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COVERING MANIFOLDS WITH CELLS

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In attempting to triangulate a topological manifold, one would like to be able to cover a manifold with closed cells whose intersections are nice. This paper is a study of minimal coverings of manifolds by open cells and a method of improving the intersections as the connectivity allows. The principal theorem is the following.

Theorem 1: If M^n is a k-connected topological n-manifold (without boundary) and q is the minimum of k and $n-3$, then M^n can be covered by p open cells if $p(q+1) > n$. Futhermore, these cells may be chosen so that the intersection of any collection of these cells is $(q-1)$-connected.

As a consequence of this theorem it is shown that a contractible open n-manifold ($n \geqq 5$) is the union of two open cells whose intersection is a contractible open manifold. One might note for instance that a 3-connected 10-manifold can be covered by 3 open cells whose intersections are 2-connected.

1. Definitions and notation. An n-manifold is a connected separable locally Euclidean metric space. Superscripts will denote dimension. If K^r is an abstract simplicial complex, we denote its carrier by $|K^r|$. The s-skeleton of K^r will be denoted by K^s. The complementary skeleton (sometimes called the dual skeleton) of K^s is defined to be the union of all simplexes in the first barycentric subdivision of K^r whose carriers do not intersect $|K^s|$. We denote the complementary skeleton of K^s by K^s_* and note that the dimension of K^s_* is $r-s-1$. If Ψ is a homeomorphism of the unit ball in E^n into a manifold, then we denote by $|\Psi|$ the image of Ψ and by $|\Psi|_\alpha$, $0 \leqq \alpha \leqq 1$ the image under Ψ of the ball of radius α.

2. Covering by cells. In what follows we shall rely heavily on the topological engulfing of Newman [3].

THEOREM 2.1. *Let X be a locally tame closed set of dimension $k \leqq n-3$ in a k-connected topological n-manifold M^n, and let U be a $(k-1)$-connected open set in M^n such that $X \sim U$ is compact. Then there is a homeomorphism $h: M^n \twoheadrightarrow M^n$ such that $X \subset h(U)$ and h is the identity on the complement of a compact set in M^n.*

Using this engulfing we shall prove

THEOREM 2.2. *Let M^n be a k-connected topological n-manifold* *and let q be the minimum of k and $n-3$, then M^n can be covered by p open n-cells if $p(q+1)>n$.*

This theorem is proven by induction using the following lemmas.

LEMMA 2.3. *Let K^r be a finite subcomplex of a triangulation T of E^n, let U, U' and U'' be open sets in E^n such that $U \subset U' \subset U''$, no simplex in $|T|$ intersects both U and $E^n \sim U'$, no simplex in $|T|$ intersects both U' and $E^n \sim U''$, and $K^s \subset U''$. Suppose $V \subset E^n$ is an open set containing $|K^s_*|$. Then there is a homeomorphism h: $E^n \twoheadrightarrow E^n$ such that* (i) *$K^r \subset h(U'') \cup V$,* (ii) *$h \mid (U \cup |K^s| \cup |K^s_*|) = 1$ and* (iii) *there is a compact set $A \subset E^n$ such that $h \mid E^n \sim A = 1$.*

LEMMA 2.4. *Let M^n be a k-connected n-manifold and let q be the minimum of k and $n-3$. Let $|\phi|$ be a cell in M^n, $0 \leqq \alpha \leqq 1$ and $|K^r|$ an r-dimensional polyhedron in $|\phi|$. Let $|\Psi_1|, |\Psi_2|, \cdots, |\Psi_m|$ be n-cells in M^n where $m(q+1)>r$, then there exist homeomorphisms $h_1, h_2, \cdots, h_m$ of M^n onto itself such that*

$$K^r \subset h_1(|\mathring{\Psi}_1|) \cup h_2|\mathring{\Psi}_2| \cup \cdots \cup h_m|\mathring{\Psi}_m|$$

and $|\Psi_1|_\alpha \subset h_i(|\Psi_i|_{[(\alpha+1)/2]})$.

