE ARENS PRODUCTS AND AN IMBEDDING THEOREM

JULIEN HENNEFELD

Let X be a separable Banach space, $B(X)$ be the algebra of all bounded linear operators on X, and $\mathscr{C}$ be the algebra of all compact linear operators. This paper centers around the general question of giving a construction of $B(X)$ as a Banach algebra starting from $\mathscr{C}$.

It is a result of Schatten and von Neumann that if H is a Hilbert space, then there is an isometric imbedding of $B(H)$ onto $\mathscr{C}^{**}$, where $\mathscr{C}^{**}$ denotes the second dual of $\mathscr{C}$. Moreover, each of the two Arens products on $\mathscr{C}^{**}$ coincides with the multiplication induced on $\mathscr{C}^{**}$ by operator multiplication on $B(H)$. The proofs of these results make strong use of the Hilbert space structure.

In this paper we generalize these results to a large class of uniformly convex spaces. Moreover, we show that even when $B(X)$ is not equal to $\mathscr{C}^{**}$ it is still possible to construct $B(X)$ as a Banach algebra starting from $\mathscr{C}$.

We now amplify the above statements. The theorem of Schatten and von Neumann is proved in [9, p. 48]. See Civin and Yood [2, p. 869] or Rickart [8, p. 289] for the result on the Arens products.

In § 2 we give basic definitions and elementary results concerning Banach space bases and linear operators. In § 3 we prove the existence of an isometric imbedding from $B(X)$ into $\mathscr{C}^{**}$, under the assumption that X has a shrinking, unconditionally monotone basis. Also, we show that under the same assumptions, a sufficient condition for the imbedding to be surjective is that X be uniformly convex. In § 4 we prove that the imbedding is surjective $\Longleftrightarrow$ the two Arens products on $\mathscr{C}^{**}$ coincide, and in that case they coincide with the multiplication on $\mathscr{C}^{**}$ induced by operator multiplication on $B(X)$. Finally, we show that for a certain class of Banach spaces, $B(X)$ is characterized as the largest subset of $\mathscr{C}^{**}$ in which $\mathscr{C}$ is a 2-sided ideal.

2. Preliminary definition and results.

DEFINITION 2.1. A basis (e_j) in a Banach space X is a sequence of elements of X, such that for each $x \in X$, there is a unique sequence of scalars (a_j) depending on x such that $\lim_{n\to\infty} \| \sum_{j=1}^{n} a_j e_j - x \| = 0$. The coefficient a_j is called the j^{th} coordinate of x. It is a theorem of Banach's that if you define e_i^* by $e_i^*(e_j) = \delta_{ij}$, then e_i^* is in X^*. A basis is called shrinking if (e_i^*) is a basis for X^*. A basis is called unconditional if for each $x \in X$, the series $\sum_{j=1}^{\infty} e_j^*(x)e_j$ is unconditionally convergent.

DEFINITION 2.2. If (e_j) is a basis for X, let $U_m x = \sum_{i \leqq m} e_i^*(x)e_i$. Then (e_j) is called a monotone basis if $\| U_m x \| \leqq \| x \|$ for all x in X and integers m.

DEFINITION 2.3. If (e_j) is an unconditional basis and D is a subset of the positive integers, let $x^D = \sum_{i=1, i \in D}^{\infty} e_i^*(x)e_i$. It is clear that x^D is convergent, since in a Banach space an unconditionally convergent series is also subseries convergent. Then (e_j) is called unconditionally monotone if $\| x^D \| \leqq \| x \|$ for all x in X and subsets $D \subset \omega$.

PROPOSITION 2.1. *If X is a Banach space with an unconditional basis (e_j), then X can be renormed isomorphically so that (e_j) is an unconditionally monotone basis.*

Proof. The norm $\| x \|' = \sup \{\| x^D \| : D$ is a finite subset of $\omega\}$ is isomorphic to the original norm, and has the property that every rearrangement of (e_j) is a monotone basis for X[4, p. 73]. Suppose that (e_j) is not unconditionally monotone with respect to the new norm. Then there exists a subset $S \subset \omega$ such that

$$\left\| \sum_{j=1}^{\infty} a_j e_j \right\|' < \left\| \sum_{j \in S}^{\infty} a_j e_j \right\|' .$$

Hence, for n large enough

$$\left\| \sum_{j \leqq n} a_j e_j \right\|' < \left\| \sum_{j \leqq n, j \in S} a_j e_j \right\|' .$$

But this contradicts the fact that if we rearrange the basis (e_j) so that we take first all the j in S and $\leqq n$, then it is a monotone basis.

