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ON AN EMBEDDING PROPERTY OF GENERALIZED CARTER SUBGROUPS

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If $\mathscr{E}$ and $\mathscr{F}$ are saturated formations, $\mathscr{E}$ is strongly contained in $\mathscr{F}$ (written $\mathscr{E} \ll \mathscr{F}$) if:

(1.1) For any solvable group G with $\mathscr{E}$-subgroup E, and $\mathscr{F}$-subgroup F, some conjugate of E is contained in F.

This paper is concerned with the problem:

(1.2) Given $\mathscr{E}$, what saturated formations $\mathscr{F}$ satisfy $\mathscr{E} \ll \mathscr{F}$?

The object of this paper is to prove two theorems. The first, Theorem 5.3, shows that if $\mathscr{T}$ is a nonempty formation, and $\mathscr{E} = \{G \mid G/F(G) \in \mathscr{T}\}$. ($F(G)$ is the Fitting subgroup of G), then any formation $\mathscr{F}$ which strongly contains $\mathscr{E}$ has essentially the same structure as $\mathscr{E}$ in that there is a nonempty formation $\mathscr{U}$ such that $\mathscr{F} = \{G \mid G/F(G) \in \mathscr{U}\}$. The second, Theorem 5.8, exhibits a large class of formations which are maximal in the partial ordering $\ll$. In particular, if $\mathscr{N}^i$ denotes the formation of groups which have nilpotent length at most i, then $\mathscr{N}^i$ is maximal in $\ll$. Since for $\mathscr{N} = \mathscr{N}^1$, the $\mathscr{N}$-subgroups of a solvable group G are the Carter subgroups, question (1.2) is settled for the Carter subgroups.

Since the theory of formations is of relatively recent origin, we give a few highlights. The theory begins with a paper [4] by Gaschütz which provides the setting in which the results of Carter [1] on the existence of nilpotent self-normalizing subgroups of solvable groups take their most natural form. He showed that given a saturated formation $\mathscr{F}$, and any finite solvable group G, one can find a conjugacy class of subgroups of G (called $\mathscr{F}$-subgroups of G) which is connected in a natural way with the formation $\mathscr{F}$. Recently, Carter and Hawkes [2] have made a major contribution to the theory by generalizing the work of Philip Hall on system normalizers in solvable groups to $\mathscr{F}$-normalizers, and investigating the relationships between the $\mathscr{F}$-subgroups of a solvable group G and the $\mathscr{F}$-normalizers of G. As is clear from (1.1), this paper proceeds in a different direction by considering the relative embedding of the $\mathscr{F}$-subgroups for two distinct saturated formations $\mathscr{E}$, $\mathscr{F}$. We consider only finite solvable groups in this paper.

The machinery used in the proof of our main theorem, Theorem 5.8, is developed in § 4. We begin by obtaining a characterization of strong containment which depends only on the two formations $\mathscr{E}$ and $\mathscr{F}$. This characterization depends on the knowledge that if $\mathscr{F}$ is a saturated formation, then $\mathscr{F}$ is a locally defined formation (see

§ 2), a result proved by Lubeseder in [7]. In certain cases, we are able to strengthen our characterization of strong containment so that it gives a complete description of the minimal local definition of $\mathscr{F}$ as a necessary condition for strong containment.

In § 6, we present an example which shows that Hypothesis II of our main theorem is not redundant. The formation which gives the example is $\mathscr{R} = \{G \mid G/F(G) \text{ is an } r'\text{-group}\}$, where r is a prime. It is apparent from Theorem 6.2 that $\mathscr{R}$ is not maximal in the partial ordering $\ll$. In fact, there are an infinite number of formations which strongly contain $\mathscr{R}$.

Preliminary results are presented in § 3. In particular, we give a cover-avoidance characterization of the $\mathscr{F}$-subgroups of a group, a result which may have some interest by itself. We remark, however, that one half of this characterization has appeared in [2].

2. Notation and quoted results.

We use the following notation:

G — a finite solvable group;
$D(G)$ — the Frattini subgroup of G, the intersection of all maximal subgroups of G;
$F(G)$ — the Fitting subgroup of G, the maximal normal nilpotent subgroup of G;
Z_p — the field of integers mod p, p a prime;
π — a set of primes;
π' — the complementary set of primes;
$O_\pi(G)$ — the maximal normal π-subgroup of G;
$O_{\pi'\pi}(G)$— the inverse image in G of $O_\pi(G/O_{\pi'}(G))$.

If $K \lhd H \leqq G$, then H/K is a *section of* G, and if $F \leqq G$ normalizes both H and K, it is an *F-invariant* section of G. If H/K is an F-invariant section of G, then $C_F(H/K)$ is the kernel of the representation of F as a subgroup of the automorphism group of H/K. $C_{H/K}(F)$ is the set of elements of H/K fixed by every element of F. The following results will be used frequently:

LEMMA 2.1. (*Covering Lemma* [6], *Theorem* 1) *If A is a group of automorphisms of the group G whose order is prime to the order of G, and if H/K is an A-invariant section of G, then $C_G(A)$ covers $C_{H/K}(A)$.*

LEMMA 2.2.[1] (*Frobenius reciprocity for modules*, [8], p. 144)

[1] The result on page 144 of [8] does not look quite like the Frobenius reciprocity theorem quoted above, but if we define the map

$$\chi: \ \mathrm{Hom}_{\mathfrak{R}(G)}(\mathfrak{R}(G), N) \to N \text{ by the rule}$$
$$\chi: \ \varphi \to \varphi(1) \qquad \varphi \in \mathrm{Hom}_{\mathfrak{R}(G)}(\mathfrak{R}(G), N) ,$$

then it is not difficult to show that χ is a $\mathfrak{R}(H)$-isomorphism from $\mathrm{Hom}_{\mathfrak{R}(G)}(\mathfrak{R}(G),N)$ onto $N|_H$.

Let G be a group, $H \leqq G$, and $\mathfrak{R}$ a field. If M is a $\mathfrak{R}(H)$-module, and N a $\mathfrak{R}(G)$-module, then $\mathrm{Hom}_{\mathfrak{R}(G)}(M|^G, N)$, *and* $\mathrm{Hom}_{\mathfrak{R}(H)}(M, N|_H)$ *are isomorphic as vector spaces over $\mathfrak{R}$. Here $M|^G$ is the $\mathfrak{R}(G)$-module induced from M to G, and $N|_H$ is the restriction of N to H.*

The final part of this section consists of a short summary of the theory of formations as presented in the papers of Gaschütz and Lubeseder [4], [5], and [7].

DEFINITION 2.3. For each prime p, let $\mathscr{F}(p)$ be a formation. Let $\mathscr{F}$ denote the collection of groups G which satisfy the following two conditions:

(a) if $\mathscr{F}(p)$ is nonvoid, and K is a p-chief factor of G, then $G/C_G(K)$ lies in $\mathscr{F}(p)$;

(b) if $\mathscr{F}(p)$ is empty, then G is a p'-group.

Then $\mathscr{F}$ is a formation called the formation *locally defined by the family* $\{\mathscr{F}(p)\}$. In general, a formation $\mathscr{F}$ is *locally defined* if there is a family $\{\mathscr{F}(p)\}$ of formations such that $\mathscr{F}$ is locally defined by $\{\mathscr{F}(p)\}$.

Since the intersection, over all p-chief factors K of G, of the groups $C_G(K)$ is the group $O_{p'p}(G)$, it is easy to see that condition (a) above is equivalent to

(2.1) if $\mathscr{F}(p)$ is nonempty, then $G/O_{p'p}(G)$ lies in $\mathscr{F}(p)$.

The family $\mathscr{F}(p)$ of formations which define $\mathscr{F}$ is not unique. If, however, $\{\mathscr{F}(p)\}$ and $\{\mathscr{F}'(p)\}$ are two families of formations which locally define the same formation $\mathscr{F}$, then the family $\{\mathscr{H}(p) \mid \mathscr{H}(p) = \mathscr{F}(p) \cap \mathscr{F}'(p)\}$ also defines $\mathscr{F}$. Thus there is a *unique* minimal local definition for any locally defined formation $\mathscr{F}$. For example, the minimal local definition of the formation of all nilpotent groups is obtained by setting $\mathscr{N}(p) = \{1\}$ for all primes p.

THEOREM 2.4. ([4], p. 302;]5], p. 198; [7]) *A formation $\mathscr{F}$ is saturated if, and only if, it is locally defined.*

In view of this theorem, we shall use the terms saturated and locally defined interchangeably from now on.

DEFINITION. 2.5. Let $\mathscr{F}$ be a formation. A subgroup F of G is an $\mathscr{F}$-subgroup of G provided:

(a) $F \in \mathscr{F}$;

(b) if $F \leqq U \leqq G$, and N is a normal subgroup of U such that U/N lies in $\mathscr{F}$, then $FN = U$, i.e., F covers U/N.

The following two lemmas appear in [4], and describe the basic properties of $\mathscr{F}$-subgroups.

LEMMA 2.6. ([4], p. 301) *If the formation $\mathscr{F}$ is saturated, then every solvable group G has an $\mathscr{F}$-subgroup. All $\mathscr{F}$-subgroups of G are conjugate.*

LEMMA 2.7. ([4], p. 301) *Let $\mathscr{F}$ be a formation, and G a group. Let F be an element of $\mathscr{F}$ such that $F \leqq G$. Then:*

(a) *if F is an $\mathscr{F}$-subgroup of G, and $F \leqq U \leqq G$, F is also an $\mathscr{F}$-subgroup of U;*

(b) *if $N \lhd G$, and F is an $\mathscr{F}$-subgroup of G, then FN/N is an $\mathscr{F}$-subgroup of G/N;*

(c) *if $N \lhd G$, F'/N is an $\mathscr{F}$-subgroup of G/N, and F is an $\mathscr{F}$-subgroup of F', then F is an $\mathscr{F}$-subgroup of G.*

3. **Preliminary results.** The first three lemmas of this section are elementary, but they are used frequently enough to justify their inclusion. The last two theorems give a cover-avoidance characterization of the $\mathscr{F}$-subgroups of a group.

LEMMA 3.1. *Let H be a normal p'-subgroup of G, $\mathfrak{K}$ a field of characteristic p, and M an indecomposable $\mathfrak{K}(G)$-module. Then $M|_H$ is a completely reducible $\mathfrak{K}(H)$-module whose nonisomorphic irreducible components form a single orbit $\mathfrak{C}$ of conjugate $\mathfrak{K}(H)$-modules under action by the elements of G. Let L, J be two $\mathfrak{K}(G)$-modules of M such that $L \subset J$. Then the distinct $\mathfrak{K}(H)$-irreducible components of $(J/L)|_H$ are precisely the elements of $\mathfrak{C}$.*

Proof. Complete reducibility of $M|_H$ is clear since H is a p'-group. Since the decomposition of $M|_H$ into its homogeneous components is unique, these components are permuted by the action of G on M. Indecomposability implies only one orbit $\mathfrak{O}$ can occur, hence the same statement holds for the nonisomorphic irreducible components of $M|_H$. The transitivity of G on the orbit $\mathfrak{C}$ and the fact that at least one element of $\mathfrak{C}$ appears as a constituent of $(J/L)|_H$ yields the last statement of the lemma.

LEMMA 3.2. *Let G be a group, and M a $\mathfrak{K}(G)$-module. M is faithful if, and only if, $M|_{F(G)}$ is faithful.*

Proof. The lemma follows *a fortiori* from the statement that if $1 < N \lhd G$, then $1 < N \cap F(G)$.

We now begin a discussion of the properties of $\mathscr{F}$-subgroups of solvable groups. If G is a group, and $\mathscr{F}$ a formation, we use $G_{\mathscr{F}}$ to denote the intersection of all normal subgroups N of G such that the factor group G/N lies in $\mathscr{F}$. It is useful to know the behavior of $G_{\mathscr{F}}$ under homomorphisms, so we prove

LEMMA 3.3. *Let $\mathscr{F}$ be a formation, G a group, and $H \lhd G$. Then.*

$$(G/H)_{\mathscr{F}} = G_{\mathscr{F}}H/H .$$

Proof. Let F be the inverse image in G of $(G/H)_{\mathscr{F}}$. Then G/F is isomophic to $(G/H)/(G/H)_{\mathscr{F}}$, hence G/F lies in $\mathscr{F}$. Therefore, $G_{\mathscr{F}}H \leqq F$.

Since $G/G_{\mathscr{F}}H$ lies in $\mathscr{F}$, it follows that $G_{\mathscr{F}}H/H$ is a normal subgroup of G/H whose corresponding factor group lies in $\mathscr{F}$. Therefore F/H is contained in $G_{\mathscr{F}}H/H$; this completes the proof.