Sketch of the proof of Lemma 2.3. There exists a homeomorphism g of K^r onto itself such that $g(U'' \cap |K^r|) \cup (V \cap |K^r|) = |K^r|$ and $g \mid (|K^r| \cap U') = 1$. (To get g push out linearly from K^s_*. See ([6], p. 570) for more details on this push.) We extend g by coneing to get a homeomorphism h of E^n onto itself so that h is the identity on the complement of the union of all n-simplexes in E^n having a face in $|K^r|$ that does not lie in U'. h is the desired homeomorphism.

Proof of Lemma 2.4. Let T be a triangulation of $|\phi|$ such that K^r is a subcomplex of T. We proceed by induction on $b=[r/(q+1)]$. If $b=0$ then $r \leqq q$. Applying the method of Connell [3] we stretch $|\Psi_1|_1$ over $|\Psi_1|_{[(\alpha+1)/2]} \cup |K^r|$ keeping $|\Psi_1|_\alpha$ fixed. Assume now that the theorem is true for every r' such that $[r'/(q+1)]<b$ and suppose $[r/(q+1)]=b$ and $m(q+1)>r$. Let T_1 be a subdivision of T such that no simplex of T_1 meets $\mathrm{Bd}\,|\Psi_m|_{[(\alpha+1)/2]}$ and $\mathrm{Bd}\,|\Psi_m|_1$ or $\mathrm{Bd}\,|\Psi_m|_\alpha$ and $\mathrm{Bd}\,|\Psi_m|_{[(\alpha+1)/2]}$. Let $|K^r_1|$ be the complex $|K^r|$ after subdivision. Assume $|\Psi_1|, |\Psi_2|, \cdots, |\Psi_m|$ are given n-cells. As in the case $b=0$ we can stretch $|\Psi_m|_1$ by h_m over $|\Psi_m|_{[(\alpha+1)/2]} \cup |K^q_1|$ keeping $|\Psi_m|_{[(\alpha+1)/2]}$ fixed.

By the inductive hypothesis there are homeomorphisms $h_1, h_2, \cdots, h_{m-1}$ such that $|K^q_1| \subset \bigcup_{i+1}^{m-1} h_i(|\Psi_i|)$ and $h_i \mid |\Psi_i|_{[(\alpha+1)/2]} = 1, i = 1, 2, \cdots,$

$m-1$. We apply Lemma 2.3 to $|K_1^q|$ with $U = |\mathring{\Psi}_m| \cap |\mathring{\phi}|$, $U' = |\mathring{\Psi}_m|_{[(\alpha+1)/2]} \cap |\mathring{\phi}|$. $U'' = |\mathring{\Psi}_m|_1 \cap |\mathring{\phi}|$ and $V = \bigcup_{i+1}^{m-1} h_i(|\Psi_i|_1)$, so get the desired homeomorphism h_m.

Proof of Theorem 2.2. Let $|\phi_1|, |\phi_2|, \cdots$ be a collection of cells in M^n such that $|\phi_1|_\alpha, |\phi_2|_\alpha, \cdots$ covers M^n for some $\alpha, 0 < \alpha < 1$. (In case M^n is compact, a finite sequence of cells would suffice.) Choose a sequence (α_i) of real numbers so that $\alpha < \alpha_1 < \alpha_2 \cdots < 1$. Using Lemma 2.4 we stretch $|\phi_1|_{\alpha_1} |\phi_2|_{\alpha_1} \cdots |\phi_p|_{\alpha_1}$, over $|\phi_{p+1}|$ by $h_{1,1} \cdots h_{p,1}$ so that $|\phi_i|_\alpha \subset h_{i,1}(|\phi_i|_{\alpha_1})$ for each $i = 1, 1, \cdots, p$. Next we stretch $\bigcup_{i=1}^p h_i(|\phi_i|_{\alpha_1})$ over $|\phi_{p+2}|_\alpha$ by $h_{1,2}, h_{2,2}, \cdots, h_{p,2}$ as before. Continuing indictively we get $|\phi_i|_\alpha \subset h_{i,1}|\phi_i|_{\alpha_1} \subset h_{i,2}h_{i,1}(|\phi_i|_{\alpha_2}) \cdots$ which is a monotone union of open n-cells. By [1] this union is an open cell for each i. Thus we have covered M^n by open cells.

Note that if M^n were compact, it could be covered by p closed cells with bicollared boundaries.