Next we use a theorem of Maddaus to investigate $\mathscr{C}$, the space of compact operators and its dual.

NOTATION 2.1. E_{ij} will denote the elementary matrix with a one in the ij^{th} coordinate and zeros elsewhere.

DEFINITION 2.4. By a matrix concentrated in the j^{th} column (row), we will mean a matrix whose entries outside the j^{th} column (row), are all zero.

THEOREM 2.1. *Let X be a Banach space with a basis (e_j). For* *each compact operator A, let A_n be the operator whose matrix consists of the first n rows of A and zeros elsewhere. Then A is the uniform limit of the A_n.*

Proof. This is proved in Maddaus [6].

PROPOSITION 2.2. *Let X be a Banach space with a basis (e_k). Then for each fixed j, the set of matrices of $\mathscr{C}$ concentrated in the j^{th} row is linearly isometric as a Banach space to X^*.*

Proof. Let R be the matrix of an operator in $\mathscr{C}$ concentrated in the j^{th} row. Define $\alpha(e_k)=R_{jk}$ and extend α linearly to finite linear combinations of (e_k). Let $x = \sum_{k=1}^{n} b_k e_k$. Then $\alpha(x) = \sum_{k=1}^{n} b_k R_{jk}$ and $R(x) = (\sum_{k=1}^{n} b_k R_{jk})e_j$. Then since $|\alpha(x)| = \|R(x)\|$ for each such x, α can be extended to a functional $\alpha \in X^*$ and the map $R \mapsto \alpha$ is isometric. This map is surjective because given $\alpha \in X^*$, define the matrix R concentrated in the j^{th} row with $R_{jk} = \alpha(e_k)$.

PROPOSITION 2.3. *Let X be a Banach space with an unconditionally monotone basis (e_k). Then for each fixed j the set of matrices of $\mathscr{C}$ concentrated in the j^{th} column is linearly isometric as a Banach space to X.*

Proof. Let C_j be a matrix in $\mathscr{C}$ concentrated in the j^{th} column. Consider the map $C_j \mapsto C_j e_j$. Clearly $\|C_j e_j\| \leqq \|C_j\|$. For the other inequality, consider $x = b_j e_j + \sum_{i \neq j} b_i e_i$ with $\|x\| = 1$. Then by unconditional monotonicity $|b_j| \leqq 1$. Hence,

$$\|C_j x\| = \|C_j(b_j e_j)\| \leqq \|C_j e_j\| .$$

PROPOSITION 2.4. *Let X be a Banach space with a shrinking basis (e_j). Then, with each f in $\mathscr{C}^*$ we can associate a matrix so that $f = g \Longleftrightarrow$ their matrices coincide.*

Proof. First, we will show that the marices with a finite number of nonzero entries span a dense linear manifold of $\mathscr{C}$.

Given a compact operator A and $\varepsilon>0$, choose n so that $\|A-A_n\| < (\varepsilon/2)$, where A_n is the matrix consisting of the first n rows of A. Let R_j be the operator A_n followed by the canonical projection onto the 1-dimensional subspace spanned by $[e_j]$, for $j = 1, 2, \cdots, n$. The matrix for R_j is simply the j^{th} row of A_n and all other rows zero. Using the fact that the map in Proposition 2.2. is isometric and the hypothesis that (e_k) is a shrinking basis, it follows that each of the matrices R_j can be approximated to within $\varepsilon/2n$ by deleting (i.e., replacing by zeros) the tail of the j^{th} row. Therefore, by the triangle inequality A can be approximated to within ε by a finite matrix.

For f in $\mathscr{C}^*$ we can define the matrix (f_{ij}) by $f_{ij} = f(E_{ij})$. Then if f and g have the same matrices they are equal.

PROPOSITION 2.5. *Suppose X is a Banach space with an unconditionally monotone basis (e_j) and T is in $B(X)$. Then the matrix obtained by deleting (i.e., replacing by zeros) any set of rows or columns from T is in $B(X)$ and has norm $\leqq \| T \|$.*

Proof. Fix a subset $D \subset \omega$. Define $Px = \sum_{j \in D}^{\infty} e_j^*(x)e_j$. Then, $\| TP(x) \| \leqq \| T \| \| Px \| \leqq \| T \| \| x \|$. Thus, $\| TP \| \leqq \| T \|$. Also note that the matrix for TP is formed by deleting the j^{th} column from T for every $j \notin D$.

Similarly, $\| PT \| \leqq \| T \|$ and the matrix for PT is formed by deleting the j^{th} row from T for every $j \notin D$.