The next theorem generalizes a remark made by Carter in [1], and provides the first half of a cover-avoidance characterization of $\mathscr{F}$-subgroups.

THEOREM 3.4. *Let $\mathscr{F}$ be a formation locally defined by the family $\{\mathscr{F}(p)\}$, G be a group, F a subgroup of G which lies in $\mathscr{F}$, and K an F-composition factor of G. Then*

(a) *F either covers, or avoids K;*

(b) *if F covers K, and $p \mid |K|$, then $F/C_F(K) \in \mathscr{F}(p)$;*

(c) *if F is an $\mathscr{F}$-subgroup of G, and $p \mid |K|$, then*

$$F/C_F(K) \in \mathscr{F}(p) \Rightarrow F \text{ covers } K . \tag{3.1}$$

Proof. Let $K = L/M$ be the F-composition factor in question. Statement (a) follows from the fact that F acts irreducibly on K, and $(L \cap F)M/M$ is an F-invariant subgroup of K.

If F covers K, then looking at F as a set of operators on K, it follows that K is operator isomorphic to $L \cap F/M \cap F$, a p-chief factor of F. Therefore the kernel of the representation of F on $L \cap F/M \cap F$ is $C_F(K)$. Since F lies in $\mathscr{F}$, $F/C_F(K)$ lies in $\mathscr{F}(p)$. This proves (b).

Now suppose F is an $\mathscr{F}$-subgroup of G, and K is a p-section of G such that $F/C_F(K)$ lies in $\mathscr{F}(p)$. To show F covers K, it suffices to show that F covers the larger section FL/M. But by Lemma 2.7, F is an $\mathscr{F}$-subgroup of FL, hence it is sufficient to show $\bar{F} = FL/M$ is an element of $\mathscr{F}$ since F, by definition, covers any such factor of FL.

If q is a prime distinct from p, then K, as a normal q'-subgroup of $\bar{F}$, is contained in $O_{q'}(\bar{F})$. Therefore $O_{q'q}(F)L/M$ is contained in $O_{q'q}(\bar{F})$, so $\bar{F}/O_{q'q}(\bar{F})$ is isomorphic to a quotient group of $FL/O_{q'q}(F)L$. But $FL/O_{q'q}(F)L$ is isomorphic to a quotient group of $F/O_{q'q}(F)$. Since $F \in \mathscr{F}$, (2.1) implies $F/O_{q'q}(F)$ lies in $\mathscr{F}(q)$, hence $\bar{F}/O_{q'q}(\bar{F})$ is also in $\mathscr{F}(q)$.

Let $U = F_{\mathscr{F}(p)}$. Since $F \in \mathscr{F}$, $F/O_{p'p}(F)$ lies in $\mathscr{F}(p)$. Therefore U is contained in $O_{p'p}(F)$. Since we have assumed $F/C_F(K) \in \mathscr{F}(p)$, it follows that K is contained in the center of UL/M. Therefore

UL/M has a normal p-complement. As a normal subgroup of $\bar{F}$, it follows that UL/M is contained in $O_{p'p}(\bar{F})$, the maximal normal subgroup of $\bar{F}$ which has a normal p-complement. Therefore $\bar{F}/O_{p'p}(\bar{F})$ is isomorphic to a quotient of F/U and must lie in $\mathscr{F}(p)$. This shows that $\bar{F}$ satisfies (2.1) for all primes p, so $\bar{F}$ lies in $\mathscr{F}$.

Our next theorem will show that (3.1) characterizes the $\mathscr{F}$-subgroups of a solvable group G. In order to obtain as weak an hypothesis as possible, we prove two lemmas. (3.1) actually applies only to specific F-composition factors of G, so when we say that (3.1) holds for an F-composition series, $G = G_0 > G_1 > \cdots > G_n = 1$, of G, we mean F satisfies that property for all factors G_i/G_{i+1} of the series for which the hypothesis of (3.1) holds.

LEMMA 3.5. *Suppose $\mathscr{F}$ is a formation locally defined by $\{\mathscr{F}(p)\}$, F lies in $\mathscr{F}$, and $F \leqq G$. Let A/B be an F-invariant section of G such that $A > C > B$ defines a fixed F-composition series of A/B. If* (3.1) *holds for this series, then* (3.1) *holds for every F-composition series of A/B.*

Proof. We may assume that a second F-composition series of A/B exists and is defined by $A > D > B$ where $D \neq C$. Then we must have $A = CD$ and $B = C \cap D$. Therefore

$$A/B \cong C/B \times D/B, A/C \cong D/B, A/D \cong C/B\ , \tag{3.2}$$

where the decomposition is an operator decomposition, and the isomorphisms are operator isomorphisms.

Suppose the decomposition (3.2) is unique. If $F/C_F(A/D)$ lies in $\mathscr{F}(p)$, it follows from (3.2) that $F/C_F(C/B)$ lies in $\mathscr{F}(p)$. Since (3.1) holds for the series $A > C > B$, F covers C/B. Therefore $(F \cap A)D \geqq (F \cap C)D \geqq CD = A$, so F covers A/D. If $F/C_F(D/B)$ lies in $\mathscr{F}(q)$, then (3.1) implies F covers A/C. Because of the uniqueness of the decomposition, and the fact that $F \cap A$ is not contained in C, either $A = (F \cap A)B$, or $D = (F \cap A)B$. In the former case, F covers all of A/B, and in the latter case, $F \cap A = F \cap D$ since $F \cap A \leqq D$. Therefore, in either case, F covers D/B.

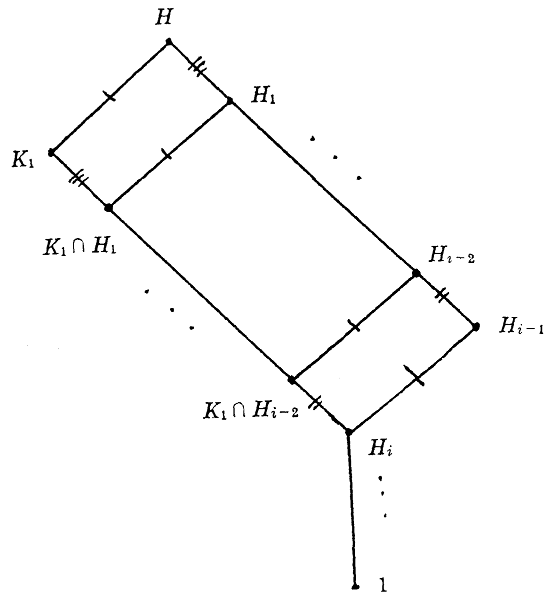
The decomposition (3.2) is unique if the orders of the factors are relatively prime, so we may assume A/B is an elementary abelian p-group for some prime p. This means that we can look at A/B as a $Z_p(F)$-module. If the factors are distinct $Z_p(F)$-modules, then the decomposition is again unique. If they are isomorphic, it follows from (3.1) for the series $A > C > B$ that F either covers, or avoids A/B. Therefore Lemma 3.5 holds in all cases.

LEMMA 3.6. *Assume F lies in $\mathscr{F}$, $H \leqq G$, and $F \leqq N_G(H)$. If*

(3.1) *holds for a fixed F-composition series of H, then it holds for every F-composition series of H.*

Proof. Let $H = H_0 > H_1 > \cdots > H_n = 1$ be the fixed F-composition series of H for which (3.1) holds. Use induction on n. If $H = K_0 > K_1 > \cdots > K_n = 1$ is a second F-composition series for H, and $K_1 = H_1$, (3.1) holds for the second series by induction.

If K_1 and H_1 are distinct, we let i be the smallest integer such that $K_1 \cap H_i = H_i$. Because $H_i \leqq K_1 \cap H_{i-1}$, we have $H_i = K_1 \cap H_{i-1}$, so that we have the following lattice diagram:



Now H_1 is F-invariant, and because of the isomorphisms indicated in the diagram, $H_1 > K_1 \cap H_1 > \cdots > K_1 \cap H_{i-1} = H_i > \cdots > H_n = 1$ is an F-composition series for H_1 which has length $n-1$. By induction, (3.1) holds for this series. Therefore, (3.1) holds for the F-composition series of $H/H_1 \cap K_1$ defined by the series $H > H_1 > H_1 \cap K$. By Lemma 3.5, (3.1) holds for the F-composition series

$$H > K_1 > H_1 \cap K_1 > \cdots > K_1 \cap H_{i-1} = H_i > \cdots > H_n = 1$$

of H. In particular, (3.1) holds, by induction, for any F-composition series of K_1. Therefore (3.1) holds for the series $K_0 > K_1 > \cdots > K_n = 1$.

THEOREM 3.7. *Let $\mathscr{F}$ be a formation locally defined by $\{\mathscr{F}(p)\}$. Let G be a group, and F a subgroup of G which lies in $\mathscr{F}$. If* (3.1) *holds for a fixed F-composition series $G = G_0 > G_1 > \cdots > G_n = 1$ of G, then F is an $\mathscr{F}$-subgroup of G.*

Proof. We use induction on $|G|$. By Lemma 3.6, we may assume that the series $G = G_0 > G_1 > \cdots > G_n = 1$ is a refinement of the chief series $G = H_0 > H_1 > \cdots > H_{m-1} > H_m = 1$. Then $H_{m-1} = G_k$ for some k. H_{m-1} is a minimal normal subgroup of G, so we set $\bar{G}_i = G_i/G_k$ for $i = 0, 1, \cdots, k$, $\bar{F} = FG_k/G_k$, and $\bar{G} = \bar{G}_0$. Our first step is to show that $\bar{F}$ is an $\mathscr{F}$-subgroup of G.

If $m = 1$, the result is trivial. If m is larger than 1, then H_{m-1} is a proper subgroup of G, and by induction, to show that $\bar{F}$ is an $\mathscr{F}$-subgroup of G, it is sufficient to verify (3.1) for the F-composition series $\bar{G}_0 > \bar{G}_1 > \cdots \bar{G}_k = \bar{1}$.

For each i, set $K_i = G_i/G_{i+1}$, and $\bar{K}_i = \bar{G}_i/\bar{G}_{i+1}$. Since $G_k \leqq G_{i+1}$ for $i < k$, G_k centralizes the section K_i for $i < k$. Therefore, $C_{\bar{F}}(\bar{K}_i) = C_F(K_i)G_k/G_k$ for all $i < k$. Thus,

$$\bar{F}/C_{\bar{F}}(\bar{K}_i) \cong FG_k/C_F(K_i)G_k \cong F/C_F(K_i)(F \cap G_k) .$$

But $F \cap G_k \leqq C_F(K_i)$, so we have

$$\bar{F}/C_{\bar{F}}(\bar{K}_i) \cong F/C_F(K_i) \text{ for } i < k . \tag{3.3}$$

Suppose $\bar{K}_i$ is a p-section of $\bar{G}$ such that $\bar{F}/C_{\bar{F}}(\bar{K}_i)$ lies in $\mathscr{F}(p)$. By (3.3), $F/C_F(K_i)$ lies in $\mathscr{F}(p)$, so F covers K_i. Therefore, $(FG_k \cap G_i)G_{i+1} = (F \cap G_i)G_kG_{i+1} = (F \cap G_i)G_{i+1} = G_i$. By taking homomorphic images, and noting that $FG_k \cap G_i/G_k = \bar{F} \cap \bar{G}_i$, we get $(\bar{F} \cap \bar{G}_i)\bar{G}_{i+1} = \bar{G}_i$. Thus $\bar{F}$ covers $\bar{K}_i$. Therefore (3.1) holds for the $\bar{F}$-composition series $\bar{G} = \bar{G}_0 > \bar{G}_1 > \cdots > \bar{G}_k = \bar{1}$ of $\bar{G}$.

Now that we know $\bar{F}$ is an $\mathscr{F}$-subgroup of $\bar{G}$, it follows from Lemma 2.7 that we can complete our proof by showing that F is an $\mathscr{F}$-subgroup of FG_k.

Suppose $FG_k < G$. We consider the series

$$FG_k = D_0 \geqq D_1 \geqq \cdots \geqq D_n = 1 ,$$

where $D_i = FG_k \cap G_i$ for each i. Suppose $D_i > D_{i+1}$ for some i. Then

$$D_i/D_{i+1} \cong (FG_k \cap G_i)G_{i+1}/G_{i+1} > 1 .$$

This is an operator isomorphism, hence because F is irreducible on K_i, we have

$$D_i/D_{i+1} \cong G_i/G_{i+1} . \tag{3.4}$$

Therefore the distinct terms of the series $D_0 \geqq D_1 \geqq \cdots \geqq D_n = 1$, form an F-composition series for FG_k which passes through G_k. Since F covers FG_k/G_k, and since $D_i = G_i$ for $i \geqq k$, (3.1) holds for this composition series. By induction, F is an $\mathscr{F}$-subgroup of FG_k.