3. Improving the intersections of the covering cells.

The following lemmas will enable us to improve the connectivity of the intersections of the covering cells. In referring to the homotopy groups we omit reference to the fixed base point even though we do not assume the sets to be path connected.

LEMMA 3.1. *If $A \subset E^n$ is compact, U is a neighborhood of A and $i_*\colon \Pi_k(A) \to \Pi_k(U)$ is the map induced by inclusion, then $i_*(\Pi_k(A))$ is finitely generated.*

Proof. Let $|K|$ be a finite polyhedron in E^n such that $A \subset |K| \subset U$. Then $\Pi_k(|K|)$ is finitely generated for each k, so factoring the map i_* through $\Pi_k(|K|)$ we see that $i_*(\Pi_k(A))$ is finitely generated.

LEMMA 3.2. *Let M^n be a q-connected n manifold ($q \leqq n-3$) and let $|\Psi_1|$ and $|\Psi_2|$ be n-cells in M and let $0 < \alpha < \beta < 1$. Then there exists a homeomorphism $h\colon M^n \to M^n$ such that $h \mid |\Psi_1|_\alpha = 1$ and if $i\colon (|\Psi_1|_\alpha \cap |\Psi_2|_\alpha) \to h(|\Psi_1|_\beta \cap |\Psi_2|_\beta)$ is the inclusion then*

$$i_*\colon \Pi_k(|\Psi_1|_\alpha \cap |\Psi_2|_\alpha) \to \Pi_k(h(|\Psi_1|_\beta \cap |\Psi_2|_\beta))$$

is trivial for $k = 1, 2, \cdots, q-1$.

Proof. Note that $|\mathring{\Psi}_1|_{[(\alpha+\beta)/2]} \cap |\mathring{\Psi}_2|_{[(\alpha+\beta)/2]}$ is a neighborhood of the compact set $|\Psi_1|_\alpha \cap |\Psi_2|_\alpha$ in $|\mathring{\Psi}|_\beta$ so, by Lemma 3.1 the image of $\Pi_k(|\Psi_1|_\alpha \cap |\Psi_2|_\alpha)$ in $\Pi_k(|\mathring{\Psi}_1|_{[(\alpha+\beta)/2]} \cap |\mathring{\Psi}_2|_{[(\alpha+\beta)/2]})$ is finitely generated. Note that the generators can be assumed to be piecewise linear in

$|\Psi_2|_\beta$. Since $|\Psi_2|_\beta$ is contractible each of these generators bounds a singular polyhedral cell. Corresponding to the groups

$$\Pi_1(|\Psi_1|_\alpha \cap |\Psi_2|_\alpha),\ \Pi_2(|\Psi_1|_\alpha \cap |\Psi_2|_\alpha),\ \cdots,\ \Pi_{q-1}(|\Psi_1|_\alpha \cap |\Psi_2|_\alpha)$$

and the groupoid $\Pi_0(|\Psi_1|_\alpha \cap |\Psi_2|_\alpha)$ we get a finite collection of polyhedral singular cells in $|\Psi_2|_\beta$ of dimension less than or equal to q. Let P^s be the union of all these polyhedral singular cells. Using the topological engulfing we get a homeomorphism $g\colon M \to M$ such that $g \mid |\Psi_1|_{[(\alpha+\beta)/2]} = 1$ and $P^s \subset g(|\Psi_1|_\beta)$. It is easy to see that

$$i_*\colon \Pi_r(g(|\Psi_1|_\alpha) \cap |\Psi_2|_\alpha) \to \Pi_r(g(|\Psi_1|_\beta) \cap |\Psi_2|_\beta)$$

is trivial for $r = 0, 1, 2, \cdots, q-1$.

Using exactly the idea of the proof of the previous lemma, one can prove the following generalization.

