Vol. 29, No. 3, 1969

EXTENDING BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES OF A POLYDISC

HERBERT ALEXANDER

Let E be a subvariety of the unit polydisc

$$U^N = \{(z_1, \cdots, z_N) \in C^N : |z_i| < 1, 1 \leqq i \leqq N\}$$

such that E is the zero set of a holomorphic function f on U^N, i.e., $E = Z(f)$ where $Z(f) = \{z \in U^N : f(z) = 0\}$. This amounts to saying that E is a subvariety of pure dimension $N-1$. In [2] Walter Rudin proved that if E is bounded away from the torus $T^N = \{(z_1, \cdots, z_N) \in C^N : |z_i| = 1, 1 \leqq i \leqq N\}$, then there is a bounded holomorphic function F on U^N such that $E = Z(F)$. Call such a subvariety E, that is, a pure $N-1$ dimensional subvariety of U^N bounded from T^N, a *Rudin variety*. We are interested in the following question: When is it possible to extend every bounded holomorphic function on a Rudin variety E to one on U^N? Examples show this is not always possible. We will say that a pure $N-1$ dimensional subvariety E of U^N is a *special Rudin variety* if there exists an annular domain $Q^N = \{(z_1, \cdots, z_N) \in C^N : r < |z_i| < 1, 1 \leqq i \leqq N\}$ for some $r (0 < r < 1)$ and a $\delta > 0$ such that

(i) $E \cap Q^N = \emptyset$ and

(ii) if $1 \leqq k \leqq N$ and $(z', \alpha, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $(z', \beta, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $\alpha \neq \beta$, then $|\alpha - \beta| \geqq \delta$. Obviously (i) implies that a special Rudin variety is a Rudin variety. We have the

THEOREM. If E is a special Rudin variety in U^N, then there exists a bounded linear transformation $T: H^\infty(E) \to H^\infty(U^N)$ (where H^∞ is the corresponding Banach space of bounded holomorphic functions under sup norm) which extends each bounded holomorphic function on E to one on U^N.

REMARK. The proof of the theorem is a modification of the proof in [2] of Rudin's theorem: the changes reflecting the fact that we are dealing with an additive problem while Rudin's was of a multiplicative nature. I am further indebted to Professor Rudin for some comments (on a preliminary version of this paper) which led to improvement in the hypothesis of the theorem.

The following lemma is well-known and easy to prove.

LEMMA 1. *If $0 < r < 1$ and $Q = \{\lambda \in C : r < |\lambda| < 1\}$ and*

$$h(\lambda) = \sum_{-\infty}^{\infty} a_n \lambda^n, \; h_1(\lambda) = \sum_{-\infty}^{-1} a_n \lambda^n$$

for $\lambda \in Q$, *then*

$$\| h_1 \|_Q \leqq K \| h \|_Q$$

where K (> 1) *is a constant depending only on* r.

If h is holomorphic on $Q^N = \{(z_1, \cdots, z_N): r < |z_i| < 1, 1 \leqq i \leqq N\}$ then h has a Laurent expansion

$$(1) \qquad h(z_1, z_2, \cdots, z_N) = \sum a(n_1, n_2, \cdots, n_N) z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N} .$$

Following [2], we define $\pi_j h$, $1 \leqq j \leqq N$, to be the holomorphic function on Q^N whose Laurent series is obtained by deleting in (1) all terms in which $n_j \geqq 0$. Lemma 1 implies

LEMMA 2. $\| \pi_j h \|_{Q^N} \leqq K \| h \|_{Q^N}$.

Proof of the theorem. Since E is a subvariety of U^N of pure dimension $N-1$, there exists by [1, p. 251] a function f holomorphic on U^N such that at each point of U^N the germ of f generates the ideal of germs of holomorphic functions which vanish on the germ of E at the given point. In particular, $E = Z(f)$. We will show that $\partial f / \partial z_k \neq 0$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leqq k \leqq N$. We give the proof for $k = 1$, the other cases are identical. Let $(\alpha, \alpha') \in (U \times Q^{N-1}) \cap E$. Now f is regular in the first coordinate [1, p. 13] at (α, α') since otherwise $f(\zeta, \alpha')$ vanishes in a neighborhood of α and hence for $|\zeta| < 1$ and so $E = Z(f) \supseteqq \{(\zeta, \alpha'): |\zeta| < 1\}$, contradicting (i) in the definition of a special Rudin variety. Thus we can apply the Weierstrass preparation theorem and write in some neighborhood of (α, α'), $f = \Omega p$ where Ω is invertible and p is a Weierstrass polynomial. Factor p into primes: $p = p_1^{e_1} \cdots p_t^{e_t}$ where p and the p_i's are of the form