PROPOSITION 2.6. *Suppose X is a Banach space with an unconditionally monotone, shrinking basis (e_j), and that f is in $\mathscr{C}^*$. Then the matrix obtained by deleting any set of rows or columns from the associated matrix for f, is the matrix associated with a functional in $\mathscr{C}^*$ with norm $\leqq \| f \|$.*

Proof. Fix a subset $D \subset \omega$. Let $d: \mathscr{C} \rightarrow \mathscr{C}$ be the linear transformation which deletes the j^{th} column for each $j \notin D$. Then its adjoint d^* has norm 1. Note that $(d^*f)A = f(dA)$. Hence, the matrix for d^*f is formed by deleting every j^{th} column for $j \notin D$.

The argument for deleting rows is similar.

PROPOSITION 2.7. *Let X be a Banach space with an unconditionally monotone, shrinking basis.*

(1) *For each fixed j, the set of matrices in $\mathscr{C}^*$ which are concentrated in the j^{th} row is linearly isometric as a Banach space to X^{**}.*

(2) *For each fixed j, the set of matrices in $\mathscr{C}^*$ which are concentrated in the j^{th} column is linearly isometric to X^*.*

Proof. (1) Let $f_j \in \mathscr{C}^*$ be concentrated in the j^{th} row. Define $\phi(e_k^*) = f_{jk}$. Extend ϕ linearly to finite linear combinations of (e_k^*). It follows from Proposition 2.2 that ϕ can be extended to a functional in X^{**}. Moreover, $\| \phi \| = \| f_j \|$ since f_j approaches its norm on compact operators of norm one, concentrated in the j^{th} row. The map $f_j \mapsto \phi$ is surjective because given $\phi \in X^{**}$, the matrix whose j^{th} row is given by $f_{jk} = (e_k^*)$ and whose other rows are zero is in $\mathscr{C}^*$.

(2) The proof is similar.

3. An imbedding theorem. We are now ready to give an isometric imbedding of $B(X)$ into $\mathscr{C}^{**}$.

THEOREM 3.1. *If (e_j) is an uncondionally monotone, shrinking basis for the Banach space X, then there is a linear isometric map from $B(X) \to \mathscr{C}^{**}$ such that each A in $\mathscr{C}$ is taken onto its usual image under the evaluation map of $\mathscr{C} \to \mathscr{C}^{**}$.*

Proof. Given T in $B(X)$ let R_j be the matrix consisting of the j^{th} row of T with zeros elsewhere. Define Φ_T in $\mathscr{C}^{**}$ by $\Phi_T(f) = \sum_{j=1}^{\infty} f(R_j)$, where f is in $\mathscr{C}^*$ and $\|f\| = 1$. We must show that the series $\sum_{j=1}^{\infty} f(R_j)$ is convergent. By Proposition 2.5.

$$|f(R_{j_1} + \cdots + R_{j_n})| \leqq \|T\|$$

for an arbitrary set of integers $\{j_1, \cdots, j_n\}$, since the left side represents f applied to a compact operator formed by deleting rows from T. It is clear then that the series $\sum_{j=1}^{\infty} f(R_j)$ is unconditionally convergent.

The map $T \mapsto \Phi_T$ is obviously linear, since matrix addition and taking limits are linear operations.

$$|\Phi_T(f)| = \left|\sum_{j=1}^{\infty} f(R_j)\right| = \lim_{n\to\infty}\left|f\left(\sum_{j=1}^{n} R_j\right)\right| \leqq \|f\|\,\|T\|\,,$$

since $\sum_{j=1}^{n} R_j$ is a compact operator of norm $\leqq \|T\|$. Hence, Φ_T is bounded and $\|\Phi_T\| \leqq \|T\|$. To prove the reverse, first, we note that $\|\sum_{j=1}^{n} R_j\|$ approaches $\|T\|$ as n approaches ∞. Then, given $\varepsilon > 0$, take $\|\sum_{j=1}^{n} R_j\| > \|T\| - \varepsilon$. Since $\sum_{j=1}^{n} R_j$ is compact, we can find by the Hahn Banach theorem a g in $\mathscr{C}^*$ of norm 1, such that

$$g\left(\sum_{j=1}^{n} R_j\right) > \|T\| - \varepsilon\,.$$

Then let g^D be the matrix formed by deleting the columns of g past the n^{th}. By Proposition 2.6., $\|g^D\| \leqq 1$, and we have that $\Phi_T(g^D) > \|T\| - \varepsilon$. Hence, $\|\Phi_T\| \geqq \|T\|$ and the imbedding is isometric.