If $G = FG_k$, then G_k is a minimal normal subgroup of G, and F acts irreducibly on G_k. Therefore F either covers, or avoids G_k. If F covers G_k, then $F = G$, so F is an $\mathscr{F}$-subgroup of G. Suppose F avoids G_k, and G_k is a p-group. Then $FG_k/C_{FG_k}(G_k) \cong F/C_F(G_k)$ cannot lie in $\mathscr{F}(p)$ since (3.1) holds for G_k. Therefore $(FG_k)_{\mathscr{F}} \geqq G_k$. Since FG_k/G_k lies in $\mathscr{F}$, $(FG_k)_{\mathscr{F}} = G_k$.

If $F \leqq U \leqq FG_k$, then $U = F$, or $U = FG_k$. The above remarks show that F covers $U/U_{\mathscr{F}}$ in both cases. Therefore F is an $\mathscr{F}$-subgroup of FG_k, and the proof is complete.

As one application of Theorem 3.7, we prove

COROLLARY 3.8. *Let $\mathscr{F}$ be a formation locally defined by $\{\mathscr{F}(p)\}$, $H \in \mathscr{F}$, and let I be a finitely generated $Z_p(H)$-module. Let $G = HI$ be the semi-direct product of I by H where the action of H on I by conjugation is the usual one. Then,*

(a) *$F = HC_I(O_{p'}(H_{\mathscr{F}(p)}))$ is an $\mathscr{F}$-subgroup of G,*

(b) *As a $Z_p(H)$-module, $I = C_I(O_{p'}(H_{\mathscr{F}(p)})) \dotplus G_{\mathscr{F}}$.*

Proof. Let $W = C_I(O_{p'}(H_{\mathscr{F}(p)}))$. Our first task is to show HW lies in $\mathscr{F}$. Suppose q is a prime distinct from p, then W is a q'-group normal in HW, so $O_{q'q}(F) = O_{q'q}(H)W$. Therefore,

$$F/O_{q'q}(F) \cong H/O_{q'q}(H) .$$

Since H lies in $\mathscr{F}$, $F/O_{q'q}(F) \in \mathscr{F}(q)$.

Let $U = H_{\mathscr{F}(p)}$. Then $O_{p'}(U)$ centralizes W. Since $H/O_{p'p}(H)$ lies in $\mathscr{F}(p)$, $U \leqq O_{p'p}(H)$. Therefore UW has a normal p-complement, and as a normal subgroup of F, must be contained in $O_{p'p}(F)$. Therefore $F/O_{p'p}(F)$ is isomorphic to a quotient group of H/U. Since $H/U \in \mathscr{F}(p)$, so is $F/O_{p'p}(F)$. Therefore, (2.1) holds for all primes r, so F lies in $\mathscr{F}$.

Now let $G = G_0 > G_1 > \cdots > G_n = 1$ be an F-composition series for G such that $G_l = I$ for some l. In order to check (3.1) for this series, we need only consider $K_i = G_i/G_{i+1}$ for $i \geqq l$, since F covers G/I. W centralizes every K_i, so we have

$$F/C_F(K_i) \cong H/C_H(K_i) . \tag{3.5}$$

If $i \geqq l$, and $F/C_F(K_i) \in \mathscr{F}(p)$, then (3.5) implies $C_H(K_i) \geqq U$. In particular, $O_{p'}(U)$ centralizes K_i. Therefore F covers K_i, and (3.1) holds for the series in question. By Theorem 3.7, F is an $\mathscr{F}$-subgroup of G.

By complete reducibility, $I = W \dotplus (I, O_{p'}(U))$, and since $O_{p'}(U)$ is normal in H, both W and $V = (I, O_{p'}(U))$ are normal in HI. Clearly HI/V is the largest factor of HI covered by F. Therefore $V = G_{\mathscr{F}}$.

REMARK. This result cannot be extended to the case where I is a p-group of class 2 because of the following example. Let I be the quaternion group. I has an automorphism h of order 3 such that h acts fixed point free on $I/D(I)$, and centralizes $D(I)$. Let H be the cyclic group of order 3 generated by h, and let $G = HI$. A Carter subgroup of G is $H \times D(I)$, but $D(I)$ has no complement in I, so no splitting is possible.

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REMARK. If $\mathscr{F}$ is a saturated formation, $H \in \mathscr{F}$, and $\{\mathscr{F}_1(p)\}$, $\{\mathscr{F}_2(p)\}$ are two local definitions for $\mathscr{F}$, then $O_{p'}(H_{\mathscr{F}_1(p)}) = O_{p'}(H_{\mathscr{F}_2(p)})$.

Proof. Clearly $H/H_{\mathscr{F}_i(p)} \in \mathscr{F} \cap \mathscr{F}_i(p)$, so $H_{\mathscr{F}_i(p)} \geqq H_{\mathscr{F} \cap \mathscr{F}_i(p)}$. Since $\mathscr{F} \cap \mathscr{F}_i(p)$ is contained in $\mathscr{F}_i(p)$, $H_{\mathscr{F}_i(p)} = H_{\mathscr{F}_i(p) \cap \mathscr{F}}$, and in the terminology of [2], we may assume the local definitions $\{\mathscr{F}_i(p)\}$ are integrated.

By Theorem 2.2 of [2], we have $\mathscr{P}\mathscr{F}_1(p) = \mathscr{P}\mathscr{F}_2(p)$, where

$$\mathscr{P}\mathscr{F}_i(p) = \{G \mid G/O_p(G) \in \mathscr{F}_i(p)\} .$$

Since $H_{\mathscr{F}_i(p)} \leqq O_{p'p}(H)$ for each i, it follows that for each i,

$$O_{p'}(H_{\mathscr{F}_i(p)}) = H_{\mathscr{P}\mathscr{F}_i(p)} .$$

Since $\mathscr{P}\mathscr{F}_1(p) = \mathscr{P}\mathscr{F}_2(p)$, the remark follows.

4. Strong containment. In this section, we shall characterize strong containment. In certain cases, we can make our characterization more precise by giving generating sets for certain of the formations $\mathscr{F}(p)$ in the minimal local definition of $\mathscr{F}$. The results of this section form the basis for our results in § 5.

LEMMA 4.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be two nonempty saturated formations; let $\mathscr{E}$ be locally defined by $\{\mathscr{E}(p)\}$. Let G be a group of minimal order satisfying:*

(4.1) *An $\mathscr{E}$-subgroup of G is not contained in any $\mathscr{F}$-subgroup of G.*

If F is an $\mathscr{F}$-subgroup of G, and E is an $\mathscr{E}$-subgroup of F, then

(a) *$G_{\mathscr{F}} = M$ is a minimal normal subgroup of G; G is the semidirect product of M by F; F acts faithfully and irreducibly on M.*

(b) *If M is a p-group, then $E^* = EC_M(O_{p'}(E_{\mathscr{E}(p)}))$ is an $\mathscr{E}$-subgroup of G, and $1 < C_M(O_{p'}(E_{\mathscr{E}(p)})) \leqq M$.*

Proof. If G is an element of $\mathscr{F}$, then $G = F$ contains every $\mathscr{E}$-subgroup of G, hence G does not satisfy (4.1). Therefore $G \notin \mathscr{F}$;

in particular, G is not the identity. Let $M \neq 1$ be a minimal normal subgroup of G. By Lemma 2.7, FM/M is an $\mathscr{F}$-subgroup of G/M. Because of the minimality of $|G|$ with respect to the property (4.1), some $\mathscr{E}$-subgroup of G/M is contained in FM/M. Since all $\mathscr{E}$-subgroups of G/M are conjugate, we can find an $\mathscr{E}$-subgroup E of G such that $EM \leqq FM$. E, as an $\mathscr{E}$-subgroup of G, is also an $\mathscr{E}$-subgroup of FM. Because G satisfies (4.1), no conjugate of E under FM can be contained in F. The minimality of G implies $G = FM$.

G/M is in $\mathscr{F}$, but G is not, so $G_{\mathscr{F}} = M$. Since $F \cap M$ is a normal subgroup of G, properly contained in M, $F \cap M = 1$, so G is semidirect product of M by F. Since M was arbitrary to begin with, and we showed $M = G_{\mathscr{F}}$, M is the unique minimal normal subgroup of G. Therefore F acts faithfully and irreducibly on M. This proves (a).

G/M is isomorphic to F, so EM/M is an $\mathscr{E}$-subgroup of G/M. By Lemma 2.7, an $\mathscr{E}$-subgroup of EM is also an $\mathscr{E}$-subgroup of G. Corollary 3.8 shows that $E^* = EC_M(O_{p'}(E_{\mathscr{E}(p)}))$ is an $\mathscr{E}$-subgroup of EM. Since E^* is not contained in F, statement (b) holds.

Before stating the characterization, we introduce some notation.

DEFINITION 4.2. If $\mathscr{E}$ and $\mathscr{F}$ are two saturated formations, and $\mathscr{E}$ is locally defined by $\{\mathscr{E}(p)\}$, set

(a) $\pi(\mathscr{E}) = \{p \mid \mathscr{E}(p)$ is nonempty$\}$. $\pi(\mathscr{E})$ is called the characteristic of $\mathscr{E}$.

(b) If $p \in \pi(\mathscr{E})$, we denote by $\Phi(p)$ the collection of all $H \in \mathscr{F}$ such that if E is an $\mathscr{E}$-subgroup of H, then H has a faithful irreducible $Z_p(H)$-module M which satisfies

$$1 < C_M(O_{p'}(E_{\mathscr{E}(p)})) \leqq M . \tag{4.2}$$

(c) If $p \in \pi(\mathscr{E})$, let $\theta(p)$ be the collection of all H in $\Phi(p)$ such that H has at least one faithful irreducible $Z_p(H)$-module satisfying

$$1 < C_M(O_{p'}(E_{\mathscr{E}(p)})) < M . \tag{4.3}$$

THEOREM 4.3. *Suppose $\mathscr{E}$ and $\mathscr{F}$ are two saturated formations locally defined by $\{\mathscr{E}(p)\}$ and $\{\mathscr{F}(p)\}$ respectively. Then $\mathscr{E} \ll \mathscr{F}$ if, and only if, for each prime p in the characteristic of $\mathscr{E}$, $\Phi(p)$ is contained in $\mathscr{F}(p)$.*

Proof. Suppose $\Phi(p)$ is contained in $\mathscr{F}(p)$ for each p in the characteristic of $\mathscr{E}$, and $\mathscr{E}$ is not strongly contained in $\mathscr{F}$. Then the class of groups satisfying (4.1) with respect to the formations $\mathscr{E}$ and $\mathscr{F}$ is nonempty, so we choose G to be an element of minimal order in this class. By Lemma 4.1, if $G_{\mathscr{F}}$ is a p-group, then p divides the order of an $\mathscr{E}$-subgroup of G, and must be an element of the characteristic of $\mathscr{E}$. By Lemma 4.1, if F is an $\mathscr{F}$-subgroup of G, then F lies in $\Phi(p)$. Therefore F is an element of $\mathscr{F}(p)$, and $O_p(F) = 1$.

Since $G_{\mathscr{F}}$ is the unique minimal normal subgroup of $G, G_{\mathscr{F}} = O_{p'p}(G)$. Therefore $F \cong G/G_{\mathscr{F}} = G/O_{p'p}(G)$ lies in $\mathscr{F}(p)$. If q is a prime distinct from p, then $G_{\mathscr{F}} \leqq O_{q'}(G)$, and it follows that $O_{q'q}(G) = G_{\mathscr{F}}O_{q'q}(F)$. Therefore,

$$G/O_{q'q}(G) \cong F/O_{q'q}(F) .$$

Since F lies in $\mathscr{F}$, we see that $G/O_{q'q}(G)$ lies in $\mathscr{F}(q)$. By (2.1), G lies in $\mathscr{F}$, a contradiction to the fact that $G_{\mathscr{F}} > 1$. Therefore $\mathscr{E} \ll \mathscr{F}$.

