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LOCAL DOMAINS WITH TOPOLOGICALLY T-NILPOTENT RADICAL

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This paper is concerned with local integral domains (no chain condition) which have the following property: for each ideal $\mathscr{A} \neq 0$ of A and for each sequence $(a_n)_{n \in N}$ of elements of $M(A)$, the maximal ideal of A, there is an $M \in N$ such that $a_0 a_1 \cdots a_k \in \mathscr{A}$. A local domain with this property is called a local domain with TTN. These rings are shown to be rings with Krull dimension 1 and local domains with Krull dimension 1 are shown to be dominated by rank 1 valuation rings. Modules over these rings are studied and results concerning divisibility and existence of simple submodules are obtained.

Noetherian integral domains with TTN are studied. Integral extensions of these rings are also studied. By localization of previous results, a characterization is given of those integral domains A with the property that every nonzero torsion A-module has a simple submodule.

H. Bass in [1] studied rings with the property that the Jacobson radical was T-nilpotent (T for transfinite), i.e., for each sequence $(a_n)_{n \in N}$ of elements of the Jacobson radical, $a_0 a_1 \cdots a_k = 0$ for some k. Local integral domains with TTN are just local domains with the property that $A/\mathscr{A}$ has T-nilpotent radical for each ideal $\mathscr{A} \neq 0$ of A.

In this paper A will denote a ring. All rings will be assumed to be commutative and have an identity. All modules will be unitary.

A will be called a local ring if A has a unique maximal ideal. If A is a local ring, $M(A)$ will denote its maximal ideal. If B is a local subring of A, A is said to dominate B if $M(A) \cap B = M(B)$. For convenience we agree that an integral domain is not a field.

If E and F are A-modules, $E \otimes F$ will mean $E \otimes_A F$.

DEFINITION. An ideal $\mathscr{A}$ of a ring A will be called topologically T-nilpotent if for each ideal $\mathscr{B}$ of A, $\mathscr{B} \subset \mathscr{A}$, $\mathscr{B} \neq 0$, and each sequence $(a_i)_{i \in N}$ of elements of $\mathscr{A}$, there is an $n \in N$ with $a_0 a_1 \cdots a_n \in \mathscr{B}$.

DEFINITION. An ideal $\mathscr{A}$ of a ring A will be called topologically nilpotent if for each ideal $\mathscr{B}$ of A, $\mathscr{B} \subset \mathscr{A}$, $\mathscr{B} \neq 0$, and each element a of $\mathscr{A}$, $a^n \in \mathscr{B}$ for some $n \in N$.

It is clear that it suffices to consider ideals $\mathscr{B}$ which are nonzero principal ideals.

DEFINITION. A local ring A will be said to have TTN (respectively Krull dimension 1) if $M(A)$ is topologically T-nilpotent (respectively topologically nilpotent.)

It is clear that "TTN" is stronger than "Krull dim 1".

EXAMPLES. If K is a field, $K[[X]]$, the ring of formal power series in one indeterminant, is a local domain with TTN. Formal series in more indeterminants are local domains with neither TTN or Krull dim 1. More generally, a discrete valuation ring has TTN.

It is easy to see that a local domain has Krull dimension 1 if and only if it has only one nonzero prime ideal, for if there are prime ideals $\mathscr{P}$ and $\mathscr{P}'$ with $x \in \mathscr{P}$, $x \notin \mathscr{P}'$, then $z^n \notin \mathscr{P} \cap \mathscr{P}' \neq 0$ for any n. Conversely, if A is a local domain with only one nonzero prime ideal, and $x \in M(A)$, any ideal $\mathscr{P}$ maximal with respect to $x^n \notin \mathscr{P}$ for any n is a prime ideal. Hence this definition of Krull dimension 1 and the standard definition agree.

An example of a local domain with Krull dim 1 but not TTN will be given after the following construction.

Let G be an ordered group, K a field. It is well known that there is a field F with a valuation v such that $v(F - \{0\}) = G$ and K is the residue field; that is, if A is the valuation ring for v, K and $[A/(M(A)]$ are isomorphic. (See McLane, [6]). This can be constructed by letting F be the set of formal power series with coefficients in K and "exponents" in G; i.e., an element in F looks like $\sum_{\alpha \in G} a_\alpha x^\alpha$ where $(a_\alpha)_{\alpha \in G}$ is a family of elements of K with well-ordered support. Multiplication and addition are as power series. The unit is $\sum_{\alpha \in G} a_\alpha x^\alpha$ where $a_0 = 1$, $a_\alpha = 0$ if $\alpha \neq 0$. This same construction can be done when G is not a group but a submonoid of the positive elements of an ordered group. One still obtains a local ring with residue field K, but it is not is general a valuation ring, much less a field. We will call this ring K^G. (For a more detailed explanation, see [6].)

