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ALGEBRAS FORMED BY THE ZORN VECTOR MATRIX

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In the Zorn vector matrix algebra the three dimensional vector algebra is replaced by a finite dimensional Lie algebra L over a field of characteristic not 2 equipped with an associative symmetric bilinear form (a, b) and having the property: $[a[bc]] = (a, c)b - (a, b)c, a, b, c \in L$. We determine all the alternative algebras $\mathfrak{A}$ obtained in this way: If the bilinear form (a, b) on L is nondegenerate then $\mathfrak{A}$ is the split Cayley algebra or a quaternion algebra. For a degenerate form (a, b), $\mathfrak{A}$ is a direct sum of its radical and a subalgebra which is either a quaternion or two dimensional separable algebra. As an immediate consequence of the first result we have shown that if the bilinear form on the Lie algebra L is nondegenerate then L is simple with dimension three or one.

Let Φ be a field of characteristic not two throughout this paper. Let A be an anti-commutative algebra over Φ with a symmetric bilinear form (a, b) which is associative, i.e., $(ac, b) = (a, cb), a, b, c \in A$, and we consider the set $\mathfrak{A}$ of 2×2 vector matrices of the form:

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \alpha, \beta \in \Phi; a, b \in A .$$

$\mathfrak{A}$ is a vector space Φ under the usual addition, $+$, and multiplication by scalars. A multiplication in $\mathfrak{A}$ ([5] and [2]) is defined to be:

$$(1) \qquad \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma - (a, d), & \alpha c + \delta a + bd \\ \gamma b + \beta d + ac, & \beta\delta - (b, c) \end{pmatrix} .$$

Then $\mathfrak{A}$ is a flexible algebra over Φ in the sense that

$$(xy)x = x(yx), x, y \in \mathfrak{A} .$$

Furthermore $\mathfrak{A}$ is an alternative algebra over Φ, i.e., $x^2y = x(xy)$ and $(yx)x = yx^2, x, y \in \mathfrak{A}$ if and only if the anti-commutative algebra A has the following property:

$$(2) \qquad a(bc) = (a, c)b - (a, b)c, a, b, c \in A .$$

This is checked easily by a comparison of entries of vector matrices x^2y and $x(xy)$. We note that this property implies the Jacobi identity: $a(bc) + b(ca) + c(ab) = 0$ and A is a Lie algebra over the field Φ.

We shall determine all the alternative algebras over Φ which are constructed from the Lie algebras with (2) by the Zorn vector matrices. First we determine all the Lie algebras with (2) and let L be a finite dimensional Lie algebra over Φ equipped with an associative symmetric bilinear form (a, b) and having the property (2). We return to writing $[a\ b]$ in place of ab. Set $L^{\perp} = \{a \in L \mid (a, b) = 0, b \in L\}$ the radical of the bilinear form. If the bilinear form (a, b) is nondegenerate, i.e., $L^{\perp} = 0$, it follows from (2) that L is a simple Lie algebra. On the other hand, if (a, b) is degenerate we have the following.

LEMMA. *If the bilinear form (a, b) is degenerate, then the Lie algebra L is nilpotent with $L^3 = 0$ or $L = \Phi u + L^{\perp}$ where $L^{\perp}$ is a nonzero abelian ideal and $(ad\ u)^2|_{L^{\perp}} = \rho I, \rho = -(u, u) \neq 0$ in Φ.*

Proof. If $L^{\perp} = L$, the condition (2) implies $L^3 = 0$. In the rest of the proof we assume that $L^{\perp} \neq L$, and $L^{\perp}$ is a nonzero proper ideal of L. There exists an element $u \neq 0$ in L which is not in $L^{\perp}$ and satisfies $(u, u) \neq 0$. Let $(y_1, y_2, \cdots, y_m)$ be a basis for $L^{\perp}$.

$$(ad\ u)^2|_{L^{\perp}} = -(u, u)I$$

because we have $(ad\ u)^2 y_i = [u[u, y_i]] = -(u, u)y_i$ for all y_i. Since

$$\rho = -(u, u) \neq 0$$

in Φ, the mapping $ad\ u$ is nonsingular on $L^{\perp}$.

$$(ad\ u)[y_i, y_j] = (u, y_j)y_i - (u, y_i)y_j = 0$$

for all i, j imply $[y_i, y_j] = 0$ which means $L^{\perp}$ abelian. Finally we show that L is the direct sum of two subspaces Φu and $L^{\perp}$. Let x be any element of L, not in $L^{\perp}$. $(ad\ u)[x, y_i] = -(u, x)y_i$ and set $\tau = -(u, x)$. Then $(ad\ u)ad(\tau u - \rho x)|_{L^{\perp}} = 0$. Since $ad\ u$ is nonsingular on $L^{\perp}$, $ad(\tau u - \rho x)|_{L^{\perp}} = 0$. We wish to show that $(y, \tau u - \rho x) = 0$ for any y of L, which is equivalent to saying that $x \in \Phi u + L^{\perp}$. Since $[\tau u - \rho x, y_i] = 0$ for all y_i of the basis for $L^{\perp}$, $0 = [y[\tau u - \rho x, y_i]] = -(y, \tau u - \rho x)y_i$. This has completed our proof.