Suppose $\mathscr{E} \ll \mathscr{F}$, $p \in \pi(\mathscr{E})$, and $F \in \Phi(p)$. Let M be the faithful irreducible $Z_p(F)$-module mentioned in the definition of $\Phi(p)$. Set $G = FM$, where the action of F on M by conjugation is the module action. By Corollary 3.8, an $\mathscr{F}$-subgroup of G is $F^* = FC_M(O_{p'}(F_{\mathscr{F}(p)}))$, hence $G = F^*M$. Let E be an $\mathscr{E}$-subgroup of F. Since EM/M is an $\mathscr{E}$-subgroup of G/M, it follows, from Lemma 2.7 and Corollary 3.8, that $E^* = EC_M(O_{p'}(E_{\mathscr{E}(p)}))$ is an $\mathscr{E}$-subgroup of G. E^* does not avoid M, and because $\mathscr{E} \ll \mathscr{F}$, E^* is contained in some $\mathscr{F}$-subgroup of G, hence F^* does not avoid M. Since F^* is irreducible on M, it follows that F^* contains M, hence $F^* = G$.

Since G lies in $\mathscr{F}$, and F acts faithfully on the p-chief factor M of G, we have F isomorphic to $G/G_G(M)$, an element of $\mathscr{F}(p)$. Therefore $\Phi(p)$ is contained in $\mathscr{F}(p)$.

Because of this characterization, if $\mathscr{E} \ll \mathscr{F}$, and p is a prime in the characteristic of $\mathscr{E}$, then $\Phi(p) \subseteqq \mathscr{F}(p)$ for any $\mathscr{F}(p)$ which lies in some local definition of $\mathscr{F}$. This leads naturally to the question:

Suppose $\{\mathscr{F}(p)\}$ is the unique minimal local definition for $\mathscr{F}$.

(4.4) If p is a prime in the characteristic of $\mathscr{E}$, is $\mathscr{F}(p)$ the smallest formation generated by the set $\Phi(p)$?

The answer to this question is yes, provided the set $\theta(p)$ is nonempty for at least two primes. We have not been able to relax the hypothesis on the $\theta(p)$'s. In order to prove this partial result, we shall, for the next few lemmas, investigate properties of the $\Phi(p)$'s and $\theta(p)$'s.

LEMMA 4.4. *Let $\mathscr{E}$ and $\mathscr{F}$ be nonempty saturated formations with local definitions $\{\mathscr{E}(p)\}$ and $\{\mathscr{F}(p)\}$ respectively. Suppose $\mathscr{E} \ll \mathscr{F}$, and G is an element of $\mathscr{F}$ with $\mathscr{E}$-subgroup E.*

(a) *Suppose* $G \in \Phi(q)$ *for some prime* q *in the characteristic of* $\mathscr{E}$. G *lies in* $\theta(q)$ *if, and only if,* $O_{q'}(E_{\mathscr{E}(q)}) > 1$.

(b) *If* $O_q(G) = 1$, *and the permutation representation on the cosets of* $O_{q'}(E_{\mathscr{E}(q)})$ *is faithful, then* G *lies in* $\langle\Phi(q)\rangle$, *the smallest formation generated by the set* $\Phi(q)$.

(c) *Let* V *be a faithful irreducible* $Z_p(K)$*-module, where* K *is a group. If* $G = KV$, *and the permutation representation on the cosets of* $O_{q'}(E_{\mathscr{E}(q)})$ *is faithful for some prime* q *in* $\pi(\mathscr{E}) - \{p\}$, *then* G *lies in* $\Phi(q)$.

(d) *For each* r, s *in* $\pi(\mathscr{E})$, $\theta(r) \leqq \langle\Phi(s)\rangle$.

Proof. Let $H = O_{q'}(E_{\mathscr{E}(q)})$. If G is in $\Phi(q)$, then G has a faithful irreducible $Z_q(G)$-module I such that $1 < C_I(H) \leqq I$. Equality holds if and only if $H = 1$, so (a) is true;

Now suppose G satisfies the hypothesis of (b). Let T be the $Z_q(G)$-module which affords the representation of G on the cosets of H. Since H is a q'-group, the principal $Z_q(H)$-module is a direct summand of the regular $Z_q(H)$-module, hence

(4.5) T is a direct sum of principal indecomposable $Z_q(G)$-modules.

We write $T = T_1 \dotplus \cdots \dotplus T_s$, where each T_i is indecomposable, and let U_i be the unique maximal proper $Z_q(G)$-submodule of T_i. Finally, we let M_i be the factor module T_i/U_i.

Since $O_q(G)$ is trivial, $F(G)$ is a q'-subgroup of G, hence by Lemma 3.1, the distinct irreducible components of $M_i|_{F(G)}$ are exactly the same as the distinct irreducible components of $T_i|_{F(G)}$. Since T is faithful, it follows that if we let M be the direct sum of all the modules M_i, then $M|_{F(G)}$ is faithful. By Lemma 3.2, M is a faithful $Z_q(G)$-module.

We now apply the Frobenius reciprocity theorem for modules, i.e., Lemma 2.2. For each $i = 1, 2, \cdots, s$

$$(0) \subset \mathrm{Hom}_{Z_q(G)}(T, M_i) \cong \mathrm{Hom}_{Z_q(H)}(1, M_i|_H) ,$$

where 1 denotes the principal $Z_q(H)$-module. Therefore, for each i,

$$1 < C_{M_i}(H) \leqq M_i .$$

Set $G_i = G/C_G(M_i)$. Then $E_i = EC_G(M_i)/C_G(M_i)$ is an $\mathscr{E}$-subgroup of G_i. By Lemma 3.3, $(E_i)_{\mathscr{E}(q)} = E_{\mathscr{E}(q)}C_G(M_i)/C_G(M_i)$. It follows from (2.1) and the definition of $E_{\mathscr{E}(q)}$ that $E_{\mathscr{E}(q)}$ has a normal q-complement. Therefore $O_{q'}((E_i)_{\mathscr{E}(q)}) = O_{q'}(E_{\mathscr{E}(q)})C_G(M_i)/C_G(M_i)$. This implies

$$1 < C_{M_i}(O_{q'}(E_{\mathscr{E}(q)})) = C_{M_i}(O_{q'}(E_i)_{\mathscr{E}(q)}) \leqq M_i .$$

Since G lies in $\mathscr{F}$, $q \in \pi(\mathscr{E})$, and M_i is a faithful irreducible

$Z_q(G_i)$-module, G_i lies in $\Phi(q)$ for each i. Therefore $G = G/\bigcap_i C_G(M_i)$ lies in $\langle\Phi(q)\rangle$, the smallest formation generated by the set $\Phi(q)$. This proves (b).

The proof of (c) is essentially the same as the proof of (b). Let $G = KV$ be the group mentioned in the hypothesis of (c). Let T be the $Z_q(G)$-module which affords the permutation representation on H. Once again, T has a decomposition into a direct sum $T = T_1 \dotplus \cdots \dotplus T_s$ of principal indecomposable $Z_q(G)$-modules. Since G is faithful on T, V is nontrivial on some T_i, say T_1. If U_1 is the unique maximal proper $Z_q(G)$-submodule of T_1, then Lemma 3.1 implies V is nontrivial on $M = T_1/U_1$. By Frobenius reciprocity, we again have $1 < C_M(H) \leqq M$.

Since K acts faithfully and irreducibly on V, it follows from Lemma 1.2 of [3] that $O_p(K) = 1$, hence $V = F(G)$. Since V is minimal normal in G, and nontrivial on M, it is faithful on M. Lemma 3.2 implies G is faithful on M. Since $q \in \pi(\mathscr{E})$, G is, by definition, an element of $\Phi(q)$. This proves (c).

Part (d) is the only statement in Lemma 4.4 which requires the assumption $\mathscr{E} \ll \mathscr{F}$. Suppose $H \in \theta(r)$, E is an $\mathscr{E}$-subgroup of H, and M is a faithful irreducible $Z_r(H)$-module such that $1 < C_M(O_{r'}E_{\mathscr{F}(r)})) < M$. Set $G = HM$. By Corollary 3.8, $F = HC_M(O_{r'}(E_{\mathscr{F}(r)}))$ is an $\mathscr{F}$-subgroup of G, and since $E \cong EM/M$ is an $\mathscr{E}$-subgroup of G/M, $E^* = EC_M(O_{r'}(E_{\mathscr{E}(r)}))$ is an $\mathscr{E}$-subgroup of G. Since $\mathscr{E} \ll \mathscr{F}$, F cannot avoid M, hence $F = G$ is an element of $\mathscr{F}$.

Let N be the intersection of all the conjugates of $O_{s'}(E^*_{\mathscr{E}(s)})$ in G. Then $N \lhd G$, and $N \cap M \leqq E^* \cap M = C_M(N_{r'}(E_{\mathscr{E}(r)})) < M$. Therefore $N \cap M = 1$. This shows that the representation of G on the cosets of $O_{s'}(E^*_{\mathscr{E}(s)})$ is faithful on M. Because $M = F(G)$, it follows from Lemma 3.2 that this representation is faithful on G. By part (c), G is an element of $\Phi(s)$ for any s in $\pi(\mathscr{E}) - \{r\}$. Therefore $H \cong G/M$ is an element of $\langle\Phi(s)\rangle$, for s in $\pi(\mathscr{E}) - \{r\}$. Since $\theta(r)$ is contained in $\Phi(r)$, it follows that $\theta(r) \subseteqq \langle\Phi(s)\rangle$ for each s in the characteristic of $\mathscr{E}$. This proves (d).

The next lemma has an elegant proof. This proof was shown to me by Professor E. C. Dade, and it shortens this part of the discussion considerably.

LEMMA 4.5. *Let A, B be two groups and assume the center of A is the identity. If M is a faithful $Z_p(A)$-module, and T is a faithful $Z_p(B)$-module, then $M \otimes T$ is a faithful $Z_p(A \times B)$-module.*

Proof. If V is a vector space over Z_p, we let $GL(V)$ denote the general linear group on V. Then $A \times B \leqq GL(M) \times GL(T) = C$, so we examine the kernel K of the representation of C on $M \otimes T$. Let $m_1, \cdots, m_r$ be a Z_p-basis for M, and $t_1, \cdots, t_s$ a Z_p-basis for T. Then $\{m_i \otimes t_j \mid 1 \leqq i \leqq r, 1 \leqq j \leqq s\}$ is a Z_p-basis for $M \otimes T$. Suppose $f \times g$ is an element of K, and

$$m_i f = \textstyle\sum_k \varphi_{ik} m_k \qquad \text{for each } i\ ,$$
$$t_j g = \textstyle\sum_l \alpha_{jl} t_l \qquad \text{for each } j\ .$$

Then $m_i \otimes t_j = m_i \otimes t_j(f \times g) = (\sum \varphi_{ik} m_k) \otimes (\sum \alpha_{jl} t_l)$. By collecting terms, and equating coefficients we see that

$$\varphi_{ik}\alpha_{jl} = \begin{matrix} 0 \text{ if } (i, j) \neq (k, l) \\ 1 \text{ if } (i, j) = (k, l)\ . \end{matrix}$$

Therefore $m_i f = \varphi m_i$ for each i, and $t_j g = \varphi^{-1} t_j$ for each j. Therefore f lies in the center of $GL(M)$, and g lies in the center of $GL(T)$.

If $a \times b \in (A \times B) \cap K$, it follows, from the assumption that $Z(A) = 1$, that we must have $a = 1$. This means that the constant φ is the identity in Z_p, so $b = 1$. Therefore $A \times B$ acts faithfully on $M \otimes T$.

LEMMA 4.6. *Suppose $A \in \Phi(p) - \theta(p)$, $B \in \theta(p)$, and either $Z(A)$ or $Z(B)$ is the identity. Then $A \times B \in \langle \theta(p) \rangle$, the smallest formation generated by the set $\theta(p)$.*

Proof. Let E be an $\mathscr{E}$-subgroup of A, and E^* an $\mathscr{E}$-subgroup of B. Since $A \in \Phi(p) - \theta(p)$, it follows from Lemma 4.4 (a) that $O_{p'}(E_{\mathscr{E}(p)}) = 1$. By (2.1), and the definition of $E_{\mathscr{E}(p)}$, we see that $E_{\mathscr{E}(p)}$ has a normal p-complement. Therefore $E_{\mathscr{E}(p)}$ is a p-group. Since $B \in \theta(p)$, $O_{p'}(E^*_{\mathscr{E}(p)}) > 1$.