$$(\zeta - \alpha)^n + a_{n-1}(\zeta')(\zeta - \alpha)^{n-1} + \cdots + a_0(\zeta')$$

for (ζ, ζ') near (α, α') with $a_j(\alpha') = 0$. Now the degree of each p_i must be equal to 1 since otherwise there would exist $\zeta'_n \to \alpha'$ with ζ'_n off the discriminant locus of some p_i and so there would exist $\alpha_n \neq \beta_n$ near α with $p_i(\alpha_n, \zeta'_n) = 0 = p_i(\beta_n, \zeta'_n)$ and thus (α_n, ζ'_n) and (β_n, ζ'_n) are in $(U \times Q^{N-1}) \cap E$, but $\zeta'_n \to \alpha'$ implies $\alpha_n \to \alpha$ and $\beta_n \to \alpha$ and so $|\alpha_n - \beta_n| \to 0$, contradicting (ii). A similar argument also using (ii) shows that there cannot be more than one p_i and so $f = \Omega p_1^{e_1}$ near (α, α'). Finally, since the germ of f generates the ideal of E at (α, α'), e_1 must be equal to 1. Thus $f(\zeta, \zeta') = \Omega(\zeta, \zeta')(\zeta - \alpha + a_0(\zeta'))$ and $\partial f / \partial \zeta (\alpha, \alpha') = \Omega(\alpha, \alpha') \neq 0$ as required.

Now by Theorem 1 of [2] applied to $E = Z(f)$ there is a bounded holomorphic function F on U^N such that $E = Z(F)$. Examination of the construction in [2] shows that $1/F$ is bounded on Q^N since $F = f_1 e^{g-g_1}$ on Q^N and $1/f_1$ and $|\operatorname{Re}(g - g_1)|$ are bounded on Q^N. We will show that there is an $\varepsilon > 0$ such that $|\partial F/\partial z_k| > \varepsilon$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leqq k \leqq N$. We do this for $k = 1$, the finitely many other cases are identical. From [2], $F = fe^g$ for some g and so $\partial f/\partial z_1 \neq 0$ on $(U \times Q^{N-1}) \cap E$ implies $\partial F/\partial z_1 \neq 0$ there. Now for $z' \in Q^{N-1}$

$$z' \longrightarrow \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\partial F/\partial z_1(\zeta, z')}{F(\zeta, z')} d\zeta$$

is a continuous integer-valued function and so is a constant m_1 giving the number of zeros for $F(\cdot, z')$ in U. Since these zeros are the points of $(U \times Q^{N-1}) \cap E$ and $\partial F/\partial z_1 \neq 0$ there, it follows that the m_1 zeros $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$ are distinct simple zeros. By (ii) then, $|\alpha_i(z') - \alpha_j(z')| \geqq \delta$ for $i \neq j$. Write $F(\cdot, z') = BH$, where B is the Blaschke product with zeros at $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$. Now since $1/F$ is bounded on Q^N $1/H$ is bounded on U. But on E, $\partial F/\partial z_1 = \partial B/\partial z_1 \cdot H$ and since

$$|\alpha_i(z') - \alpha_j(z')| \geqq \delta, \partial B/\partial z_1$$

is bounded from zero on E by some constant depending on δ, and as H is also bounded from zero independently of z', it follows that $\partial F/\partial z_1$ is bounded from zero on $(U \times Q^{N-1}) \cap E$.

Let $d = \operatorname{dist}(E, Q^N)$ which we may assume is positive by increasing r if need be. Let g be a bounded holomorphic function on E. We shall extend g to a bounded function on U^N. By the general Oka-Cartan theory [1], there is a holomorphic extension G of g to U^N; G need not be bounded. Since $F \neq 0$ on Q^N, we may define a function h_1 on $U \times Q^{N-1}$ as follows: Let $(z_1, z') \in U \times Q^{N-1}$. Choose a circle Γ about 0 lying in Q and enclosing z_1 with positive orientation and set

$$h_1(z_1, z') = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta .$$

h_1 is clearly independent of the choice of Γ and holomorphic on $U \times Q^{N-1}$. We claim that $G/F - h_1$ is bounded on Q^N. Let $(z_1, z') \in Q^N$ where $z_1 \in Q$, $z' \in Q^{N-1}$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_{m_1}$ be small circles about $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$, the zeros of $F(\cdot, z')$. Then the Cauchy integral formula reads

$$(G/F)(z_1, z') = \frac{1}{2\pi i} \int_{\Gamma - \Gamma_1 - \cdots \Gamma_{m_1}} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta .$$