LEMMA 3.3. *Let M^n be a q-connected n-manifold* ($q \leqq n-3$), *let* $0 < \alpha < \beta < 1$ *and let $|\Psi_1|, \cdots |\Psi_m|$ be n-cells in M^n. Then there exist homeomorphisms $g_1, \cdots, g_{m-1}$ of M^n onto itself such that $g_i \mid |\Psi_i|_\alpha = 1$ and for any subset $k_1, k_2, \cdots, k_r$ of distinct integers between* 1 *and m the map $i_*\colon \Pi_k(\bigcap_{i=1}^r (|\Psi_{k_i}|_\alpha)) \to \Pi_k(\bigcap_{i=1}^r g_{k_i}(|\Psi_{k_i}|_\beta)$ is trivial, where i_* is induced by the inclusion map.*

Proof of Theorem 1. Our proof is essentially a refinement of the proof of Lemma 2.3 interlacing the steps of the proof of 2.3 with the improvements of the intersection given by Lemma 3.3. Let $|\phi_1|, |\phi_2|, |\phi_3|, \cdots$ be a collection of cells in M^n such that $\bigcup_{i=1}^\infty |\phi_i|_\alpha$ covers M^n for some α, $0 < \alpha < 1$. Let (α_i) and (β_i) be sequences of real numbers such that $\alpha < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$ and $\alpha_i < 1$ for each i. As in the proof of Lemma 2.3 we stretch $\bigcup_{i=1}^p |\phi_i|_{\alpha_1}$ over $|\phi_{p+1}|_\alpha$ by $h_{1,1}, h_{2,1}, \cdots h_{p,1}$ so that $h_{i,1} \mid |\phi_i|_\alpha = 1$.

Next using Lemma 3.3 we get homeomorphisms $g_{1,1}, g_{2,1}, \cdots, g_{p,1}$ of M^n onto itself so that $g_{i,1} \mid |\phi_i|_{\alpha_1} = 1$ and for any integers $k_1, k_2, \cdots, k_r$ between 1 and p, $i_*\colon \Pi_k(\bigcap_{i=1}^r h_{k_i}(|\phi_{k_i}|_{\alpha_1})) \to \Pi_k(\bigcap_{i=1}^r g_{k_i,1} h_{k_i,1}(|\phi_{k_1}|_\beta)$ is trivial for $0 \leqq k < q$. We continue this process first engulfing $|\phi_{p+2}|_\alpha$ then improving the intersections. For each $i = 1, 2, \cdots, p$ we get an increasing sequence of open cells $|\phi_i|_\alpha$, $g_{i,1}h_{i,1}(|\phi_i|_{\beta_1})$, $g_{1,2}h_{i,2}g_{i,1}h_{i,1}(|\phi_i|_{\beta_2})$, $\cdots$, whose direct limit is an open n-cell, call it C_i. For any collection $k_1, k_2, \cdots, k_r$ of integers between 1 and p we see, using the fact that the groups of the intersection $\bigcap_{i=1}^r C_{k_i}$ are the direct limits of

$$\bigcap_{i=1}^r |\phi_{k_i}|_\alpha,\ \bigcap_{i=1}^r g_{k_i} h_{k_i}(|\phi_{k_1}|_{\beta 1}) \cdots \quad \text{that} \quad \Pi_k(\bigcap_{i=1}^r C_{k_i}) = 0$$

for $k = 0, 1, 2, \cdots, q-1$.

We should point out that a special case of Theorem 1 was proved by Zeeman in [6].

COROLLARY 3.4. *A contractible open n-manifold, $n \leqq 5$ is the union of two open cells whose intersection is a contractible open manifold.*

Proof. By Theorem 1 we can cover such a manifold by two open cells C_1 ahd C_2 whose intersection is 1-connected. Using the Meyer-Victoris sequence we get the exact sequence

$$\cdots \to H_{k+1}(C_1 \cup C_2) \to H_k(C_1 \cap C_2) \to H_k(C_1) \oplus H_k(C_2) \to \cdots .$$

But $H_{k+1}(C_1 \cup C_2) = H_k(C_1) = H_k(C_2) = 0$ for every k, so $H_k(C_1 \cap C_2) = 0$. The Hurewics isomorphism shows $\Pi_k(C_1 \cap C_2) = 0$ for each k. This implies that $C_1 \cap C_2$ is contractible.

One might hope that the groups of the intersections of the covering cells might be improved to give trivial groups in dimension q. The following example shows that this may not be possible without using more cells to cover the manifold.