Now $E \times E^*$ is an $\mathscr{E}$-subgroup of $A \times B$. We wish to examine $O_{p'}((E \times E^*)_{\mathscr{E}(p)})$. Since $(E \times E^*)/(E_{\mathscr{E}(p)} \times E^*_{\mathscr{E}(p)})$ lies in $\mathscr{E}(p)$, $(E \times E^*)_{\mathscr{E}(p)}$ is a normal subgroup of $E \times E^*$ contained in $E_{\mathscr{E}(p)} \times E^*_{\mathscr{E}(p)}$. We define a subgroup W of $E^*_{\mathscr{E}(p)}$ by:

$$\mathrm{W} = \{e \in E^*_{\mathscr{E}(p)} \mid \exists\, t \in (E \times E^*)_{\mathscr{E}(p)} \ni t = d \times e, \text{ and } d \in E_{\mathscr{E}(p)}\}\ .$$

In other words, W is just the collection of all elements of $E^*_{\mathscr{E}(p)}$ which appear as components of elements of $(E \times E^*)_{\mathscr{E}(p)}$. W is clearly a normal subgroup of E^* which is contained in $E^*_{\mathscr{E}(p)}$. By construction, $(E \times E^*)_{\mathscr{E}(p)}$ is a subgroup of $E \times W$, hence it follows that E^*/W lies in $\mathscr{E}(p)$. Therefore $W = E^*_{\mathscr{E}(p)}$.

Now if e is any element of $O_{p'}(W)$, then there is an element d in $E_{\mathscr{E}(p)}$ such that $t = d \times e$ lies in $(E \times E^*)_{\mathscr{E}(p)}$. Since $E_{\mathscr{E}(p)}$ is a p-group, by taking an appropriate power of t, we see that e lies in

$(E \times E^*)_{\mathscr{E}(p)}$. Therefore,

$$(4.6) \qquad O_{p'}(E^*_{\mathscr{E}(p)}) \leqq O_{p'}((E \times E^*)_{\mathscr{E}(p)}) \leqq O_{p'}(E_{\mathscr{E}(p)} \times E^*_{\mathscr{E}(p)}) = O_{p'}(E^*_{\mathscr{E}(p)}) .$$

By assumption, A has a faithful irreducible $Z_p(A)$-module M, and B has a faithful irreducible $Z_p(B)$-module T such that $1 < C_T(O_{p'}(E^*_{\mathscr{E}(p)})) < T$. By Lemma 4.5, $M \otimes T$ is a faithful $Z_p(A \times B)$-module.

Since the restriction of $M \otimes T$ to B is isomorphic to a multiple of T, if we let U be any $Z_p(A \times B)$-composition factor of $M \otimes T$, then the restriction of U to B is also a multiple of T. Because of (4. 6), we have

$$(4.7) \qquad 1 < C_U(O_{p'}(E^*_{\mathscr{E}(p)})) = C_U(O_{p'}((E \times E^*)_{\mathscr{E}(p)})) < U ,$$

for each U.

Let $G = (A \times B)/C_{A\times B}(U)$, then $\bar{E} = (E \times E^*)C_{A\times B}(U)/C_{A\times B}(U)$ is an $\mathscr{E}$-subgroup of G. It follows from Lemma 3.3, and (4.7) that $O_{p'}(\bar{E}_{\mathscr{E}(p)}) > 1$. By (4.7) and the fact that G is an element of $\mathscr{F}$, it follows from Lemma 4.4 (a) that G lies in $\theta(p)$.

Let V be the direct sum of all $Z_p(A \times B)$-composition factors occurring in a composition series of $M \otimes T$. By Lemma 1.2 of [3], $F(A \times B) = F(A) \times F(B)$ is a p'-group, so the fact that $M \otimes T$ is faithful implies the restriction of V to $F(A \times B)$ is also faithful. By Lemma 3.2, V is a faithful completely reducible $Z_p(A \times B)$-module. Therefore $A \times B = (A \times B)/\bigcap_U C_{A\times B}(U)$ lies in $\langle\theta(p)\rangle$, and this completes the proof.

COROLLARY 4.7. *If $\mathscr{E} \ll \mathscr{F}$, and there is an element B in $\theta(p)$ such that $Z(B) = 1$, then $\langle\Phi(p)\rangle \subseteqq \langle\Phi(q)\rangle$ for each q in $\pi(\mathscr{E})$.*

Proof. By Lemma 4.7, if $A \in \Phi(p) - \theta(p)$, then $A \times B$ lies in $\langle\theta(p)\rangle$. Therefore A is an element of $\langle\theta(p)\rangle$, so $\langle\Phi(p)\rangle = \langle\theta(p)\rangle$. By Lemma 4.4 (d), if q is a prime in the characteristic of $\mathscr{E}$, then $\theta(p) \subseteqq \langle\Phi(q)\rangle$, hence $\langle\Phi(p)\rangle \subseteqq \langle\Phi(q)\rangle$.

THEOREM 4.8. *Suppose $\mathscr{E} \ll \mathscr{F}$, and $\theta(p)$, $\theta(r)$ are nonempty for two primes p, r in the characteristic of $\mathscr{E}$. Let $\{\mathscr{F}(q)\}$ be the unique minimal local definition of $\mathscr{F}$. Then*

$$\mathscr{F}(q) = \langle\Phi(q)\rangle \qquad \textit{for each } q \textit{ in } \pi(\mathscr{E}) .$$

Proof. We define a new formation $\mathscr{F}^*$ by setting

$$\langle\Phi(q)\rangle = \mathscr{F}^*(q) \qquad \text{for } q \in \pi(\mathscr{E}) ,$$
$$\mathscr{F}(q) = \mathscr{F}^*(q) \qquad \text{for } q \in \pi(\mathscr{E})' .$$

Since $\mathscr{E} \ll \mathscr{F}$, $\mathscr{F}^*(q)$ is contained in $\mathscr{F}(q)$ for each q, by Theorem 4.3. Therefore $\mathscr{F}^* \subseteqq \mathscr{F}$.

Let $\Phi^*(q)$ be the set specified in Definition 4.2 for the formation $\mathscr{F}^*$. Since $\mathscr{F}^* \subseteqq \mathscr{F}$, $\Phi^*(s) \subseteqq \Phi(s) \subseteqq \mathscr{F}^*(s)$ for each s in $\pi(\mathscr{E})$. Therefore Theorem 4.3 implies $\mathscr{E}$ is strongly contained in $\mathscr{F}^*$.

Suppose $\mathscr{F}^* \subset \mathscr{F}$. If G is an element of minimal order in $\mathscr{F} - \mathscr{F}^*$, then G is a semi-direct product, $G = F^*M$, where F^* is an $\mathscr{F}^*$-subgroup of G. F^* acts faithfully and irreducibly on the elementary abelian t-group M. Since G lies in $\mathscr{F} - \mathscr{F}^*$, $F^* \cong G/C_G(M)$ lies in $\mathscr{F}(t) - \mathscr{F}^*(t)$. For t in $\pi(\mathscr{E})'$, this contradicts the definition of $\mathscr{F}^*(t)$, hence t is a prime in the characteristic of $\mathscr{E}$.

Since $\mathscr{E} \ll \mathscr{F}^*$, F^*, as an $\mathscr{F}^*$-subgroup of G, must contain some $\mathscr{E}$-subgroup E of G. Thus for any prime q, the permutation representation on the cosets of $O_{q'}(E_{\mathscr{E}(q)})$ is faithful. By Lemma 4.4 (c), G lies in $\Phi(q)$ for each q in $\pi(\mathscr{E}) - \{t\}$.

By Lemma 4.4 (d), $\theta(q) \subseteqq \mathscr{F}^*(s)$ for each q, s in $\pi(\mathscr{E})$, hence if G lies in $\theta(q)$ for some q in $\pi(\mathscr{E}) - \{t\}$, then G lies in $\mathscr{F}^*(t)$. Suppose, therefore, that $G \in \Phi(q) - \theta(q)$ for each q in $\pi(\mathscr{E}) - \{t\}$. One of the primes p, r is unequal to t, say p. Then G is an element of $\Phi(p) - \theta(p)$ such that $Z(G) = 1$. Since $\theta(p)$ is nonempty, there is a group H in $\theta(p)$, so by Lemma 4.6, $G \times H$ is an element of $\mathscr{F}^*(t)$, hence in each case F^*, as a factor group of G, must lie in $\mathscr{F}^*(t)$, a contradiction.

Therefore $\mathscr{F}^* = \mathscr{F}$. Since $\{\mathscr{F}^*(q)\}$ forms a local definition for $\mathscr{F}^*$, we have $\Phi(q) \subseteqq \mathscr{F}(q) \subseteqq \mathscr{F}^*(q)$ for each q in the characteristic of $\mathscr{E}$, so the proof of Theorem 4.8 is complete.

Because we could not relax the hypothesis on the $\theta(p)$'s, we thought it appropriate to include

THEOREM 4.9. *Suppose $\mathscr{E} \ll \mathscr{F}$, and $p \in \pi(\mathscr{E})$. $\theta(p)$ is empty if, and only if, for each element F of $\mathscr{F}$, an $\mathscr{E}$-subgroup E of F either covers or avoids each p-chief factor of F.*

Proof. Suppose an $\mathscr{E}$-subgroup of F either covers or avoids each p-chief factor of F for every F in $\mathscr{F}$. Let $F \in \Phi(p)$, and let E be an $\mathscr{E}$-subgroup of F. Let M be a faithful irreducible $Z_p(F)$-module such that $C_M(O_{p'}(E_{\mathscr{E}(p)})) > 1$. By Corollary 3.8, and the fact that $\mathscr{E} \ll \mathscr{F}$, $F^* = FC_M(O_{p'}(F_{\mathscr{F}(p)}))$ is an $\mathscr{F}$-subgroup of FM, acts irreducibly on M, and does not avoid M. Therefore $F^* = FM$; M is a p-chief factor of $G = FM$ which is not avoided by the $\mathscr{E}$-subgroup $E^* = EC_M(O_{p'}(E_{\mathscr{E}(p)}))$ of G. Therefore $O_{p'}(E_{\mathscr{E}(p)})$ centralizes M, so $\theta(p)$ is empty.

Suppose $\theta(p)$ is empty, F lies in $\mathscr{F}$, and E is an $\mathscr{E}$-subgroup of

F which does not avoid the p-chief factor $K = L/N$ of F. Let $\bar{F} = F/C_F(K)$. Our first assertion is that the semi-direct product $\bar{F}K$ lies in $\mathscr{F}$ (the action of $\bar{F}$ on K is the action induced by the action of F on K). By Corollary 3.8, $\bar{F}^* = \bar{F}C_K(O_{p'}(\bar{F}_{\mathscr{F}(p)}))$ is an $\mathscr{F}$-subgroup of $\bar{F}K$. Therefore $\bar{F}^*$ acts irreducibly on K, and $\bar{F}^*/C_{\bar{F}^*}(K)$ is isomorphic to $\bar{F}$. Since F is in $\mathscr{F}$, $\bar{F}$ is in $\mathscr{F}(p)$. By Theorem 3.4, $\bar{F}^*$ covers K, hence $\bar{F}K$ is an element of $\mathscr{F}$.

$\bar{E} = EC_F(K)/C_F(K)$ is an $\mathscr{E}$-subgroup of $\bar{F}$. By Lemma 3.3, $\bar{E}_{\mathscr{E}(p)} = E_{\mathscr{E}(p)}C_F(K)/C_F(K)$. Because $E_{\mathscr{E}(p)}$ has a normal p-complement, it follows that $O_{p'}(\bar{E}_{\mathscr{E}(p)}) = O_{p'}(E_{\mathscr{E}(p)})C_F(K)/C_F(K)$. Therefore

$$C_K(O_{p'}(E_{\mathscr{E}(p)})) = C_K(O_{p'}(\bar{E}_{\mathscr{E}(p)})) .$$

$O_{p'}(E_{\mathscr{E}(p)})$ centralizes every p-section of E, hence it centralizes $(L \cap E)N/N$, a nonidentity subgroup of K. Therefore,

$$1 < C_K(O_{p'}(\bar{E}_{\mathscr{E}(p)})) \leqq K .$$

Thus $\bar{F}$ lies in $\Phi(p)$. $\theta(p)$ is empty, so it follows from Lemma 4.4 that $\bar{E}_{\mathscr{E}(p)}$ is a p-group. If U is any $\bar{E}$-composition factor K, then $\bar{E}_{\mathscr{E}(p)}$ centralizes U since it is contained in $O_p(\bar{E})$. Upon taking inverse images in E, we see that $C_E(U)$ contains $E_{\mathscr{E}(p)}$, so that $E/C_E(U)$ lies in $\mathscr{E}(p)$. By Theorem 3.4, E covers U, hence E also covers all of K.

5. Structure theorems.

Throughout this section we shall make the following assumptions:

Hypothesis I. $\mathscr{E}$ and $\mathscr{F}$ are saturated formations such that
(a) $\mathscr{N} \subseteqq \mathscr{E} \ll \mathscr{F}$;

(b) there is a nonempty formation $\mathscr{T}$ such that $\mathscr{E} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{T}\}$.

Our first theorem says that the structure of $\mathscr{F}$ is essentially the same as the structure of $\mathscr{E}$ in that there exists a formation $\mathscr{U}$ such that $\mathscr{F} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{U}\}$.