EXAMPLE. Let K be a field, G the set of nonnegative real numbers under addition. Then K^G is a valuation ring with Krull dim 1 but not TTN.

2. Relationships to valuation rings. The following theorem is well-known.

THEOREM. *Let A be a local domain, K its field of fraction. Then there is a valuation v on K such that A is dominated by the valuation ring of v.* (See [4], p. 92).

In this section it is proved that if A is a local domain with Krull dim 1 the group of values can be picked as a subgroup of the additive real numbers.

DEFINITION. A subgroup H of an ordered group G is called isolated if whenever $x, y \in G, x \geqq y \geqq 0$ and $x \in H$, then $y \in H$.

DEFINITION. An ordered group G is called Archimedean if whenever $x, y \in G$ and $x > 0, y \geqq 0$ there is a positive integer n such that $nx > y$.

It is easily shown that any Archimedean ordered group is order and group isomorphic to a subgroup of the additive real numbers. (See [8], p. 45).

A valuation $v: K \to G_\infty$ is said to be of rank 1 if $v(K - \{0\})$ is Archimedean.

THEOREM. *If a local domain A has Krull* dim 1 *then there is a rank* 1 *valuation w on K, the field of fractions of A, such that A is dominated by the valuation ring of w. If A also has TTN, there is an $s \in H$, the group of values of w, $s > 0$, such that $w(x) \geqq s$ if $x \in M(A)$.*

Proof. A is dominated by a valuation ring V which is a subring of K. Let $G = v(K - \{0\})$ where $v: K \to G_\infty$ is a valuation on K which has V as its valuation ring. Consider the set L of isolated proper subgroups of G. If L is empty we are through. If not, $\Gamma = \bigcup_{C \in L} C$ is an isolated subgroup of G, for it is easy to see Γ is a subgroup: and if $x \in \Gamma, x \geqq 0$, then $x \in C$ for some $C \in L$. So if $y \in G, x \geqq y > 0$, then $y \in C$ so $y \in \Gamma$. Γ is also a proper subgroup, for if $a \in M(A)$, $a \neq 0$, then $v(a) \notin C$ for any $C \in L$, for if $c/d \in K$, $c, d \in A$, then $v(c/d) = v(c) - v(d)$. But $a^n \in (c)$ for some n. Hence $v(a^n) = nv(a) \geqq v(c) \geqq 0$. So $nv(a) \geqq v(c/d) > 0$. If $v(a) \in C$, then $nv(a) \in C$ and C is isolated, so $v(c/d) \in C$. But c/d is arbitrary so $C = G$, a contradiction. Thus if $v(a) \notin C$ for any $C \in L, v(a) \notin \bigcup_{C \in L} C$ so Γ is a proper subgroup of G. Thus Γ is a—in fact the only—maximal isolated subgroup of G. Then G/Γ can be made into an ordered group by setting $x + \Gamma \geqq y + \Gamma$ in G/Γ if $x \geqq y$ in G. It is easily verified that G/Γ with this order is an Archimedean ordered group. If $\phi: G \to G/\Gamma$ is the canonical surjection, we can extend ϕ to a map $\phi^*: G_\infty \to (G/\Gamma)_\infty$ by defining $\phi^*(\infty) = \infty$. Then $\phi^* \circ v: K \to (G/\Gamma)_\infty$ is a valuation on K. Also if $x \in K$ and $v(x) \geqq 0$, then $\phi^* \circ v(x) \geqq 0$. By construction if $x \in M(A)$, $v(x) \notin \Gamma$ so then $\phi^* \circ v(x) \neq 0$. So $\phi^* \circ v(x) > 0$ if $x \in M(A)$. Thus W, the valuation ring of $w = \phi^* \circ v$, dominates A.

If A has TTN, suppose that the elements of the form $w(x)$, $x \in M(A)$, are not bounded away from zero. Then, identifying G/Γ with a subgroup of R, the additive real numbers, there is a sequence $(a_n)_{n \in N}$ of elements of $M(A)$ with $w(a_n) \leqq 1/2^n$, and a $b \in M(A)$ with $w(b) > 2$. Then $w(a_0 a_1 \cdots a_n) = w(a_0) + \cdots + w(a_n) < 2$. So

$$a_0 a_1 a_2 \cdots a_n \notin (b)$$

for any integer n, a contradiction. Thus the theorem is proved.