Then as we noted in Proposition 2.4., the finite matrices form a dense manifold of $\mathscr{C}$. It is clear that Φ and the evaluation map agree on all finite matrices in $\mathscr{C}$ and hence on all of $\mathscr{C}$.

PROPOSITION 3.1. *Let X be a Banach space with an unconditionally monotone, shrinking basis. Then $B(X) = \mathscr{C}^{**}$ under the previous imbedding $\langle = \rangle$ the set of finite matrices in $\mathscr{C}^*$ is a dense* *linear manifold. Moreover, in that case X is reflexive.*

Proof. If the set of finite matrices is not dense in $\mathscr{C}^*$, then there exists a nonzero F in $\mathscr{C}^{**}$, which is 0 on all finite matrices. However no Φ_T for nonzero T in $B(X)$ can have this property, since if T has the entry $T_{ij} \neq 0$, then $\Phi_T(f_{ij}) = T_{ij}$ where f_{ij} is an elementary matrix in $\mathscr{C}^*$.

Assume the finite matrices are dense in $\mathscr{C}^*$. Let π be an arbitrary functional in X^{**}. Then by Proposition 2.7., π can be identified with an $f \in \mathscr{C}^*$ which is concentrated in the j^{th} row. Since the finite matrices are dense in $\mathscr{C}^*$, $\sum_{k=1}^{\infty} f_{jk}\hat{e}_k$ converges in norm to π and hence X is reflexive.

Given $F \in \mathscr{C}^{**}$, define the matrix (F_{ij}) by $F_{ij} = F(f_{ij})$. F is determined by this associated matrix. By reflexivity and Proposition 2.7., it follows that each column of F represents an element of X with respect to (e_j). Then let T_n be the matrix consisting of the first n columns of F. It is the matrix of a compact operator. Furthermore $\Phi_{T_n}(f) = F(f^D)$ for each $f \in \mathscr{C}^*$, where f^D is the matrix formed from f by deleting all the columns past n^{th}. Hence, $\| T_n \| = \| \Phi_{T_n} \| \leqq \| F \|$. Define the operator T by $T(\sum_{j=1}^{n} a_j e_j) = T_n(\sum_{j=1}^{n} a_j e_j)$. T is well defined on the set of all finite linear combinations of the (e_j), and has norm $\leqq \| F \|$. Hence, it can be extended uniquely to a bounded operator on all of X. It is clear that $F = \Phi_T$, since F and Φ_T agree on all finite matrices in $\mathscr{C}^*$.

The next proposition puts Proposition 3.2. into a more workable form for applications.

PROPOSITION 3.2. *Let X be a Banach space with an unconditionally monotone shrinking basis (e_j). Then, $B(X) = \mathscr{C}^{**} \Longleftrightarrow$ for each f in $\mathscr{C}^*$, $\| f^N \| \to 0$, where f^N is the matrix formed from f by deleting the first N rows and N columns.*

Proof. We will show that the condition on the right is satisfied $\Longleftrightarrow$ the set of finite matrices in $\mathscr{C}^*$ span a dense manifold.

Suppose that the finite matrices are norm dense in $\mathscr{C}^*$. Given $\varepsilon > 0$ and $f \in \mathscr{C}^*$ there exists a finite g such that $\| f - g \| < \varepsilon$. Then since g is finite we can pick N large enough so that $f^N = (f - g)^N$. By Proposition 2.6. $\| (f - g)^N \| \leqq \| f - g \| < \varepsilon$.

Conversely, suppose $\| f^N \| \to 0$. Given $\varepsilon > 0$ choose N large enough: $\| f^N \| = \| f - (f - f^N) \| < \varepsilon/2$. The matrix for $f - f^N$ is not finite, but can be approximated to within $\varepsilon/2$ by a finite matrix.

The next proposition shows that if $B(X) \neq \mathscr{C}^{**}$, then the Banach space X behaves very much like (c_0), the space of sequences which converge to 0.

PROPOSITION 3.3. *Let X be a Banach space with an unconditionally monotone shrinking basis (e_j). If $B(X) \neq \mathscr{C}^{**}$, then for every $\varepsilon > 0$, and integer n, we can find an x of norm 1, such that $x = x_1 + \cdots + x_n$, where each x_i is a finite linear combination of distinct sets of basis vectors and $\| x_i \| \geqq 1 - \varepsilon$.*