First we prove two lemmas.

LEMMA 5.1. *Let $\mathscr{T}$ be a nonempty formation. Let $\mathscr{G}$ be the formation locally defined by setting $\mathscr{G}(p) = \mathscr{T}$ for each p. Let $\mathscr{E} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{T}\}$. Then $\mathscr{G} = \mathscr{E}$.*

Proof. Suppose $G \in \mathscr{E}$. Because $O_{p'p}(G)$ contains $F(G)$, $G/F(G) \in \mathscr{T}$ implies that, for each p, $G/O_{p'p}(G)$ lies in $\mathscr{T}$. By (2.1), G is an element of $\mathscr{G}$.

If G is in $\mathscr{G}$, then $G/O_{p'p}(G)$ is in $\mathscr{T}$ for each prime p. Since $\mathscr{T}$ is a formation, and $F(G) = \bigcap_p O_{p'p}(G)$, $G/F(G)$ lies in $\mathscr{T}$. From this it follows that $\mathscr{G} = \mathscr{E}$.

LEMMA 5.2. *If G is a group with $\mathscr{E}$-subgroup E, and E lies in $\mathscr{T}$, then $E = G$. If $\{\mathscr{F}(q)\}$ is any local definition for $\mathscr{F}$, and G is an element of $\mathscr{T}$ such that $O_q(G) = 1$, then G lies in $\mathscr{F}(q)$.*

Proof. We prove our first statement by induction on the nilpotent length of G. If G is nilpotent, then G is already in $\mathscr{E}$, so there is nothing to prove. Since E lies in $\mathscr{T}$, $EF(G)$ lies in $\mathscr{E}$. Since E is an $\mathscr{E}$-subgroup of G, E covers $U/U_{\mathscr{E}}$ for any subgroup U of G which contains E. Therefore E contains $F(G)$. Set $\bar{G} = G/F(G)$, then $\bar{E} = E/F(G)$ is an $\mathscr{E}$-subgroup of $\bar{G}$. By induction, $\bar{E} = \bar{G}$, hence $E = G$.

Let $\{\mathscr{F}(q)\}$ be any local definition for $\mathscr{F}$. Suppose $G \in \mathscr{T}$, and $O_p(G) = 1$. Let M be the regular $Z_p(G)$-module, and form the semidirect product $G_1 = GM$. Since G lies in $\mathscr{T}$, G_1 lies in $\mathscr{E}$. It is a simple consequence of strong containment that $\mathscr{E} \subseteqq \mathscr{F}$, hence $G_1 \in \mathscr{F}$. Since $O_p(G) = 1$, and G acts faithfully on M, $M = O_{p'p}(G_1)$. Therefore G_1/M is an element of $\mathscr{F}(p)$. Since G is isomorphic to G_1/M, G lies in $\mathscr{F}(p)$. This completes the proof of the lemma.

THEOREM 5.3. *Suppose $\mathscr{E}$ and $\mathscr{F}$ satisfy Hypothesis* I. *Then there is a formation $\mathscr{U}$ containing $\mathscr{T}$, such that*

$$\mathscr{F} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{U}\} .$$

Proof. If $\mathscr{E} = \mathscr{F}$, the formation $\mathscr{T}$ satisfies the requirements of the theorem. Assume $\mathscr{E} \subset \mathscr{F}$. By Lemma 5.1, the family $\{\mathscr{E}(p) \mid \mathscr{E}(p) = \mathscr{T}$ for each $p\}$ is a local definition for $\mathscr{E}$. We shall use this family for the local definition of $\mathscr{E}$ throughout the remainder of the proof. Let $\{\mathscr{F}(q)\}$ be the unique minimal local definition of $\mathscr{F}$. A second application of Lemma 5.1 says that we need only show $\mathscr{F}(r) = \mathscr{F}(s)$ for each pair of primes r, s. In view of Theorem 4.8 and Corollary 4.7, we begin by examining the set $\theta(s)$ for various primes s. Since $\mathscr{N} \subseteqq \mathscr{E}$, $\pi(\mathscr{E})$ contains all primes, so $\theta(s)$ and $\Phi(s)$ are defined for each s.

Let G be an element of minimal order in $\mathscr{F} - \mathscr{E}$. By minimality, if N is any normal nonidentity subgroup of G, then G/N lies in $\mathscr{E}$. Therefore $G_{\mathscr{E}}$ is the unique minimal normal subgroup of G. If E is an $\mathscr{E}$-subgroup of G, then $EG_{\mathscr{E}} = G$, and $E \cap G_{\mathscr{E}} = 1$. Furthermore, E acts faithfully and irreducibly on $G_{\mathscr{E}}$. We set $M = G_{\mathscr{E}}$, and note that M is an elementary abelian p-group for some prime p.

Since G is not in $\mathscr{E}$, Lemma 5.2 implies E is not an element of $\mathscr{T}$. Therefore $F(E) \geqq E_{\mathscr{T}} > 1$. But it follows from Lemma 1.2 of [3] that $F(E)$ is a p'-group, so for some prime r distinct from p, $E_{\mathscr{T}}$ has a nonidentity normal Sylow r-subgroup R. If s is a prime distinct from r, then

$$O_{s'}(E_{\mathscr{E}(s)}) = O_{s'}(E_{\mathscr{T}}) \geqq R > 1 \ .$$

Because M is the unique minimal normal subgroup of G, and $E \cap M = 1$, the permutation representation on the cosets of $O_{s'}(E_{\mathscr{E}(s)})$ is faithful for each s. By Lemma 4.4, G lies in $\theta(s)$ for each prime s distinct r and p. Since E is faithful and irreducible on M, the center of G is trivial.

Now fix a prime $s \neq r$, p. Then G is in $\theta(s)$, so there exists a faithful irreducible $Z_s(G)$-module J such that $1 < C_J(O_{s'}(E_{\mathscr{E}(s)})) < J$. We let G^* be the semi-direct product GJ. Since E is isomorphic to an $\mathscr{E}$-subgroup of G^*/J, it follows from Lemma 2.7 and Corollary 3.8 that $E^* = EC_J(O_{s'}(E_{\mathscr{E}(s)}))$ is an $\mathscr{E}$-subgroup of G^*. An $\mathscr{F}$-subgroup of G^* covers G^*/J since G lies in $\mathscr{F}$; it cannot avoid J because $\mathscr{E} \ll \mathscr{F}$. Therefore G^* lies in $\mathscr{F}$. Because E is a quotient group of E^*, and E is not in $\mathscr{T}$, E^* is not in $\mathscr{T}$, hence

$$1 < (E^*)_{\mathscr{T}} = (E^*)_{\mathscr{E}(p)} \leqq E_{\mathscr{T}} C_J(O_{s'}(E_{\mathscr{E}(s)})) \ .$$

$(E^*)_{\mathscr{T}}$ is a p'group because $E_{\mathscr{T}}$ is a subgroup of the p'-group $F(E)$, and s is not equal to p. The permutation representation on the cosets of $(E^*)_{\mathscr{T}}$ is faithful since J is the unique minimal normal subgroup of G^*, and $(E^*)_{\mathscr{T}} \cap J \leqq C_J(O_{s'}(E_{\mathscr{E}(s)})) < J$. It follows from parts (a) and (c) of Lemma 4.4 that G^* lies in $\theta(p)$. By construction, the center of G^* is trivial, hence we have established:

(5.1) If $s \neq r$, then there is a group X in $\theta(s)$ such that, $Z(X) = 1$.

We can now apply the results of § 4. The characteristic of $\mathscr{E}$ contains all primes, so by Theorem 4.8, and (5.1), $\mathscr{F}(s) = \langle \Phi(s) \rangle$ for each prime s. By Corollary 4.7, we have

$$(5.2) \qquad \begin{aligned} \mathscr{F}(s) &= \mathscr{F}(q) && \text{for } s,\ q \text{ in } r' \ , \\ \mathscr{F}(s) &\subseteqq \mathscr{F}(r) && \text{for each } s \ . \end{aligned}$$

For $s \neq r$, we set $\mathscr{U} = \mathscr{F}(s)$. The final step in the proof will be to show $\Phi(r) \subseteqq \mathscr{U}$.

By part (d) of Lemma 4.4, $\theta(r) \subseteqq \mathscr{F}(s)$ for each s, so $\theta(r) \subseteqq \mathscr{U}$. Suppose $H \in \Phi(r) - \theta(r)$, and E is an $\mathscr{E}$-subgroup of H. Then it follows from Lemma 4.4 that $E_{\mathscr{T}}$ is an r-group.

If $E_{\mathscr{T}} = 1$, then E is in $\mathscr{T}$. By Lemma 5.2, $E = H$, and if s is any prime not dividing $r|H|$, $O_s(H) = 1$, so H lies in $\mathscr{F}(s) = \mathscr{U}$.

Suppose $E_{\mathscr{S}} > 1$. Since H has a faithful irreducible $Z_r(H)$-module, $O_r(H) = 1$, so the permutation representation on the cosets of $E_{\mathscr{S}}$ is faithful. Since H is in $\Phi(r)$, H lies in $\mathscr{F}$. Thus if s is a prime which does not divide the order of H, it follows from part (b) of Lemma 4.4 that H lies in $\langle\Phi(s)\rangle$. Therefore

$$\Phi(r) \subseteqq \mathscr{U} \subseteqq \mathscr{F}(r) = \langle\Phi(r)\rangle .$$

Since $\mathscr{U} = \mathscr{F}(s)$ for each s, Lemma 5.1 says that

$$\mathscr{F} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{U}\} .$$

The fact that $\mathscr{U}$ contains $\mathscr{T}$ is a consequence of part (b) of Lemma 4.4.

We are interested in finding formations which are maximal in the partial ordering $\ll$. Since $\mathscr{E} \ll \mathscr{F}$ implies $\mathscr{E} \subseteqq \mathscr{F}$, we shall assume $\mathscr{E} \subset \mathscr{F}$, as well as Hypothesis I. Since $\mathscr{E} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{T}\}$, we fix our local definition for $\mathscr{E}$ by setting $\mathscr{E}(p) = \mathscr{T}$ for each p. We assume that $\{\mathscr{F}(p)\}$ is the minimal local definition for $\mathscr{F}$. By the proof of Theorem 5.3, there is a formation $\mathscr{U}$, containing $\mathscr{T}$, such that $\mathscr{F}(p) = \mathscr{U}$ for each p. Since $\mathscr{E} \subset \mathscr{F}$, we must have $\mathscr{T} \subset \mathscr{U}$.

Before stating our main theorem, we prove several lemmas. The proof of Lemma 5.5 contains the essential construction used in the proof of the main theorem.

LEMMA 5.4, *Let G be a group, and $1 < H \leqq G$. Assume that the permutation representation of G on the cosets of H is faithful. If M is the $Z_p(G)$-module which affords this representation, set $U = \bigcap_{g \in G} C_M(H)g$. Then U is a $Z_p(G)$-submodule of M, and M/U is a faithful $Z_p(G)$-module.*

Proof. We can let the cosets of H in G be a Z_p-basis for M, i.e., let $M = Z_p \cdot H + Z_p \cdot Hg_2 + \cdots + Z_p Hg_s$, where $s = [G : H]$, and the operation of G on M is by right multiplication.

For each g in $G, C_M(H)g = C_M(H^g)$, hence $U = C_M(\bigcup_{g \in G} H^g)$. In other words, if N is the normal closure of H in G, then $U = C_M(N)$. Since N is normal in G, U is a $Z_p(G)$-submodule of M.

Let $\mathfrak{O}_1, \cdots, \mathfrak{O}_m$ be the orbits of the cosets Hg_i under action by N. Since $N \lhd G$, G permutes these orbits transitively, thus all orbits have the same number of elements $[N:H]$. Since H is not normal in G, it follows that $[N:H] \geqq 3$.

For each i, let $\mathfrak{O}_i = \{Hg_{i1}, \cdots, Hg_{ir}\}$ where $r = [N:H]$. Set $u_i = \sum_{j=1}^{r} Hg_{ij}$. It is a standard result that the elements u_i of M form a Z_p-basis for $C_M(N)$. Hence a Z_p-basis for M/U consists of the cosets:

$$\{U + Hg_{ij} \mid 1 \leqq i \leqq m,\ 1 \leqq j \leqq r-1\} .$$

Suppose x lies in the kernel of the representation of G on M/U. Then for $1 \leqq i \leqq m$, and $1 \leqq j \leqq r-1$ we have

$$Hg_{ij}x - Hg_{ij} = \sum_{k=1}^{m} \alpha_k u_k ,$$

where the α_k are suitably chosen elements of Z_p. Since $Hg_{ij}x$ is a coset, and each u_k is a sum of at least three distinct cosets, we must have $\alpha_k = 0$ for each k. Since x permutes the orbits of N, it follows from the fact that x fixes Hg_{i1} that x fixes each orbit $\mathfrak{O}_i$. This, together with our above remarks show that x lies in the kernel of M. Since M is faithful, so is M/U.