COROLLARY. *$\bigcap_{n \in N} M(A)^n = 0$ if A has TTN.*

Proof. Let s be as above. Then if $x \in M(A)^n$, $w(x) \geqq ns$. No x except 0 can satisfy this for all n.

The above theorem shows that a valuation ring has TTN if and only if it is a discrete valuation ring.

From the proof of the theorem it is not hard to see the following. Let A be a local domain with Krull dim 1. Let a valuation ring V have these properties:

(1) V is a subring of K, the field of fractions of A.

(2) V has rank 1.

(3) V dominates A.

Then V is maximal with respect to these conditions. V is not unique as is shown in § 5.

Many examples are furnished by the following easy proposition.

PROPOSITION. *Let V be a rank 1 valuation ring, K its field of fractions, $v: K \to R_\infty$ the valuation. Let A be a local subring of V such that there is an $s \in R, s > 0$ such that if $x \in M(A)\, v(x) \geqq s$. Also suppose that there is a $p \in R$ such that $M(A)$ contains $\{x \in V \mid v(x) \geqq p\}$. Then A has TTN.*

Proof. Let $(a_i)_{i \in N}$ be a sequence of elements of $M(A)$, $a \in M(A)$, $a \neq 0$. Let n be such that $ns > p + v(a)$. Then $v(a_0 a_1 a_2 \cdots a_n \cdot 1/a) = ns - v(a) \geqq p$. So $a_0 a_1 a_2 \cdots a_n \cdot 1/a \in M(A)$ so $a_0 a_1 a_2 \cdots a_n \in (a)$.

This proposition does not allow a converse for we have the following example of a local domain A with TTN, and a rank 1 valuation v on K, the field of fractions of A, such that there are elements of K of arbitrarily large valuation which are not in A. Let $S = \{x \in R \mid x = a\sqrt{2} + b, a, b \in z$ and $a\sqrt{2} + b \geqq \sqrt{|a| + |b|} + 1\} \cup \{0\}$. Consider $F^S = A$ for F a field. Let K be the field of fractions of A. S is a submonoid of the nonnegative real numbers under addition. Then A is a local domain with $M(A)$ being $\{x \in A \mid v(x) > 0\}$ where $v: K \to R \cup \{\infty\}$ is the obvious valuation.

It is easy to see that any $n + 1$ elements of $M(A)$, $a_0, a_1, \cdots a_n$, are such that if $a\sqrt{2} + b \in \operatorname{supp}(x)$, where $x = a_0 a_1 \cdots a_n$, then $a\sqrt{2} + b > \sqrt{|a| + |b|} + 1 + n$. This fact will be used often subsequently without direct reference.

To show that $M(A)$ is topologically T-nilpotent it suffices to show that if $(a_i)_{i\in N}$ is a sequence of elements of $M(A)$, $b \in M(A)$, $b \neq 0$, then for some $n \in N$, $a_0 a_1 \cdots a_n b^{-1} \in A$, for then $a_0 a_1 \cdots a_n \in (b)$. To do this we show that for some n, $\operatorname{supp}(a_0 a_1 \cdots a_n b^{-1}) \subset S$. Let $\operatorname{supp}(b) = (d_i)_{i\in I}$ where $(d_i)_{i\in I}$ is a well-ordered family of elements of S with d_0 as least element. Then an element in $\operatorname{supp}(b^{-1})$ looks like

$$- d_0 + \sum_{i\in I} n_i(d_i - d_0)$$

where $(n_i)_{i\in I}$ is a family of positive integers with finite support. If we can show that there is an $m \in N$ such that for any sequence $(n_i)_{i\in I}$ of nonnegative integers of finite support

$$- d_0 + \sum_{i\in I} n_i(d_i - d_0) + m \in S \,,$$

we are finished.

Let $d_i = s_i\sqrt{2} + t_i$ for each $i \in I$. Let $M \in R$ be such that if $M_1 > M$

$$2\sqrt{M_1} - \sqrt{2M_1} > 2(|s_0\sqrt{2}| + |t_0|) + 2\sqrt{|s_0| + |t_0|} + 1 \,.$$