Proof. By the previous proposition there exists an f in $\mathscr{C}^*$ such that $\| f^N \|$ does not approach 0. The f^N decrease in norm, since f^{N+1} is formed by deleting a row and a column from f^N. We can assume without loss of generality that $\| f^N \| \rightarrow 1$ and never achieve it as $N \rightarrow \infty$. Then, given $\lambda > 0$, there exists an integer N_1: $\| f^{N_1} \| < 1 + \lambda$. Since the finite operators are dense in the compact operators there exists an integer $N_1' > N_1$, and a finite operator T_1 of norm 1: T_1 is concentrated on the manifold X_1 spanned by $[e_{N_1}, \cdots, e_{N_1'}]$ and $f^{N_1}(T_1) > 1$. Let $N_2 = N_1' + 1$. For f^{N_2} there exists a finite operator T_2 of norm 1, concentrated on the manifold $X_2 = [e_{N_2}, \cdots, e_{N_2'}]$: $f^{N_2}(T_2) > 1$. Repeating this process n times, we can construct $T_1, \cdots, T_n$ such that $f^{N_k}(T_k) > 1$, and the T_k are concentrated on disjoint basic blocks of X. Hence

$$\begin{aligned} n < f^{N_1}(T_1) + \cdots + f^{N_n}(T_n) = f^{N_1}(T_1 + \cdots + T_n) \\ \leqq \| f^{N_1} \| \, \| T_1 + \cdots + T_n \| , \end{aligned}$$

and $n/1 + \lambda < \| T_1 + \cdots + T_n \|$. This means that there exists an x of norm 1, where $x = x_1 + \cdots + x_n$, each x_i is in X_i, and such that

$$\frac{n}{1 + \lambda} < \| (T_1 + \cdots + T_n)x \| \leqq \| T_1 x_1 \| + \cdots + \| T_n x_n \| .$$

However, $\lambda > 0$ was arbitrary. By picking $\lambda > 0$ small enough, we can find $T_1, \cdots, T_n$: the sum $\| T_1 x_1 \| + \cdots + \| T_n x_n \|$ is as close to n as we wish. By unconditional monotonicity, each $\| x_i \| \leqq 1$. Thus, $\| T_i x_i \| \leqq 1$. Hence, each $\| T_i x_i \|$ and $\| x_i \|$ will be close to 1.

LEMMA 3.1. *A uniformly convex Banach space is reflexive.*

Proof. See Wilansky [10, p. 109].

LEMMA 3.2. *If X is a reflexive Banach space with a basis, then the basis is shrinking.*

Proof. See [10, p. 213].

THEOREM 3.2. *If X has an unconditionally monotone basis (e_j)*

and X is isomorphic to a uniformly convex Banach space Z, then $B(X) = \mathscr{C}^{**}$.

Proof. For each x in X call its norm $||x||$, and for its image in Z call its norm $|x|$. Uniform convexity means that for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if x, x' are in the unit ball of Z, and $|x - x'| > \varepsilon$, then $|x + x'|/2 \leqq 1 - \delta(\varepsilon)$. Clearly, if we renorm Z by multiplying the old norm by some constant, the renormed Z will still be uniformly convex. Hence, we may assume without loss of generality that there exists a constant M: $||x|| \leqq |x| \leqq M||x||$. Let $t = \delta(1/2M)$. Choose r large enough so that, $(1/1 - t)^r(1/2M) > 1$. Suppose $B(X) \neq \mathscr{C}^{**}$. By Proposition 3.3. there exists an x of norm 1, such that $x = x_1 + \cdots + x_{2^r}$, where each $||x_i|| \geqq 1/2$ and where each x_i is a linear combination of distinct (e_j). We want to construct an element v: $||v|| > 1$ and $|v| \leqq 1$. This will contradict the fact that $||v|| \leqq |v|$.

Consider the following system of elements like the seeding chart of a tennis tournament. In the first round put the elements $w_1, \cdots, w_{2^r}$ where $w_k = (x_1 + \cdots + x_k)/M$ and x_i as above. Then we construct the second round consisting of 2^{r-1} elements by letting the n^{th} element of the second round be $u_n = (w_{2n-1} + w_{2n})/2(1 - t)$. To form the n^{th} element y_n of the third round, let

$$y_n = \frac{1}{2(1-t)}(u_{2n-1} + u_{2n}) .$$

The elements for the other rounds are formed in the same manner.

We claim that every element in this system lies in the unit ball of Z. For the first round, each w_k is in the unit ball of Z, because $||w_k|| \leqq 1/M$ by unconditional monotonicity. We can assume that two paired elements u and u' from the n^{th} round are in the unit ball of Z. Note that there exists an x_k: $u' = (1/M(1 - t)^{n-1})x_k +$ other terms not involving x_k, whereas u does not involve any of the (e_i) used in expressing (x_k). By unconditional monotonicity

$$||u - u'|| \geqq \frac{1}{M}||x_k|| \geqq \frac{1}{2M} .$$

Hence,

$$|u - u'| \geqq \frac{1}{2M} \quad \text{and} \quad \left|\frac{1}{2(1-t)}(u + u')\right| \leqq 1 .$$

Thus an arbitrary element of the $(n+1)^{st}$ round is in the unit ball of Z. Let v be the element in the r^{th} round. Then, $v = \{1/(1-t)^r M\}x_1 +$ other terms not involving x_1. Hence $||v|| > 1$. This is impossible since $|v| \leqq 1$.