Therefore

$$(G/F - h_1)(z_1, z') = -\sum_{1}^{m_1} \frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta .$$

Clearly for r_k = radius of Γ_k,

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1}\,d\zeta$$
$$= \frac{1}{2\pi i}\int_{|\zeta-\alpha_k(z')|=r_k}\frac{G(\zeta, z')}{\zeta - z_1}\,\frac{\zeta - \alpha_k(z')}{F(\zeta, z') - F(\alpha_k(z'), z')}\,\frac{d\zeta}{\zeta - \alpha_k(z')}$$
$$\longrightarrow \frac{g(\alpha_k(z'), z')}{(\alpha_k(z') - z_1)\dfrac{\partial F}{\partial \zeta_1}(\alpha_k(z'), z')} \qquad \text{as } r_k \to 0\,.$$

So letting the radii of the Γ_k go to zero we get

$$(G/F - h_1)(z_1, z') = -\sum_{k=1}^{m_1}\frac{g(\alpha_k(z'), z')}{(\alpha_k(z') - z_1)\dfrac{\partial F}{\partial \zeta_1}(\alpha_k(z'), z')}\,.$$

Since $(\alpha_k(z'), z') \in (U \times Q^{N-1}) \cap E$, recalling the significance of d and ε we get

$$\| G/F - h_1 \|_{Q^N} \leqq \frac{m_1 \| g \|_E}{d\varepsilon}\,.$$

In the same way for each $i, 1 < i \leqq N$ we have an integer m_i and a function h_i holomorphic on $Q^{i-1} \times U \times Q^{N-i}$ such that

$$\| G/F - h_i \|_{Q^N} \leqq \frac{m_i \| g \|_E}{d\varepsilon}\,.$$

Now let $m = \max\{m_i\colon 1 \leqq i \leqq N\}$ and let $A = m/d\varepsilon$. Subtracting in the above, we get $\| h_1 - h_i \|_{Q^N} \leqq 2A \| g \|_E$. Now following [2] closely, set $h = (1-\pi_1)(1-\pi_2)\cdots(1-\pi_N)h_1$. Since $\pi_i h = 0$, h extends (uniquely) to a holomorphic function on U^N. Since h_j is holomorphic on

$$Q^{j-1} \times U \times Q^{N-j},\ \pi_j h_j = 0$$

and so $\pi_j h_1 = \pi_j(h_1 - h_j)$ and therefore by Lemma 2,

$$\| \pi_j h_1 \|_{Q^N} = \| \pi_j(h_1 - h_j \|_{Q^N} \leqq K \| h_1 - h_j \|_{Q^N} \leqq 2KA \| g \|_E\,.$$

Now, since $h - h_1 = -\sum \pi_i h_1 + \sum \pi_i \pi_j h_1 - + \cdots$ and since we get by induction and by use of Lemma 2 that $\| \pi_{i_1}\pi_{i_2}\cdots\pi_{i_S} h_1 \|_{Q^N} \leqq 2K^S A \| g \|_E$, *it* follows that $\| h - h_1 \|_{Q^N} \leqq BA \| g \|_E$ where B depends only on K. Now consider $\bar{G} = G - Fh$. $\bar{G}$ is holomorphic on U^N and extends g since G does. On $Q^N, \bar{G} = F(G/F - h_1) + F(h_1 - h)$. Therefore $\| \bar{G} \|_{Q^N} \leqq \| F \|_{U^N} A \| g \|_E + \| F \|_{U^N} BA \| g \|_E$. Thus $\bar{G}$ is bounded on U^N and $\| \bar{G} \|_{U^N} \leqq \gamma \| g \|_E$ where $\gamma = A(1 + B) \| F \|_{U^N}$ is independent of g.