There are only a finite number elements of S, $a\sqrt{2} + b$, with

$$|a| + |b| < M \,.$$

If $a\sqrt{2} + b$ is such that $a\sqrt{2} + b - (s_0\sqrt{2} + t_0) > 0$ there is an $n \in N$, $n > 0$, such that $n(a\sqrt{2} + b - (s_0\sqrt{2} + t_0)) \in S$. For if

$$(a - s_0)\sqrt{2} + (b - t_0) = \varepsilon \,,$$

let n be such that

$$\sqrt{n} > \frac{\sqrt{|a - s_0| + |b - t_0|} + 1}{\varepsilon} \,.$$

Notice that if $s\sqrt{2} + t \in s$ there is a $p \in N$ such that

$$s\sqrt{2} + t - (s_0\sqrt{2} + t_0) + p \in S \,.$$

So there is a $q \in N$ such that $- d_0 + \sum n_i(d_i - d_0) + q \in S$, where the sum runs over all d_i with $|s_i| + |t_i| < M$, for there are only a finite number of these in S and for each one there are only a finite number of integers q with $q(d_i - d_0) \notin S$. On the other hand, if $|s_i| + |t_i| \geqq M$, $|s_j| + |t_j| \geqq M$, then $(d_i - d_0) + (d_j - d_0) \in S$; for

$$d_i > \sqrt{|s_i| + |t_i|} = \sqrt{M_1} \geqq \sqrt{M}$$
$$d_j > \sqrt{|s_j| + |t_j|} = \sqrt{M_1} \geqq \sqrt{M}$$

so $d_i + d_j - 2d_0 > \sqrt{M_1} + \sqrt{M_2} - 2d_0$

$$> \sqrt{M_1} + \sqrt{M_2} + 2\sqrt{|s_0| + |t_0|} + 1 + 2(|s_0\sqrt{2}| + |t_0|) - 2d_0$$
$$> \sqrt{|s_i + s_j - 2s_0| + |t_i + t_j - 2t_0|} + 1 \,.$$

So for large enough m, $\sum_{i \in I} n_i(d_i - d_0) - d_0 + m \in S$. So A has TTN.

There are elements in K with arbitrarily large values whose values are not in S.

It is not hard to see that this is, up to isomorphism, the only rank 1 valuation on K such that elements of A have nonnegative order and elements of $M(A)$ have strictly positive order, for if $a, b \in Z$, $a\sqrt{2} + b > 0$, then $n(a\sqrt{2} + b) \in S$ for some $n \in N$, $n > 0$.

Notice that in each example given of a local domain A with TTN, for each nonzero ideal $\mathscr{A}$ of A there is an $n \in N$ such that $M(A)^n \subset \mathscr{A}$. Whether this is true in general is doubtful, but at present there are no examples to the contrary.

3. In this section the relationship of "A has TTN" to divisibility and other concepts is explored.

DEFINITION. An A-module E is called divisible if for all $x \in E$, $a \in A$, $a \neq 0$, there is $ay \in E$ such that $ay = x$.

THEOREM. *If A is a local domain with TTN, an A-module E is divisible if and only if $M(A)E = E$.*

Proof. Suppose $M(A)E = E$. Let $a \in A$, $a \neq 0$, $x \in E$. Then $A/(a)$ has T-nilpotent radical $M(A)/(a)$, and $M(A)/(a) \cdot E/(a)E = E/(a)E$. Thus by Bass [1 p. 473] $E/(a)E = 0$ or $E = (a)E$. So for some $y \in E$, $x = ay$. So E is divisible. The opposite implication is trivial.

THEOREM. *Let A be a local domain such that $M(A)E = E$ implies E is divisible. Then A has TTN.*

Proof. Let $a \in A$, $a \neq 0$, $(a_i)_{i \in I}$ be a sequence of element of $M(A)$. If $a_k = 0$ for some k $a_0 a_1 \cdots a_k = 0 \in (a)$ so suppose no $a_k = 0$.

Consider the submodule E of the field of fractions of A consisting of fractions which can be written as $b/a_0 a_1 \cdots a_n$ for some $b \in A$, $n \in N$. $M(A)E = E$ so E is divisible by hypothesis. Thus there is an element $b/a_0 a_1 \cdots a_n$ such that $a \cdot b/a_0 a_1 \cdots a_n = a_0/a_0$. Then $a_0 a_1 \cdots a_n = ab \in (a)$. Thus A has TTN.

Note that E is in this case equal to the field of fractions of A.

DEFINITION. An A-module E is called torsion free if whenever $a \in A$, $x \in E$ and $ax = 0$, then $a = 0$ or $x = 0$.

DEFINITION. An A-module S is called simple if $S \neq 0$ and 0 and S are its only submodules.

It is easy to see that if S is simple, S is isomorphic to A/M for some maximal ideal M of A. In the local case there is only one simple module up to isomorphism, $A/M(A)$. In this case it is clear that a cyclic module is simple if and only if it has annihilator $M(A)$.