LEMMA 5.5. *Let $\mathscr{E}$ and $\mathscr{F}$ satisfy Hypothesis* I, $\mathscr{E} \subset \mathscr{F}$, *and suppose there is an element H in $\mathscr{U} \cap \mathscr{E} - \mathscr{F}$ such that $O_p(H) = 1$. Then*

$$\mathscr{U} \supseteqq \{G \in \mathscr{F} \mid F(G) \textit{ is a } p\textit{-group}\} .$$

Proof. Let G be an element of $\mathscr{F}$ such that $F(G)$ is a p-group. Let E be an $\mathscr{E}$-subgroup of G, and assume $O_{p'}(E_{\mathscr{F}}) > 1$. Since $F(G)$ is a p-group, $O_{p'}(G) = 1$, so the permutation representation on the cosets of $O_{p'}(E_{\mathscr{F}})$ is faithful. Let M be the $Z_p(G)$-module which gives this representation, and let $U = C_M(N)$, where N is the normal closure in G of $O_{p'}(E_{\mathscr{F}})$. By Lemma 5.4, the $Z_p(G)$-module $M^* = M/U$ is faithful. Let $X = GM^*$. Then $F(X) = F(G)M^*$, so $X/F(X)$ is isomorphic to $G/F(G)$. Since G is in $\mathscr{F}$, so is X. Now $E^* = EC_{M^*}(O_{p'}(E_{\mathscr{F}}))$ is an $\mathscr{E}$-subgroup of X. Since U centralizes $O_{p'}(E_{\mathscr{F}})$, we have $C_{M^*}(O_{p'}(E_{\mathscr{F}})) = C_M(O_{p'}(E_{\mathscr{F}}))/U$. Let T be the intersection of all conjugates of E^* in X. Since $E^* \cap M^* = C_{M^*}(O_{p'}(E_{\mathscr{F}}))$, it follows that

$$T \cap M^* = C_{M^*}(N) = 1 .$$

But if K is a normal subgroup of X, whose intersection with M^* is trivial, then K centralizes M^*. $C_X(M^*) = C_G(M^*)M^*$, so the fact that G is faithful on M^* says that M^* is self-centralizing in X, consequently $K = 1$. From this we have $T = 1$, so the representation of X on the conjugates of E^* is faithful. Certainly it also follows that the representation of X on $E^*_{\mathscr{F}}$ is faithful, so if t is any prime which does not divide the order of X, then $X \in \langle \Phi(t) \rangle = \mathscr{U}$, by Lemma 4.4. Therefore G, as a factor group of X, also lies in $\mathscr{U}$.

We may now assume $O_{p'}(E_{\mathscr{F}}) = 1$, so $E_{\mathscr{F}}$ is a p-group. It is time to use H. If $R = I_1 \dotplus \cdots \dotplus I_t$ is a decomposition of the regular $Z_p(H)$-module into its principal indecomposable constituents, we let

$K_j = I_j/U_j$ for each j. Here U_j is the unique maximal submodule of I_j. Since R is faithful, and $F(H)$ is a p'-group, it follows from Lemmas 3.1 and 3.2 that $R^* = K_1 \dotplus \cdots \dotplus K_t$ is faithful. Since H is not an element of $\mathscr{T}$, it follows that for some j, $B = H/C_H(K_j)$ does not lie in $\mathscr{T}$. Let $K = K_j$, then B is an element of $\mathscr{U} \cap \mathscr{E} - \mathscr{T}$ which has K as a faithful irreducible $Z_p(B)$-module.

Let S be the regular $Z_p(G)$-module, and set $W = (B \times G)(K \otimes S)$, where the action of $B \times G$ on $K \otimes S$ by conjugation is the canonical action of $B \times G$ on the module $K \otimes S$. To show G lies in $\mathscr{U}$, it is sufficient to show W is in $\mathscr{U}$, since G is a factor group of W.

Since B has a faithful irreducible $Z_p(B)$-module, $F(B)$ is a p'-group. Therefore if N is the kernel of the representation of $F(B \times G)$ on $K \otimes S$, then $N = N \cap F(B) \times N \cap F(G)$. Since B and G act faithfully on $K \otimes S$, $F(B \times G)$ is faithful on $K \otimes S$. By Lemma 3.2, $K \otimes S$ is faithful. Therefore $O_{p'}(W) = 1$, so we have $F(W) = F(G)(K \otimes S)$. Since $W/F(W)$ is isomorphic to $B \times (G/F(G))$, an element of $\mathscr{U}$, it follows that W lies in $\mathscr{F}$.

An $\mathscr{E}$-subgroup of $B \times G$ is $B \times E$, so

$$E^* = (B \times E)C_{K\otimes S}(O_{p'}((B \times E)_{\mathscr{T}}))$$

is an $\mathscr{E}$-subgroup of W. Since $B \in \mathscr{U} \cap \mathscr{E} - \mathscr{T}$, $1 < B_{\mathscr{T}} \leqq F(B)$, so $B_{\mathscr{T}}$ is a p'-group. By assumption $E_{\mathscr{T}}$ is a p-group. Let V be the collection of elements of $B_{\mathscr{T}}$ which appear as components of elements of $(B \times E)_{\mathscr{T}}$. Then V is a normal subgroup of B, and $(B \times E)_{\mathscr{T}} \leqq V \times E$. Since B/V is isomorphic to $(B \times E)/(V \times E)$, B/V lies in $\mathscr{T}$, hence $V = B_{\mathscr{T}}$. If $v \in V$, then for some u in $E_{\mathscr{T}}$, $v \times u$ lies in $(B \times E)_{\mathscr{T}}$. Since $B_{\mathscr{T}}$ is a p'-group, and $E_{\mathscr{T}}$ is a p-group, v is equal to a power of $v \times u$. Therefore

$$B_{\mathscr{T}} = O_{p'}((B \times E)_{\mathscr{T}}) . \tag{5.3}$$

Now the restriction of $K \otimes S$ to $B_{\mathscr{T}}$ is a multiple of the restriction of K to $B_{\mathscr{T}}$, so it follows from Lemma 3.1 that $C_{K\otimes S}(B_{\mathscr{T}}) = 1$. By (5.3), $B \times E$ is an $\mathscr{E}$-subgroup of W.

Let t be a prime which does not divide $|W|$. The fact that the representation of W on the cosets of $B \times E$ is faithful implies that the same is true of the representation of W on the cosets of $(B \times E)_{\mathscr{T}}$. By part (b) of Lemma 4.4, W is an element of $\langle \Phi(t) \rangle = \mathscr{U}$. Therefore G lies in $\mathscr{U}$ in every case, so the proof of Lemma 5.5 is complete.

Because of the preceeding lemma, we give

DEFINITION 5.6. Let $\eta = \{p \mid \mathscr{U} \cap \mathscr{E} - \mathscr{T}$ contains a group H with $O_p(H) = 1\}$. We call a prime p *special* if p is an element of η'.

LEMMA 5.7. *If $\mathscr{E}$ and $\mathscr{F}$ satisfy Hypothesis* I, *and* $\mathscr{E} \subset \mathscr{F}$, *then there is at most one special prime.*

Proof. Let G be an element of minimal order in $\mathscr{F} - \mathscr{E}$. Then G is the semi-direct product EM where E is an $\mathscr{E}$-subgroup of G, and M is the unique minimal normal subgroup of G.

Since E acts faithfully and irreducibly on M, $M = F(G)$. By Lemma 5.2, E is not an element of $\mathscr{T}$, and since $G \in \mathscr{F}$, $G/F(G)$ lies in $\mathscr{U}$, so $E \in \mathscr{U} \cap \mathscr{E} - \mathscr{T}$.

Since $O_r(E) \cap O_s(E) = 1$ for two distinct primes r, s, $E/O_t(E)$ lies in $\mathscr{T}$ for at most one prime t. If $s \neq t$, then $E/O_s(E) \in \mathscr{U} \cap \mathscr{E} - \mathscr{T}$, so $\eta' \subseteqq \{t\}$.

REMARK. In, general, we cannot control the choice of G enough to be certain that there are no special primes. This is the basis for the example in § 6, and the reason behind

Hypothesis II. Let $G = EM$ be a fixed element of minimal order in $\mathscr{F} - \mathscr{E}$. If r is any prime such that $E/O_r(E)$ lies in $\mathscr{T}$, we assume that $\mathscr{S}(r')$, the formation of all r'-groups, is not contained in $\mathscr{T}$. (Such a prime does not necessarily exist.)

THEOREM 5.8. *Suppose $\mathscr{E}$ and $\mathscr{F}$ satisfy Hypotheses* I *and* II. *If $\mathscr{E} \subset \mathscr{F}$, then $\mathscr{F} = \mathscr{S}$, the collection of all solvable groups.*

Proof. Our first step is to show that $\mathscr{U}$ contains the collection, $\mathscr{S}(\eta)$, of all solvable η-groups. By Lemma 5.5, the fact that $\mathscr{N} \subseteqq \mathscr{E} \subseteqq \mathscr{F}$ shows that $\mathscr{U}$ contains the collection of all nilpotent η-groups. Proceeding by induction, we assume that $\mathscr{U}$ contains the collection, $\mathscr{N}^i(\eta)$, of all solvable η-groups of nilpotent length at most i. Since

$$\mathscr{N}^{i+1}(\eta) = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{N}^i(\eta)\} ,$$

$\mathscr{F}$ contains all solvable η-groups of nilpotent length at most $i + 1$.

Let $G \in \mathscr{N}^{i+1}(\eta)$, and $F(G) = P_1 \times \cdots \times P_s$, where P_i is the Sylow p_i-subgroup of $F(G)$. Set $N_i = \prod_{k \neq i} P_k$, and let R_i be the regular $Z_{p_i}(G/N_i)$-module for each $i = 1, \cdots, s$. We allow G to act on the direct product $R = R_1 \times \cdots \times R_s$ by conjugation according to the rule

$$(5.4) \qquad (r_1 \times r_2 \times \cdots \times r_s)^g = r_1(N_1 g) \times r_2(N_2 g) \times \cdots \times r_s(N_s g) .$$

Then we form the semi-direct product $X = GR$. By construction, N_i centralizes the p_i-group R_i, hence the group $F(G)R$ is nilpotent. Since $F(X)/R$ is a normal nilpotent subgroup of X/R, and X/R is isomorphic to G, $F(X) \leqq F(G)R$. Therefore $F(G)R$ is the Fitting subgroup of X, and $X/F(X)$ is isomorphic to $G/F(G)$. Since $G/F(G)$ lies in $\mathscr{U}$, it follows that X lies in $\mathscr{F}$.

For each i, set $\bar{X}_i = X/N_i(\prod_{k\neq i}R_k)$, $\bar{R}_i = N_iR/N_i(\prod_{k\neq i}R_k)$, and $\bar{G} = G(\prod_{k\neq i}R_k)/N_i(\prod_{k\neq i}R_k)$. By modularity

$$G(\textstyle\prod_{k\neq i}R_k) \cap N_iR = N_i(\prod_{k\neq i}R_k) .$$

Thus $\bar{X}_i$ is the semi-direct product of $\bar{R}_i$ by $\bar{G}_i$, hence

$$C_{\bar{X}_i}(\bar{R}_i) = C_{\bar{G}_i}(\bar{R}_i)\bar{R}_i .$$

Because $\bar{G}_i$ acts faithfully on $\bar{R}_i$, it follows that $\bar{R}_i$ is a self-centralizing normal p_i-subgroup of $\bar{X}_i$. Therefore $O_{p_i'}(\bar{X}_i) = 1$, so $F(\bar{X}_i)$ is a p_i-group. But p_i lies in η, so by Lemma 5.5, $\bar{X}_i$ is an element of $\mathscr{U}$ for each i. Since the intersection of the groups $N_i(\prod_{k\neq i}R_k)$ over all i is the identity, X is an element of $\mathscr{U}$. Therefore G lies in $\mathscr{U}$, and by induction it follows that $\mathscr{S}(\eta) \subseteqq \mathscr{U}$.