EXAMPLE. $S^3 \times S^3$ can be covered by three open cells whose intersections are 1-connected. We show that these intersections cannot be improved to be 2-connected. Suppose, to the contrary that

$$S^3 \times S^3 = C_1 \cup C_2 \cup C_3$$

and that $C_1 \cap C_2, C_1 \cap C_3$ and $C_2 \cap C_3$ are 2-connected. Using the Meyer-Victoris sequence we get $H_3(C_1 \cup C_2) \oplus H_3(C_2) \to H_3(S^3 \times S^3) \to H_2((C_1 \cup C_2) \cap C_3)$ and $H_3(C_1) \oplus H_3(C_2) \to H_3(C_1 \cup C_2) \to H_2(C_1 \cap C_2)$ which implies $H_3(C_1 \cup C_2) = 0$. Furthermore we get the exact sequence

$$H_2(C_1 \cap C_3) \oplus H_2(C_2 \cap C_3) \to H_2((C_1 \cup C_2) \cap C_3) \to H_1(C_1 \cap C_2 \cap C_3)$$

but $H_2(C_1 \cap C_3) = H_2(C_2 \cap C_3) = 0 = H_1(C_1 \cap C_2 \cap C_3)$. This implies that $H_2((C_1 \cup C_2) \cap C_3) = 0$. In the first given exact sequence we have all groups being trivial except $H_3(S^3 \times S^3)$. This contradicts exactness. Note that if we are willing to use six open cells to cover $S^3 \times S^3$ we can arrange it so that the intersections are open cells.

COROLLARY 3.5. *Every topological n-manifold can be covered by $(n + 1)$ open n-cells.*

COROLLARY 3.6. *If M^n is a compact manifold ($n \leqq 5$) that is a homotopy n-sphere then $M^n = S^n$.*

Proof. M^n can be covered by two open cells hence $M^n = S^n$.

4. Covering manifolds with boundary with closed cells. Using the methods of § 2 we can get the following theorem.

THEOREM 4.1. *If M^n is a compact manifold with boundary and M^n and* Bd M^n *are q connected, $q \leqq n - 4$ then M^n can be covered by p closed cells if $p(q+1) > n$.*

We note that if A^n is the closure of region between two tame $(n-1)$ spheres in S^n then the annulus conjecture says that $A^n = S^{n-1} \times [0, 1]$. Note that the annulus conjecture is equivalent with the assertion that A^n can be covered by two closed cells. It is the stumbling block presented by the annulus conjecture that prevents us from weakening the hypothesis of Theorem 4.1 to require only that each component of Bd M^n be q-connected.

5. An equivalence for the 3-dimensional Poincare conjecture. The Poincare conjecture says that a compact n-manifold without boundary that has the same homotopy groups as a sphere is a sphere. This conjecture is known to be true for $n \neq 3, 4$. We prove the following:

THEOREM 4.1. *The 3-dimensional Poincare conjecture is true if and only if every contractible open 3-manifold that is 1-connected at infinity is the union of two open cells.*

Proof. According to Wall [5] if the Poincare conjecture is true, then an open 3-manifold that is 1-connected at infinity is a compact manifold minus a point. In the case of a contractible open manifold that is 1-connected at infinity, the 1-point compactification would be a homotopy 3-sphere, which again by the Poincare conjecture is a 3-sphere.

Conversely, suppose that each contractible open 3-manifold that is 1-connected at infinity is the union of two open cells. Let S be a homotopy 3-sphere and $p \in S$. Then $M = S \sim \{p\}$ is a contractible open manifold that is 1-connected at infinity. By hypothesis $M = E_1 \cup E_2$ where E_1 and E_2 are open 3-cells. Thus S can be covered by three open cells. McMillan and Hempel [2] have shown that such a manifold is a 3-sphere with handles. But the only simply connected 3-sphere with handles is S^3.

REFERENCES

1. M. Brown, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814.
2. J. Hempel and D. R. McMillan, Jr., *Covering 3-manifolds with open cells.*

3. M. H. A. Newman, *The engulfing theorem for topological manifolds*, Ann. of Math. **84** (1966), 555-571.

4. J. L. Stern, *Contractible open manifolds and cellular maps.*

5. C. T. C. Wall, *Open 3-manifolds which are 1-connected at infinity*, Quart. J. Math **16** (1965), 263-268.

6. E. C. Zeeman, *"The Poincare conjecture for $n \geqq 5$," topology of 3-manifolds and related topics*, Prentice-Hall, Englewood Cliffs, N. J., 1963.

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