PROPOSITION. *If A is a local domain with TTN and E an A-module which is not torsion free, E contains a simple submodule.*

Proof. Let $x \in E, x \neq 0$ be such that there is an $a \in A, a \neq 0$ and $ax = 0$. Then $a \in M(A)$. If $bx = 0$ for all $b \in M(A)$, (x) is simple. If not let $a_1 \in M(A)$ be such that $a_1 x \neq 0$. If $b \cdot a_1 x = 0$ for all $b \in M(A)$ we are finished. So suppose $a_2 \in M(A)$ is chosen such that $a_2 a_1 x \neq 0$. Having chosen $a_1, a_2, \cdots, a_n$ such that $a_1 a_2 \cdots a_n \in M(A)$ and

$$a_n \cdots a_2 a_1 \cdot x \neq 0 ,$$

if b is such that $b \in M(A)$ and $b \cdot a_n a_{n-1} \cdots a_1 x \neq 0$, let $a_{n+1} = b$. But this process must end, for there is a p such that $a_p a_{p-1} \cdots a_1 \in (a)$ and $ax = 0$.

PROPOSITION. *Suppose A is a local domain such that every A-module which is not torsion free contains a simple submodule. Then A has TTN.*

Proof. Let $\mathscr{A}$ be an ideal of A, $\mathscr{A} \neq 0$ and $(a_i)_{i \in N}$ a sequence of elements of $M(A)$. Suppose $a_0 a_1 a_2 \cdots a_n \notin \mathscr{A}$ for any n. Let L be the set of all ideals C of A such that $a_0 a_1 a_2 \cdots a_n \notin C$ for any n. L is inductively ordered by inclusion for if S is a chain of elements of L, then $\bigcup_{C \in E} C$ is an element of L. Let $\mathscr{B}$ be a maximal element in this set. Then $M(A)/\mathscr{B}$ is not torsion free so let T be a submodule of $M(A)$ containing $\mathscr{B}$ such that $T/\mathscr{B}$ is a simple submodule of $M(A)/\mathscr{B} \cdot a_0 a_1 a_2 \cdots a_n \notin T$ for any $n \in N$ for if so $a_0 a_1 \cdots a_{n+1} \in \mathscr{B}$, for $T/\mathscr{B}$ is simple so $M(A)$ annihilates it and $a_{n+1} \in M(A)$. But then $T \neq \mathscr{B}$ and $T \supset \mathscr{B}$ and $T \in L$ which is a contradiction.

PROPOSITION. *Let E and F be A-modules, A a local domain with TTN. If $E \otimes F = 0$, then E or F is torsion and E or F is divisible.*

Proof. If neither of them is divisible, then $E \neq M(A)E$, $F \neq M(A)F$ and there is a surjective homomorphism $E \otimes F \to E/M(A)E \otimes F/M(A)F$ so $E/M(A)E \otimes F/M(A)F = 0$. But these are $A/M(A)$ modules and $E/M(A)E \otimes_A F/M(A)F$ is isomorphic to $E/M(A) \otimes_{A/M(A)} F/M(A)F$. (See [2, p. 123]). But $A/M(A)$ is a field and the tensor product of two modules over $A/M(A)$ is not 0 unless one of them is 0. Thus either E or F is divisible. We may as well suppose E is divisible. If E is not torsion, then $E/t(E)$ is divisible and torsion free and not 0 (where $t(E)$ is the torsion submodule of E). So $E/t(E)$ is isomorphic to a direct sum of copies of K, the field of fractions of A (see [6, p. 10]). But if F is not torsion, $F/t(F)$ is torsion free and not 0. But we have epimorphism $E \otimes F \to E/t(E) \otimes F/t(F)$. But $E/t(E) \otimes F/t(F) \neq 0$ if $E/t(E)$ is isomorphic to a nontrivial sum of copies of K and $F/t(F)$ is torsion free and not 0, for if $K \otimes F/t(F) = 0$, $K \otimes M = 0$ for all submodules M of $F/t(F)$ as K is flat (see [2, p. 115]). But $F/t(F)$ is torsion free and thus has a submodule isomorphic to A and $K \otimes A \neq 0$. So either E or F is torsion.

This unfortunately is not nearly as complete a proposition as would be desired. The proper conjecture may be: $E \otimes F = 0$ if and only if one is divisible and the other torsion, or one is divisible and torsion and (supposing E to be the torsion divisible one) $F/t(F)$ is divisible. One can easily show that a local domain satisfying this property has TTN.