Next we show that $\bar{G}$ does not depend on the choice of G made at the beginning of the construction. Suppose $\tilde{G}$ were another (not necessarily bounded) extension of g to U^N. As above we get

$$\tilde{h}_1 = \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{G}/F}{\zeta - z_1}\,d\zeta\,.$$

But then on $U \times Q^{N-1}$

$$h_1 - \tilde{h}_1 = \frac{1}{2\pi i}\int \frac{(G - \tilde{G})/F}{\zeta - z_1}\,d\zeta\,. \tag{2}$$

Since for $z' \in Q^{N-1}$, $(G - \tilde{G})(\cdot, z')$ vanishes at $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$ and since $F(\cdot, z')$ has simple zeros and only at these points, $(G - \tilde{G})/F(\cdot, z')$ is holomorphic on U and the right hand side of (2) equals $(G - \tilde{G})/F$ and so on $U \times Q^{N-1}$

$$h_1 - \tilde{h}_1 = (G - \tilde{G})/F\,. \tag{3}$$

Since the left hand side of (3) is holomorphic on $U \times Q^{N-1}$, so is the right and consequently $(G - \tilde{G})/F = (1 - \pi_1)((G - \tilde{G})/F)$ on Q^N. In the same way we see that for each j, $(G - \tilde{G})/F = (1 - \pi_j)((G - \tilde{G})/F)$ on Q^N. Therefore on Q^N we have

$$(G - \tilde{G})/F = \prod_{j=1}^{N}(1 - \pi_j)(G - \tilde{G})/F = \prod_{j=1}^{N}(1 - \pi_j)(h_1 - \tilde{h}_1) = h - \tilde{h}\,.$$

Thus $G - Fh = \tilde{G} - F\tilde{h}$ on Q^N and so on U^N. Since the extensions thus coincide, we have a well-defined map $T\colon H^\infty(E) \to H^\infty(U^N)$ such that $\| T(g) \|_{U^N} \leqq \gamma \| g \|_E$.

To see that T is linear, let g and $\tilde{g}$ be bounded holomorphic functions on E and let λ be a complex number. Let G and $\tilde{G}$ respectively be arbitrary holomorphic extensions to U^N. Let $\tilde{\tilde{h}}_1, h_1, \tilde{h}_1$ and $\tilde{\tilde{h}}, h, \tilde{h}$ be the h_1 and the h for $G + \lambda\tilde{G}, G$ and $\tilde{G}$ respectively. Then

$$\begin{aligned}\tilde{\tilde{h}}_1 &= \frac{1}{2\pi i}\int \frac{(G + \lambda\tilde{G})/F}{\zeta - z_1}\,d\zeta \\ &= \frac{1}{2\pi i}\int\frac{G/F}{\zeta - z_1}\,d\zeta + \lambda\cdot\frac{1}{2\pi i}\int\frac{\tilde{G}}{\zeta - z_1}\,d\zeta = h_1 + \lambda\tilde{h}_1\end{aligned}$$

and $\tilde{\tilde{h}} = \Pi(1 - \pi_j)\tilde{h}_1 = [\Pi(1 - \pi_j)](h_1 + \lambda\tilde{h}_1) = h + \lambda\tilde{h}$. Therefore

$$\begin{aligned}T(g + \lambda\tilde{g}) &= (G + \lambda\tilde{G}) - F(h + \lambda\tilde{h}) \\ &= (G - Fh) + \lambda(\tilde{G} - F\tilde{h}) = T(g) + \lambda T(\tilde{g})\,.\end{aligned}$$

EXAMPLE. Let E be the Rudin variety in U^2 given by $E = Z((z_2 - \frac{1}{2})(z_1z_2 - \frac{1}{2}))$. Then E is the disjoint union of $Z(z_2 - \frac{1}{2})$ and $Z(z_1z_2 - \frac{1}{2})$. Let $g \in H^\infty(E)$ be given by

$$g \mid Z\Big((z_2 - \frac{1}{2}\Big) = 0 \quad\text{and}\quad g \mid Z\Big(z_1z_2 - \frac{1}{2}\Big) = 1\,.$$

Then g admits no bounded holomorphic extension to U^2. For if G were a bounded extension of g to U^2 we would have for $z \in U, z$ near 1,

$$1 = G\left(z, \frac{1}{2z}\right) - G\left(z, \frac{1}{2}\right) = \frac{1}{2\pi i}\int_{|\zeta|=1} G(z, \zeta)\left(\frac{1}{\zeta - \frac{1}{2z}} - \frac{1}{\zeta - \frac{1}{2}}\right)d\zeta$$

$$= \left(\frac{1}{2z} - \frac{1}{2}\right)\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{G(z, \zeta)}{\left(\zeta - \frac{1}{2z}\right)\left(\zeta - \frac{1}{2}\right)} d\zeta .$$

But as $z \to 1$, the integral is bounded and $(1/2z) - (1/2) \to 0$, a contradiction.

References

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Received January 8, 1968. The research for this paper war partially supported by the following contracts: NONR 222 (85) and NONR 3656 (08).

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