By Lemma 5.7, if EM is the minimal element of $\mathscr{F} - \mathscr{E}$ mentioned in Hypothesis II, then there is at most one prime r^* such that $E/O_{r^*}(E)$ lies in $\mathscr{T}$, thus η contains $(r^*)'$. Therefore,

$$\mathscr{S}((r^*)') \subseteqq \mathscr{S}(\eta) \subseteqq \mathscr{U} \subseteqq \mathscr{F} .$$

Suppose $\mathscr{E}$ does not contain $\mathscr{S}((r^*)')$, and let $G^* = E^*M^*$ be an element of minimal order in $\mathscr{S}((r^*)') - \mathscr{E}$. By Lemma 5.2, E^* is an element of $\mathscr{U} \cap \mathscr{E} - \mathscr{T}$, and since $E^* \in \mathscr{S}((r^*)')$, $O_{r^*}(E^*) = 1$. Therefore η contains all primes.

Now suppose $\mathscr{E}$ contains $\mathscr{S}((r^*)')$. By assumption $\mathscr{T}$ does not contain $\mathscr{S}((r^*)')$, so we can choose H in $\mathscr{S}((r^*)') \subseteqq \mathscr{U}$, H is an element of $\mathscr{U} \cap \mathscr{E} - \mathscr{T}$ with $O_{r^*}(H) = 1$. Therefore η contains all primes in every case, so we have

$$\mathscr{S} = \mathscr{S}(\eta) \subseteqq \mathscr{U} \subseteqq \mathscr{F} \subseteqq \mathscr{S} ,$$

which completes the proof of Theorem 5.8.

COROLLARY 5.9. *Let $\mathscr{N}^i$ be the collection of groups of nilpotent length at most i. Then $\mathscr{N}^i$ is maximal with respect to the partial ordering $\ll$.*

Proof. If we set $\mathscr{N}^0 = \{1\}$, then for $i \geqq 1$,

$$\mathscr{N}^i = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{N}^{i-1}\} .$$

For each prime p, $\mathscr{S}(p')$ is not contained in $\mathscr{N}^{i-1}$, hence the hypothesis of Theorem 5.8 is satisfied. The result follows from

Theorem 5.8.

6. **An example.** Let r be a prime. Throughout this section, we let $\mathscr{R}$ be the formation of all group G such that $G/F(G)$ is an r'-group. For each prime p, we set $\mathscr{R}(p) = \mathscr{S}(r')$; $\{\mathscr{R}(p)\}$ forms a local definition for $\mathscr{R}$ because of Lemma 5.1. In this section, we shall characterize the formations which strongly contain $\mathscr{R}$. The formation $\mathscr{R}$ provides an example which shows that Hypothesis II is not redundant.

LEMMA 6.1. *Let G be a group with Sylow r-subgroup R. Then $N_G(R)$ is an R-subgroup of G.*

Proof. Clearly $N_G(R)$ lies in $\mathscr{R}$. Suppose $N_G(R) \leqq U \leqq G$. We need to show $N_G(R)$ covers $U/R_{\mathscr{R}}$. Clearly $U_{\mathscr{R}}$ is the smallest normal subgroup of U whose factor group has a normal Sylow r-subgroup. If V is the smallest normal subgroup of U whose factor group U/V is an r'-group, then $R \leqq V$, so V is transitive on the Sylow r-subgroups of U. Consequently $N_G(R)V = U$. Since R covers every r-section of U, it follows that $N_G(R)$ covers $U/U_{\mathscr{R}}$. By definition, $N_G(R)$ is an $\mathscr{R}$-subgroup of G.

Suppose $\mathscr{F} \gg \mathscr{R}$, and $\mathscr{F} \supset \mathscr{R}$. If $\{\mathscr{F}(q)\}$ is the minimal local definition of $\mathscr{F}$, it follows from Theorem 5.3 that $\mathscr{F}(q) = \mathscr{F}(s)$ for each q, s. We set $\mathscr{U} = \mathscr{F}(q)$. If H lies in $\mathscr{U} \cap \mathscr{R}$, then H has a normal Sylow r-subgroup, so $H/O_r(H)$ lies in $\mathscr{S}(r')$. Therefore, Hypothesis II is violated for the prime r. It follows from Lemma 5.7 that r is the unique special prime associated with $\mathscr{F}$ and $\mathscr{R}$. The next theorem gives a class of formations which strongly contain $\mathscr{R}$.

THEOREM 6.2. *Let $\mathscr{T}$ be a nonempty formation. Let*

$$\mathscr{U} = \{G \in \mathscr{S} \mid G/O_{r'}(G) \in \mathscr{T}\} ,$$

then $\mathscr{U}$ is a formation. If

$$\mathscr{F} = \{G \in \mathscr{S} \mid G/F(G) \in \mathscr{U}\} ,$$

then $\mathscr{F}$ strongly contains $\mathscr{R}$.

Proof. Suppose $G \in \mathscr{U}$, and $N \lhd G$. Then $O_{r'}(G)N/N \leqq O_{r'}(G/N)$. Since $G/O_{r'}(G) \in \mathscr{T}$, the same is true of $(G/N)/O_{r'}(G/N)$. Therefore G/N is an element of $\mathscr{U}$.

Now let N_1, N_2 be two normal subgroups of G such that G/N_i lies in $\mathscr{U}$ for each i. For each i, let $M_i/N_i = O_{r'}(G/N_i)$, then G/M_i

lies in $\mathscr{T}$ for each i. Since $\mathscr{T}$ is a formation, $G/M_1 \cap M_2$ is in $\mathscr{T}$. For each i, $(M_1 \cap M_2)N_i/N_i$ is an r'-group, so it follows that the factor group of $G/N_1 \cap N_2$ by $O_{r'}(G/N_1 \cap N_2)$ lies in $\mathscr{T}$. Therefore $\mathscr{U}$ is a formation.

To show $\mathscr{R} \ll \mathscr{F}$, it is sufficient to show that $\Phi(p) \subseteqq \mathscr{U}$ for each prime p. Suppose $G \in \Phi(r)$, then G has a faithful irreducible $Z_r(G)$-module. This means that $O_r(G) = 1$. Since G lies in $\mathscr{F}$, $G/F(G)$ lies in $\mathscr{U}$. But $F(G)$ is an r'-group, so it follows that $G/O_{r'}(G)$ lies in $\mathscr{T}$. Therefore G lies in $\mathscr{U}$.

Suppose $G \in \Phi(p)$ for p distinct from r. An $\mathscr{R}$-subgroup of G is $N_G(R)$ where R is a Sylow r-subgroup of G. Since $p \neq r$, $O_{p'}(N_G(R)_{\mathscr{S}(r')}) = R$. Therefore G has a faithful irreducible $Z_p(G)$-module J such that $1 < C_J(R) \leqq J$. By Lemma 3.1, either $C_J(O_r(G)) = J$, or it is the identity. The latter possibility cannot occur because $1 < C_J(R) \leqq C_J(O_r(G))$. Therefore the fact that J is faithful says that $O_r(G) = 1$, so $F(G)$ is an r'-group. G lies in $\mathscr{F}$, so the same argument that was used in the preceeding paragraph shows that $G/O_{r'}(G)$ is in $\mathscr{T}$. Therefore $G \in \mathscr{U}$. By Theorem 4.3, $\mathscr{R}$ is strongly contained in $\mathscr{F}$.

Since our choice of $\mathscr{T}$ is arbitrary, it follows that we can choose an infinite number of distinct formations which strongly contain $\mathscr{R}$. Our last theorem shows that we have actually found all formations which strongly contain $\mathscr{R}$.

THEOREM 6.3. *Suppose $\mathscr{F} \gg \mathscr{R}$, and $\{\mathscr{F}(q)\}$ is the minimal local definition for $\mathscr{F}$. Then there is a nonempty formation $\mathscr{T}$ such that*

$$\mathscr{F}(q) = \{G \in \mathscr{S} \mid G/O_{r'}(G) \in \mathscr{T}\} .$$

Proof. Suppose $\mathscr{F} \supset \mathscr{R}$. By Theorem 5.3, there is a formation $\mathscr{U}$ such that $\mathscr{F}(q) = \mathscr{U}$ for each q. Our first step is to show that $\mathscr{U}$ is the smallest formation generated by the set $\{H \in \mathscr{F} \mid O_r(H) = 1\}$. Let $\mathscr{U}^*$ be the smallest formation generated by this set.

Suppose $H \in \mathscr{F}$, and $O_r(H) = 1$. Let $K = I_1 \dotplus \cdots \dotplus I_s$ be the decomposition of the regular $Z_r(H)$-module K into principal indecomposable submodules. By Lemmas 3.1, and 3.2, and the fact that $F(H)$ is an r'-group, it follows that H acts faithfully on $J = J_1 \dotplus \cdots \dotplus J_s$, where for each k, J_k is the quotient of I_k by its unique maximal submodule. For each k, set $H_k = H/C_H(J_k)$. Then J_k is a faithful irreducible $Z_r(H_k)$-module. If R_k is a Sylow r-subgroup of H_k, then $N_{H_k}(R_k)$ is an $\mathscr{R}$-subgroup of H_k, and by definition, it follows that H_k lies in $\Phi(r)$ for each k. Since $\mathscr{R} \ll \mathscr{F}$, we have $H_k \in \mathscr{F}(r) = \mathscr{U}$. Since H is faithful on J, H lies in $\mathscr{U}$. We have just shown that all generators of $\mathscr{U}^*$ lie in $\mathscr{U}$, therefore $\mathscr{U}^*$ is contained in $\mathscr{U}$. We know that $\mathscr{U}$ is the smallest formation generated by $\Phi(r)$, from the proof of Theorem 5.3. Thus if we show $\Phi(r) \subseteqq \mathscr{U}^*$, we have shown $\mathscr{U} \subseteqq \mathscr{U}^*$. If G lies in $\Phi(r)$, then G has a faithful irreducible $Z_r(G)$-module, and G lies in $\mathscr{F}$. Then $O_r(G) = 1$, so by definition G lies in $\mathscr{U}^*$. This shows $\mathscr{U} = \mathscr{U}^*$.

Let $\mathscr{T}$ be the smallest formation generated by the set $\{H/O_{r'}(H) \mid H \in \mathscr{U}\}$. Set $\mathscr{U}' = \{G \in \mathscr{S} \mid G/O_{r'}(G) \in \mathscr{T}\}$. We want to show $\mathscr{U} = \mathscr{U}'$. By construction $\mathscr{U} \subseteqq \mathscr{U}'$.

Since the generators of $\mathscr{T}$ are elements of $\mathscr{U}$, we must have $\mathscr{T} \subseteqq \mathscr{U}$. Therefore, if $G \in \mathscr{U}'$, then $G/O_{r'}(G)$ lies in $\mathscr{U}$. To show G lies in $\mathscr{U}$, we use induction on the nilpotent length of $O_{r'}(G)$. If $O_{r'}(G)$ is nilpotent, then it follows that $G/F(G)$ lies in $\mathscr{U}$. Thus $G \in \mathscr{F}$. By our first paragraph, $G/O_r(G)$ lies in $\mathscr{U}$, so G also lies in $\mathscr{U}$ since $O_r(G) \cap O_{r'}(G) = 1$.

We note that $O_{r'}(G/F(O_{r'}(G))) = O_{r'}(G)/F(O_{r'}(G))$, hence by induction, if G is in $\mathscr{U}'$, then $G/F(O_{r'}(G))$ is in $\mathscr{U}$. Therefore G lies in $\mathscr{F}$. By our first paragraph $G/O_r(G)$ is in $\mathscr{U}$, so once again it follows that G lies in $\mathscr{U}$. Therefore $\mathscr{U} = \mathscr{U}'$. This completes the proof in the case when $\mathscr{R} \subset \mathscr{F}$.

If $\mathscr{R} = \mathscr{F}$, we let $\mathscr{T}$ be the formation consisting only of the identity. We must then show that $\{\mathscr{R}(q)\}$ is the minimal local definition for $\mathscr{R}$.

Let $\{\mathscr{R}^*(q)\}$ be the minimal local definition for $\mathscr{R}$. Suppose p is an arbitrary prime, $G \in \mathscr{S}(r') = \mathscr{R}(p)$, and t is a prime which does not divide $rp\,|G|$. Let K be the regular $Z_t(G)$-module. Set $G^* = GK$. Let K_1 be the regular $Z_p(G^*)$-module. Let $G' = G^*K_1$. Since G acts faithfully on K, and G^* acts faithfully on K_1, $O_{p'p}(G') = K_1$. Depending on the choice of p, G' is either an r'-group, or has K_1 as a normal Sylow r-subgroup. Therefore $G' \in \mathscr{R}$, hence $G'/O_{p'p}(G') = G'/K_1$ lies in $\mathscr{R}^*(p)$. Therefore $\mathscr{S}(r') \subseteqq \mathscr{R}^*(p)$. This completes the proof.

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