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A CONTINUOUS PARTIAL ORDER FOR PEANO CONTINUA

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A theorem of R. J. Koch states that a compact continuously partially ordered space with some natural conditions on the partial order is arcwise connected. L. E. Ward, Jr., has conjectured that Koch's arc theorem implies the well-known theorem of R. L. Moore that a Peano continuum is arcwise connected. In this paper Ward's conjecture is proved.

1. Preliminaries. If Γ is a partial order on a set X we will write $x \leqq_{\Gamma} y$ or $x \leqq y$ for $(x, y) \in \Gamma$. We will let $L(a) = \{x: (x, a) \in \Gamma\}$. If X is a topological space, then Γ is a *continuous* partial order on X provided the graph of Γ is closed in $X \times X$. If Γ is a continuous partial order on the space X, then $L(x)$ is a closed set for every $x \in X$. A *zero* of a continuously partially ordered space X is an element 0 such that $0 \in L(x)$ for all $x \in X$. An *arc* is a locally connected continuum with exactly two noncutpoints. A *real arc* is a separable arc. A *Peano continuum* is a locally connected metric continuum.

We will use the following statement of Koch's arc theorem.

THEOREM 1. *If X is a compact continuously partially ordered space with zero such that $L(x)$ is connected for each $x \in X$, then X is arcwise connected.*

We will show that Peano continua admit such partial orders by proving the following:

THEOREM 2. *If X is a compact connected locally connected metric space, then X admits a continuous partial order with a zero such that $L(x)$ is connected for all $x \in X$.*

The proof of this theorem will use some definitions and results due to R. H. Bing [1]. An ε-partition $\mathscr{P}_{\varepsilon}$ of a subspace K of a metric space M is a finite set of closed subsets of M, each with diameter less than ε, the union of which is K, and such that the interiors in M of all the elements of $\mathscr{P}_{\varepsilon}$ are nonempty, connected, dense in the closed subset, and are pairwise disjoint. The subspace K is *partitionable* if for each positive number ε, there exists an ε-partition of K.

LEMMA 1. *Let M be a compact connected locally connected metric space. For each positive number ε there exists an ε-partition $\mathscr{P}_{\varepsilon}$ of*

M such that each element of $\mathscr{P}_\varepsilon$ *is partitionable.*

Bing proves this lemma in [1].

The proof of the Theorem 2 will follow in two parts. In the first part a relation Δ will be constructed on the Peano continuum X. The second part will be concerned with proving that Δ is the desired partial order on X. We will let d denote the metric on X.

2. The construction of the relation Δ. We will define inductively a sequence $\{\mathscr{F}(i)\}_{i=1}^{\infty}$ of finite partitions of X. With each partition we will associate a relation δ_i. The set $\{\delta_i\}_{i=1}^{\infty}$ will be a nest of closed subsets of $X \times X$ and $\Delta = \cap \delta_i$ will be the desired partial order on X.

First choose an arbitrary element of X. Call this element 0. This will be the 0 of the partial order to be constructed on X.

We will now construct the relation δ_1 as the first step of the induction.

Let $\mathscr{F}(1)$ be a finite partition on X such that for $F \in \mathscr{F}(1)$, diam $(F) < 1/2$, and such that F is partitionable. We will classify the elements of $\mathscr{F}(1)$ according to how "far away" they are from 0. Let $\mathscr{F}(1,0)$ be the set $\{F \in \mathscr{F}(1) \colon 0 \in F\}$. If $\mathscr{F}(1, i)$ has been defined for $i = 1, 2, \cdots, t-1$, let

$$(1) \quad \mathscr{F}(1, t) = \{F \in \mathscr{F}(1) - \textstyle\bigcup_{i=0}^{t-1} \mathscr{F}(1, i) \colon F \cap (\cup \mathscr{F}(1, t-1)) \neq \emptyset\} .$$

If F is an element of $\mathscr{F}(1, t)$ we will say F has order t. Because $\mathscr{F}(1)$ is a cover of the connected set X with connected sets, there is a chain of elements of $\mathscr{F}(1)$ between any two points of X. That is, if F is an element of $\mathscr{F}(1)$ then there exists some integer t and a set $\{F_i\}_{i=0}^{t} \subset \mathscr{F}(1)$ such that $0 \in F_0$, $F = F_t$ and for $i, j \in \{0, 1, \cdots, t\}$ $F_i \cap F_j \neq \emptyset$ if and only if $|i - j| \leqq 1$. This is the condition necessary for F to have order t. Thus order is defined for all elements of $\mathscr{F}(1)$.