REMARK. E. Matlis in [7] proves that if an integral domain A has the property that its field of fractions K is a countably generated A-module, then: every divisible A module is the image of a surjective homomorphism from a direct sum of copies of K; projective dimension of $K = 1$; and the torsion submodule of a divisible module is a direct summand. If A is a local domain with Krull dim 1, its field of fractions is countably generated by elements of the form $(1/a^n)_{n \in N}$ for any $a \in M(A)$, $a \neq 0$, so these propositions apply.

4. The Noetherian case.

If A is a local domain and $M(A)$ is is finitely generated, the situation is simplified.

PROPOSITION. *Let A be a local domain with Krull dimension 1. Let $M(A)$ be finitely generated. Then:*

(a) *A has TTN and in fact, if $\mathscr{A} \neq 0$ is an ideal of A, there is an integer n such that $M(A)^n \subset \mathscr{A}$.*

(b) *A is Noetherian.*

Proof. (a) Let $M(A)$ be generated by $x_1, x_2, \cdots, x_k$. Let $\mathscr{A} \neq 0$ be an ideal of A. Let p be such that $x_i^p \in \mathscr{A}$, $i = 1, \cdots, k$. Then

$M(A)^{kp} \subset \mathscr{A}$, for since any element of $M(A)$ can be written as

$$a_1x_1 + \cdots + a_xx_x$$

for some $a_1, a_2, \cdots, a_k$, a product of kp elements of $M(A)$ must contain $(x_q)^p$ for some q. Hence any product of kp elements of $M(A)$ is in $\mathscr{A}$, and hence any sum of elements of this type, so $M(A)^{kp} \subset \mathscr{A}$. Clearly this implies A has TTN.

(b) Let $\mathscr{A} \neq 0$ be an ideal of A. Then $M(A)^n \subset \mathscr{A}$ for some $n \in N$. $A/M(A)$ is a Noetherian A-module, $M(A)/M(A)^2$ is a Noetherian A-module since it is finitely generated and an $A/M(A)$-module, hence a direct sum of simple modules which must be finite since $M(A)$ is finitely generated. So $A/M(A)^2$ is Noetherian as $0 \to M(A)/M(A)^2 \to A/M(A)^2 \to A/M(A) \to 0$ is exact and an extension of a Noetherian module by a Noetherian module is Noetherian. Continuing by induction, we see that $A/M(A)^n$ is a Noetherian A-module for all n. But since $M(A)^k \subset \mathscr{A}$ for some k, $0 \to M(A)^k \to \mathscr{A} \to \mathscr{A}/M(A)^k \to 0$ is exact. $\mathscr{A}/M(A)^k$ is finitely generated as it is a submodule of the Noetherian module $A/M(A)^k$, $M(A)^k$ is finitely generated, so $\mathscr{A}$ is finitely generated. Therefore every ideal of A is finitely generated so A is Noetherian.

5. Integral extensions.

DEFINITION. If B is a ring, A a subring of $B, x \in B$ is called integral over A if x satisfies a unitary polynomial with coefficients in A. B is said to be an integral extension of A if every element of B is integral over A.

THEOREM. *Let A be a local domain with TTN. Let $A \subset K \subset F$ where K is the field of fractions of A and F a field containing K. Suppose $x \in F$ is integral over A. Let x satisfy the unitary polynomial*

$$f = X^{n+1} + a_nX^n + \cdots + a_0$$

with coefficients in A. Then if $a_i \in M(A), i = 0, \cdots n, A[x]$ is a local domain with TTN.

Proof. $A[x]$ is a local domain, for it is an integral domain and

$$M = \{y \in A[x] \mid y = c_0 + c_1x + \cdots + c_nx^n \quad \text{and} \quad c_0 \in M(A)\}$$

is the maximal ideal. To see this, we can suppose f is the unitary polynomial of least degree with all but the leading coefficient in $M(A)$ that x satisfies. Then $M \neq A[x]$, for if $1 \in M, 1 = c_0 + c_1x + \cdots + c_nx^n$ with $c_0 \in M(A)$. $1 - c_0$ is invertible in A so x is invertible in $A[x]$ and

$x^{-1} = (c_1 + c_2x + \cdots + c_nx^{n-1})(1 - c_0)^{-1}$. Then

$$x^{-1}(a_0 + a_1x + \cdots + a_nx^n + x^{n+1}) = 0$$

so $a_0(c_1 + c_2x + \cdots + c_nx^{n-1})(1 - c_0)^{-1} + a_1 + \cdots + x^n = 0$. But this produces a unitary polynomial of degree n with all but the leading coefficient in $M(A)$ which x satisfies, a contradiction. So $M \neq A[x]$. then let $\bar{M}$ be a maximal ideal of $A[x]$. Then $\bar{M} \cap A$ is a maximal ideal of A (see [4, p. 36]). Therefore $\bar{M} \cap A = M(A)$. So

$$a_0 + a_1x + \cdots + a_nx^n \in \bar{M}$$

so $x^{n+1} \in \bar{M}$. As $\bar{M}$ is maximal, it is prime, so $x \in \bar{M}$, so $M \subset \bar{M}$. But M is clearly maximal and hence the only maximal ideal of $A[x]$.