COROLLARY 3.1. *If X is isomorphic to a uniformly convex space and has an unconditional basis, then $B(X)$ is isomorphic to $\mathscr{C}^{**}$.*

Proof. Renorm X so that the basis is unconditionally monotone.

EXAMPLE 3.1. The canonical basis for l^p for $1 < p < \infty$ is unconditionally monotone and l^p is uniformly convex, see Clarkson [3]. $L^p[0, 1]$ for $1 < p < \infty$, has an unconditional basis and is uniformly convex. See Pelczynski [7].

4. The Arens products. The two Arens products are defined in stages according to the following rules. Let $\mathscr{A}$ be a Banach algebra. Let $A, B \in \mathscr{A}$; $f \in \mathscr{A}^*$; $F, G\varepsilon^{**}$.

DEFINITION 4.1.
$(f{}_1^*A)B = f(AB)$. This defines $f{}_1^*A$ as an element of $\mathscr{A}^*$.
$(G{}_1^*f)A = G(f{}_1^*A)$. This defines $G{}_1^*f$ as an element of $\mathscr{A}^*$.
$(F{}_1^*G)f = F(G{}_1^*f)$. This defines $F{}_1^*G$ as an element of $\mathscr{A}^{**}$.
We will call $F{}_1^*G$ the first Arens product, or the m_1 product.

DEFINITION 4.2.
$(A{}_2^*f)B = f(BA)$. This defines $A{}_2^*f$ as an element of $\mathscr{A}^*$.
$(f{}_2^*F)A = F(A{}_2^*f)$. This defines $f{}_2^*F$ as an element of $\mathscr{A}^*$.
$(F{}_2^*G)f = G(f{}_2^*F)$. This defines $F{}_2^*G$ as an element of $\mathscr{A}^{**}$.
$F{}_2^*G$ is the second Arens product or the m_2 product.

It is proved in Arens [1] that m_1 and m_2 are both Banach algebra products on $\mathscr{A}^{**}$, which extend the original multiplication on $\mathscr{A}$ when it is imbedded in $\mathscr{A}^{**}$.

DEFINITION 4.3. A Banach algebra $\mathscr{A}$ is called Arens regular if the two Arens products coincide on $\mathscr{A}^{**}$.

DEFINITION 4.4. Let E_α be a net of elements in the unit ball of $\mathscr{A}$. Then E_α is a weak identity if for every $A \in \mathscr{A}, f \in \mathscr{A}^*$, both $f(E_\alpha A) \to f(A)$ and $f(AE_\alpha) \to f(A)$.

LEMMA 4.1. *If $\mathscr{A}$ has a weak identity E_α, then there exists an element $I \in \mathscr{A}^{**}$, which is simultaneously* (1) *a right identity for m_1* (2) *a left identity for m_2. Call such an element I a simultaneous identity.*

Proof. (1) is proved in [2, p. 855]. The proof of (2) is similar. A subnet of the $\{E_\alpha\}$ converges to I in the weak star topology.

DEFINITION 4.5. Let X be a normed space. Then, $f_\alpha \to f$ in the bounded weak star topology $\langle = \rangle$ the $\{f_\alpha\}$ consititute a bounded set and $f_\alpha \to f$ in the weak star topology.

LEMMA 4.2. *$\mathscr{A}$ is Arens regular $\langle = \rangle$ there is a multiplication m_3 on $\mathscr{A}^{**}$ which extends the multiplication on $\mathscr{A}$ to $\mathscr{A}^{**}$ in a way such that* (1) *F_3^*G is weak star bounded continuous in F for each fixed G and* (2) *F_3^*G is weak star bounded continuous in G for each fixed F.*

Proof. Arens [1, p. 843].