Now we first take up the case the bilinear form (a, b) on the Lie algebra L is nondegenerate. It is known ([2]) that (a, b) on L is nondegenerate if and only if the algebra $\mathfrak{A}$ constructed from L is simple. Since the alternative algebra $\mathfrak{A}$ is simple, $\mathfrak{A}$ is the split Cayley algebra or an associative algebra ([1]). We consider the latter case and follow Sagle's argument in [3]. Let

$$x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \ y = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix}, \ z = \begin{pmatrix} \lambda & g \\ h & \mu \end{pmatrix}$$

be any elements of $\mathfrak{A}$. By a comparison of (1,1)-entries of $(xy)z =$

$x(yz)$ we have $([b\ d], h) = (a, [c\ g])$. Without loss of generality we may take $a = 0$ and we have $([b\ d], h) = 0$ for all $h \in L$. It follows from the nondegeneracy that $[b\ d] = 0$ for all b, d of L, i.e., $L^2 = 0$. From $0 = [a[b\ c]] = (a, c)b - (a, b)c$, we have dim $L = 1$ and therefore $\mathfrak{A}$ is a quaternion algebra. Hence we have the following

THEOREM 1. *Let L be a finite dimensional Lie algebra over a field Φ of characteristic $\neq 2$ equipped with an associative symmetric bilinear form (a, b) and having the property* (2). *If (a, b) is nondegenerate, then $\mathfrak{A}$ is the split Cayley algebra or a quaternion algebra.*

A similar consideration to this theorem is given in [3]. As an immediate consequence of the theorem we have

COROLLARY. *Let L be as in Theorem* 1. *If the bilinear form (a, b) is nondegenerate L is simple with dimensionality three or one.*

Next we consider the remaining case, that is, (a, b) on L is degenerate. Let $(u_1, u_2, \cdots, u_n)$ be a basis for L over Φ and we set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$e_{12}^{(s)} = \begin{pmatrix} 0 & u_s \\ 0 & 0 \end{pmatrix}, \quad e_{21}^{(s)} = \begin{pmatrix} 0 & 0 \\ u_s & 0 \end{pmatrix}, \qquad s = 1, 2, \cdots, n .$$

These form a basis for the algebra $\mathfrak{A}$ over Φ. Let $L = \Phi u + L^{\perp}$ be as in lemma and take the basis for L to be $u_1 = u$ and $(u_2, \cdots, u_n)$ a basis for the abelian ideal $L^{\perp}$. We have the following multiplication table for $\mathfrak{A}$:

$$\begin{aligned}
&e_i e_j = \delta_{ij} e_i ,\\
&e_i e_{ik}^{(s)} = e_{ik}^{(s)} e_k = e_{ik}^{(s)} ,\\
&e_k e_{ik}^{(s)} e_i = e_{ik}^{(s)} e_i = 0 ,\\
&e_{ik}^{(s)} e_{ki}^{(t)} = \begin{cases} \rho e_i \text{ if } (s, t) = (1,1) , \\ 0 \text{ otherwise,} \end{cases}\\
&e_{ik}^{(s)} e_{ik}^{(t)} = -e_{ik}^{(t)} e_{ik}^{(s)} = \begin{cases} 0 \text{ if } s, t = 2, 3, \cdots, n, \\ x_{ki} \text{ otherwise} \end{cases}
\end{aligned}$$

where $i, j, k = 1, 2$; $i \neq k$; $s, t = 1, 2, \cdots, n$ and x_{ki} is a 2×2 vector matrix with 0 for all entries except for (k, i)-entry $[u_s\ u_t]$. The $e_{12}^{(s)}$ and $e_{21}^{(s)}$, $s = 2, 3, \cdots, n$ are all properly nilpotent and therefore generate the radical $\mathfrak{N}$ of $\mathfrak{A}$ (Zorn Theorem 3.7 in [4]). It follows that $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ (direct sum) where $\mathfrak{S}$ is a quaternion subalgebra with basis

$(e_1, e_2, e_{12}^{(1)}, e_{21}^{(1)})$. We note that this quaternion subalgebra $\mathfrak{S}$ is the same as one given in Theorem 1. Now we consider the remaining case: $L^\perp = L$ and L is nilpotent with $L^3 = 0$. Take a basis

$$(u_1, \cdots, u_m, \cdots, u_n)$$

for L such that $(u_{m+1}, \cdots, u_n)$ is a basis for the abelian ideal L^2 of L. We have

$$[u_i\ u_j] \in L^2, 1 \leqq i, j \leqq m \text{ and}$$
$$[u_i\ u_j] = 0 \text{ otherwise.}$$

The multiplication table for $\mathfrak{A}$ is as follows:

$$\begin{aligned} e_i e_j &= \delta_{ij} e_i\ , \\ e_i e_{ik}^{(s)} &= e_{ik}^{(s)} e_k = e_{ik}^{(s)}\ , \\ e_k e_{ik}^{(s)} &= e_{ik}^{(s)} e_i = 0\ , \\ e_{ik}^{(s)} e_{ki}^{(t)} &= -(u_s, u_t) e_i = 0\ , \\ e_{ik}^{(s)} e_{ik}^{(t)} &= x_{ki} \end{aligned}$$

where $i, j, k = 1, 2$; $i \neq k$; $s, t = 1, 2, \cdots, n$ and x_{ki} is as before. The $e_{ik}^{(s)}$, $i \neq k$, $s = 1, 2, \cdots, n$ are all properly nilpotent and generate the radical $\mathfrak{R}$ of $\mathfrak{A}$. Hence $\mathfrak{A}$ is a direct sum of $\mathfrak{R}$ and a separable subalgebra $\Phi e_1 + \Phi e_2$. We have proved the following

THEOREM 2. *Let L be as in Theorem 1. If the bilinear form (a, b) is degenerate, then the algebra $\mathfrak{A}$ constructed from L is a direct sum of its radical $\mathfrak{R}$ and a subalgebra $\mathfrak{S}$ where $\mathfrak{S}$ is either a quaternion or 2-dimensional separable algebra.*

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