Now let $(f_i)_{i \in N}$ be a sequence of elements of $M(A[x])$ and $\mathscr{B}$ an ideal of $A[x]$, $\mathscr{B} \neq 0$. Then $\mathscr{B} \cap A \neq 0$ as $A[x]$ is an integral extension of A, (see [4, p. 14]). Let $\mathscr{B} \cap A = \mathscr{A}$. Then $\mathscr{A}[x] \subset \mathscr{B}$ so it would suffice to prove that for some n $f_0f_1 \cdots f_n \in \mathscr{A}[x]$. First notice that there is an m such that $x^m \in \mathscr{A}[x]$, for if p is such that $a_i^p \in \mathscr{A}$ for $i = 0 \cdots n$, let $m = p(n + 1)(n + 1)$. Then

$$x^m = (-(a_0 + a_1x + \cdots + a_nx^n)^{P(n+1)} .$$

When this is expanded, each term will be a product of $p(n + 1)$ factors, so one of $a_0, \cdots, a_n$ must be repeated at least p times so $x^m \in \mathscr{A}[x]$. Then if $f_0f_1 \cdots f_n \notin \mathscr{A}[x]$ for any n, when each f_i is written as a sum

$$f_i = c_{i0} + c_{i1}x + \cdots + c_{in}x^n$$

and the products $f_0f_1 \cdots f_n$ are expanded, there must be an infinite number of terms $c_{0k_0}x^{k_0} \cdot c_{1k_1}x^{k_1} \cdots c_{nk_n}x^{k_n}$ which are not in $\mathscr{A}[x]$. But, as before, we can find a sequence of $(c_{ik_i})_{i \in N}$ such that

$$c_{0k_0}x^{k_0} \cdots c_{rk_r}x^{k_r}$$

is not an element of $\mathscr{A}[x]$ for any r. But then, as only a finite number of the k_i could be nonzero ($x^m \in \mathscr{A}[x]$), there is a $q \in N$ such that $k_r = 0$ if $r \geqq q$. Then $(c_{i+q0})_{i \in N}$ is a sequence of elements of $M(A)$ such that $c_{0+q0} \cdots c_{i+q0} \notin \mathscr{A}$ for any i, a contradiction.

This proof could easily be modified to show that the theorem is still true if "*TTN*" is replaced by "Krull dimension 1", in its statement.

The stringent requirements on the polynomial that x must satisfy in the above theorem are not superfluous, at least to obtain a local domain; for $Z_3[\sqrt{-5}]$ is not local, as $2 + \sqrt{-5}$ and $1 + \sqrt{-5}$ are neither invertible and cannot be in the same maximal ideal. It is true however that $M(A)$ generates a topologically T-nilpotent ideal in $A[x]$ if x is integral over A, the proof being similar to the above.

If A is a discrete valuation ring, $A[x]$ is not necessarily a discrete valuation ring, even if x satisfies the type polynomial above, as $Z_3[3\sqrt{3}]$ shows.

It is not hard to show that an integral extension of a local domain A with Krull dimension 1, such that each element of the extension satisfies a unitary polynomial with all but the leading coefficient in $M(A)$, is a local domain with Krull dim 1. This is not true if "Krull dimension 1" is replaced by "TTN" for if $S = R^+ - [0, 1]$, K a field, R^+ the additive nonnegative real numbers, then K^{R^+} is an extension of the proper type of K^S, but K^{R^+} does not have TTN and K^S does.

Since the integral closure of a local domain is the intersection of all the valuation rings dominating it and contained in its field of fractions, (see [4, p. 93]), one might guess that an integrally closed local domain with TTN is a valuation ring, but this is not true.

EXAMPLE. Let $Z \times Z$ be given the product group structure and be ordered lexicographically. Then $Z \times Z$ is an ordered group. Let $S \subset Z \times Z$, $S = \{(x, p) \mid x = 0, y = 0 \text{ or } x > 0\}$. Then $A = K^S$ is a local subring of $F = K^{Z \times Z}$, A is integrally closed in F which is the field of fractions of A, A has TTN, but is not a valuation ring for F.