THEOREM 4.1. *If X is a Banach space with an unconditionally monotone, shrinking basis (e_j), then $B(X) = \mathscr{C}^{**}$ $\langle = \rangle$ $\mathscr{C}$ is Arens regular.*

Proof. Assume $B(X) = \mathscr{C}^{**}$. We claim that ordinary matrix multiplication satisfies (1) and (2) of the above lemma. Let S_α, S, and T all be in the unit ball of $B(X)$ and $S_\alpha \to S$ weak star. Let f_{ij} be the matrix in $\mathscr{C}^*$ with a 1 in the ij^{th} coordinate and zeros elsewhere. First, we claim that $(S_\alpha T)f_{ij} \to (ST)f_{ij}$. Clearly, only the i^{th} rows of S_α and S and the j^{th} column of T are relevant. By Proposition 2.3. given $\varepsilon > 0$, there exists an integer n such that the tail of the j^{th} column of T after the first n terms has norm $< \varepsilon/2$.

Since $S_\alpha \to S$ weak star, it is clear that S_α approaches S coordinatewise. Let α be large enough so that each of the first n entries of the i^{th} row of S are within $\varepsilon/2n$ of the corresponding entry of S. Then $|(S_\alpha T)f_{ij} - (ST)f_{ij}| \leqq \varepsilon$. Hence, $(S_\alpha T)f_{ij} \to (ST)f_{ij}$. Since $B(X) = \mathscr{C}^{**}$ implies that the finite matrices are norm dense in $\mathscr{C}^*$, it follows that for arbitrary $g \in \mathscr{C}^*$, $(S_\alpha T)g \to (ST)g$. The argument that (2) is satisfied is similar.

Now assume $B(X) \neq \mathscr{C}^{**}$. Then the finite matrices do not span a dense manifold of $\mathscr{C}^*$. Hence, there exists a nonzero F in $\mathscr{C}^{**}$ which is 0 on all finite matrices. Let E_n be the matrix in $\mathscr{C}$ with ones down the first n entires of the diagonal and zeros elsewhere. Then, (E_n) is a weak identity since it is actually an approximate identity by the fact that finite matrices are norm dense in $\mathscr{C}$.

Let I be the simultaneous identity in Lemma 4.1., and $f \in \mathscr{C}^*$. By Theorem 3.2. [1]

$$\begin{aligned}(F_2^*I)f &= \lim\,[(F_2^*E_n)f] = \lim\,[E_n(f_2^*F)] \\ &= \lim\,[(f_2^*F)E_n] = \lim\,[F(E_{n2}^*f)]\,.\end{aligned}$$

However, $E_{n2}^{*}f$ is the matrix in $\mathscr{C}^*$ which consists of the first n columns of f, and thus can be approximated in norm by a finite matrix, since the basis is shrinking. Hence $(F_2^*I) = 0$ whereas $F_1^*I = F$.

LEMMA 4.3. *If there is a continuous homomorphism of the Banach algebra $\mathscr{A}_1$, onto the Banach algebra $\mathscr{A}_2$, and if the multiplication in $\mathscr{A}_1$ is regular, then so is the multiplication in $\mathscr{A}_2$.*

Proof. Civin and Yood [2], Corollary 6.4.

COROLLARY 4.1. *If X is a Banach space with an unconditional basis (e_j), and which is isomorphic to a uniformly convex space, then its space of compact operators is Arens regular.*

Proof. By Proposition 2.1., X can be renormed isomorphically to X' so that (e_j) is an unconditionally monotone basis for X'. Let i be an isomorphic map from X to X'. Then the map $A \mapsto i^{-1}Ai$, where $A \in \mathscr{C}'$, is a continuous homomorphism from $\mathscr{C}'$ onto $\mathscr{C}$.

THEOREM 4.2. *Let X be a Banach space with an unconditionally monotone, shrinking basis, and for which the matrices in $\mathscr{C}^*$ with a finite number of rows are norm dense. Then $B(X) = \{F \in \mathscr{C}^{**}$: F_1^*A and A_1^*F are both in $\mathscr{C}$ for all $A \in \mathscr{C}\}$. Furthermore, each of the Arens products coincides with operator multiplication on $B(X)$.*

Proof. Let F be in $\mathscr{C}^{**}$. Let D_j denote the elementary matrix E_{jj}. Call D_{j1}^*F the j^{th} row of F. Note that D_{j1}^*F is concentrated on the j^{th} row of matrices in $\mathscr{C}^*$. In fact,

$$(D_{j1}^*F)f = D_j(F_1^*f) = (F_1^*f)D_j = F(f_1^*D_j) .$$

But the matrix for $f_1^*D_j$ is easily seen to be the matrix formed from f by deleting all but the j^{th} row. By Proposition 2.7., the j^{th} row of F can be identified with a functional in X^{***}.

Call $F_1^*D_j$ the j^{th} column of F. It is concentrated on the j^{th} column of matrices in $\mathscr{C}^*$, because D_{j1}^*f is the matrix formed by deleting all but the j^{th} column of f. Then by Proposition 2.7. it can be identified with an element of X^{**}.