Proof. Let $w: F \to (Z \times Z)_\infty$ be the obvious valuation. Let v: $F \to (Z \times Z)_\infty$ be the valuation $v(0) = \infty$, $v(x) = (a, -b)$ where $w(x) = (a, b)$. Then $A = W \cap V$ where W is the valuation ring of w, V that of v. Hence A is integrally closed and it is easily seen that A is not a valuation ring.

6. Some of the above theorems can be used to obtain characterizations of those integral domains whose localizations with respect to maximal ideals have TTN. First an easy internal characterization of these rings.

PROPOSITION. *Let A be an integral domain, M a maximal ideal of A. Then A_m has TTN if and only if for all sequences $(a_i)_{i \in N}$ of elements of M and all subideals $\mathscr{A}$ of M, $\mathscr{A} \neq 0$, there is a $t \notin M$ such that $t \circ a_0 a_1 \circ \cdots \circ a_n \in \mathscr{A}$.*

Proof. Suppose A_M has TTN, $\mathscr{A}$ is a subideal of M, $\mathscr{A} \neq 0$ and $(a_i)_{i \in N}$ is a sequence of elements of M. Then $(a_i/1)_{i \in N}$ is a sequence of elements of $M(A_M)$, $\mathscr{A} \circ A_M$ is a nonzero ideal of A_M, so

$$\frac{a_0}{1} \cdot \frac{a_1}{1} \cdots \frac{a_n}{1} \in \mathscr{A} \circ A_M$$

for some n. So $a_0/1 \cdot a_1/1 \cdots a_n/1 = a/t$ for some $a \in \mathscr{A}$, $t \notin M$. So

$a_0 a_1 \cdots a_n t = a$, as required. Conversely, suppose the condition is satisfied and that $\mathscr{A}$ is an ideal of A_M, $\mathscr{A} \neq 0$ and $(a_i/b_i)_{i \in N}$ a sequence of elements of $M(A_M)$. Then $(a_i)_{i \in N}$ is a sequence of elements of M, $\mathscr{A}' = \mathscr{A}^{\mathscr{A}} \cap A$ is a subideal of M and $\mathscr{A}' \neq 0$, so there is a $t \in A$, $t \notin M$ such that $a_0 a_1 \cdots a_n t = a \in \mathscr{A}'$. Then $a_0/b_0 \cdot a_1/b_1 \cdots a_n/b_n \in \mathscr{A}$.

EXAMPLE. Z, the integers is an example of an integral domain with the property that all its localizations have TTN.

THEOREM. *The following are equivalent for an integral domain* A.

(1) A_M *has* TTN *for every maximal ideal* M *of* A.

(2) *Every* A-*module which is not torsion free has a simple submodule.*

(3) *An* A-*module is divisible if an only if* $ME = E$ *for each maximal ideal* M *of* A.

Proof. (1) $\Rightarrow$ (2) Let E be an A-module which is not torsion free, $x \in E$, $a \in A$, $ax = 0$, $a \neq 0$, $x \neq 0$. Let M be a maximal ideal of A which contains a. Then the map $x \to x/1: E \to E_M$ is an injection, for $Ann(x) \subset M$. Then there is a simple A_M-module $S \subset A_M \cdot x/1$ by a previous theorem. S is then a simple A-module contained in Ax, for if $s/a \in S$, $s/a = rs/1$ where $ra + b = 1$ for some $b \in M$. (2) $\Rightarrow$ (1). Let E be an A_M-module for some maximal ideal M of A, E not a torsion free A_M-module. Then E is not a torsion free A-module, so E contains a simple A-module which must be isomorphic to $A_m/M(A_M)$, hence a simple A_M-module. Hence A_M has TTN by a previous theorem. (1) $\Rightarrow$ (3) E is divisible if and only if E_M is for each maximal ideal M of A by [2, p. 111]. If $ME = E$, $M(A_M)E_M = E_M$ so if A_M has TTN E_M is divisible. (3) $\Rightarrow$ (1) Let M be a maximal ideal of A, E an A_M-module such that $M(A_M)E = E$. Then E is an A-module. If M' is a maximal ideal of A distinct from

$$M, M \circ E = E \text{ as } M' \cap \subset M \neq \varnothing \ .$$

Thus $\bar{M}E = E$ for all maximal ideals $\bar{M}$ of A. So E is a divisible A-module and hence a divisible A_M-module. Thus by a previous theorem A_M has TTN.

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