We claim $F \in B(X) \iff$ each of its rows is in X^* and each of its columns is in X. Suppose $F \in \mathscr{C}^{**}$ with each of its rows in X^* and columns in X. Let T be the actual matrix formed by writing down the columns of F as elements in X with respect to the basis (e_j). Let T_n be the first n columns of T. It is a compact operator since each column is in X. Also by Proposition 2.6.

$$\|T_n\| = \|\Phi_{T_n}\| \leqq \|F\|$$

where Φ is the isometry defined in Theorem 3.1. Hence, the $\{T_n\}$ define a single bounded operator on the dense linear manifold of finite linear combinations of (e_j). This bounded operator has the same matrix as T.

Clearly Φ_T and F agree on any elementary matrix in $\mathscr{C}^*$. Hence they agree on any matrix in $\mathscr{C}^*$ concentrated in a single row, since each row of F is in X^* and the (e_j^*) form a basis for X^*. Then by the hypothesis that the matrices in $\mathscr{C}^*$ with a finite number of rows are dense, $\Phi_T = F$.

Conversely, if $F \in B(X)$ it is clear that its generalized rows and columns will be in X^* and X respectively.

Using this characterization of $B(X)$ as a subspace of $\mathscr{C}^{**}$, it is clear that if $F \notin B(X)$, then for some j either $D_{j1}^* F$ or $F_1^* D_j$ lies outside $B(X)$ and hence outside $\mathscr{C}$. But D_j is a compact operator.

To finish the proof we will show that on $B(X)$, m_1 is equal to operator multiplication. The proof for m_2 is similar.

Clearly it is enough to show that $(ST)f_j = (S_1^* T)f_j$ for f_j a matrix in $\mathscr{C}^*$ concentrated in the j^{th} row and where $\|S\| = \|T\| = \|f_j\| = 1$. Given $\varepsilon > 0$, we can approximate the j^{th} row of S in norm to within ε by deleting after the first n terms for n large enough.

Then

$$\begin{aligned}(ST)f_j &= (S_{j1}T_{11} + S_{j2}T_{21} + \cdots + S_{jn}T_{n1})f_{j1} \\ &\vdots \\ &+ (S_{j1}T_{1k} + S_{j2}T_{2k} + \cdots + S_{jn}T_{nk})f_{jk} \\ &\vdots \\ &+ (\text{error term } < \varepsilon)\ .\end{aligned}$$

We claim that $(T_1^* f_j)$ is concentrated in the j^{th} row. In fact,

$$(T_1^* f_j)E_{mk} = T(f_{j1}^* E_{mk}) = 0 \text{ if } m \neq j\ ,$$

whereas $(T_1^* f_j)E_{jk}$ = dot product of k^{th} row of T with j^{th} row of f_j.

Then,

$$\begin{aligned}S(T_1^* f_j) &= (T_{11}f_{j1} + T_{12}f_{j2} + \cdots +)S_{j1} \\ &\vdots \\ &+ (T_{n1}f_{j1} + T_{n2}f_{j2} + \cdots +)S_{jn} \\ &+ (\text{error term } < \varepsilon)\ .\end{aligned}$$

Hence $|(ST)f_j - (S_1^* T)f_j| < 2\varepsilon$, since for a finite collection of convergent series

$$\sum_{k=1}^{\infty} (a_k^1 + \cdots + a_k^n) = \sum_{k=1}^{\infty} a_k^1 + \cdots + \sum_{k=1}^{\infty} a_k^n .$$

DEFINITION 4.6. A shrinking basis (e_j) for a Banach space is called boundedly growing if there exists an $\varepsilon > 0$ and an integer n, such that $x_1 + \cdots + x_n < n - \varepsilon$ whenever the x_i's have norm 1 and are linear combinations of distinct basic vectors. For example the canonical bases for c_0 or $l^p, p > 1$ are boundedly growing. Finite direct sums of boundedly growing Banach spaces are boundedly growing. Also $l^p(X_i)$ for $p > 1$ is boundedly growing if the X_i have a common n and ε.

COROLLARY 4.2. *If a Banach space X has an unconditionally monotone, boundedly growing basis then $B(X)$ is the largest subset in $\mathscr{C}^{**}$ in which $\mathscr{C}$ is a two sided ideal.*

Proof. In proving Proposition 3.3. we showed that if the finite matrices are not dense in $\mathscr{C}^*$ then the basis is not boundedly growing. Similarly, if the matrices with a finite number of rows are not dense in $\mathscr{C}^*$, then the basis is not boundedly growing.

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