We now define sets $J(F)$, for $F \in \mathscr{F}(1)$, which will be in a sense "predecessors" of the elements of F. For $F \in \mathscr{F}(1, 0)$ let $J(F) = F$. If $J(F)$ has been defined for $F \in \mathscr{F}(1, t-1)$ and if $F_t \in \mathscr{F}(1, t)$ let

$$(2.1) \qquad J(F_t) = F_t \cup \cup \{J(F) \colon F \cap F_t \neq \emptyset, F \in \mathscr{F}(1, t-1)\} .$$

We now define the relation δ_1 on X by defining for all $x \in X$ the set $L_1(x) = \{y \colon (y, x) \in \delta_1\}$. Set

$$L_1(x) = \cup \{J(F) \colon x \in F \in \mathscr{F}(1)\} .$$

The relation δ_1 is reflexive but not anti-symmetric or transitive.

In order to define the relations $\delta_2, \cdots, \delta_n$, it will be useful to introduce some additional notation. Let F be an arbitrary fixed element of $\mathscr{F}(1, t)$ for some nonnegative integer t. Let ∂F denote the boundary of F. For $t = 0$, let $\mathscr{E}_*(F) = \{0\}$, and for $t > 0$, let

$$\mathscr{E}_*(F) = \{E \in \mathscr{F}(1, t-1) : E \cap F \neq \varnothing\} . \tag{3.1}$$

Notice that $\mathscr{E}_*(F)$ is not empty by (1) since $F \in \mathscr{F}(1, t)$. Let

$$\partial_* F = F \cap [\cup \mathscr{E}_*(F)] . \tag{4.1}$$

Except for the case when $t = 0$ and $\partial_* F = \{0\}$, $\partial_* F$ is that part of the boundary of F which is also part of the boundary of sets of order $t - 1$. Let

$$\mathscr{E}(F) = \{E \in \mathscr{F}(1) : E \neq F \quad \text{and} \quad E \cap F - \partial_* F \neq \varnothing\} .$$

That is, $\mathscr{E}(F)$ is the set of elements of $\mathscr{F}(1)$, other than F itself and the sets of order $t - 1$, whose intersection with $F - \partial_* F$ is not empty. Note that the elements of $\mathscr{E}(F)$ either have order t or order $t + 1$. Let $\mathscr{E}^*(F)$ be the set $\{E \in \mathscr{E}(F) : F \in \mathscr{E}_*(E)\}$ and let

$$\partial^* F = \cup \{F \cap E : E \in \mathscr{E}^*(F)\} .$$

Then $\mathscr{E}^*(F)$ is the set of sets in $\mathscr{F}(1)$ which have order $t + 1$ and have a nonempty intersection with F. The sets $\mathscr{E}(F)$ and $\mathscr{E}^*(F)$ may be empty. For $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$ let $\partial_E(F) = E \cap F$.

If $\mathscr{E}^*(F)$ is not empty, let $\rho(F)$ be $d(\partial_* F, \partial^* F)$. Thus $\rho(F)$ is the infimum of the distances between the points of F which are also in the sets of order $t - 1$ and those points of F which are also in sets of order $t + 1$. This distance is positive since, by (1), for each $E \in \mathscr{E}^*(F)$, $\partial_E F$ and $\partial_* F$ are disjoint closed sets. If $\mathscr{E}^*(F)$ is empty, let $\rho(F)$ be $\operatorname{diam}(F)$.

The remainder of the construction of δ_2 generalizes directly to the construction of δ_n. Thus we will assume that $\mathscr{F}(n)$, a partition of X, and the sets $\mathscr{F}(n, t)$ have been defined for $t = 0, 1, \cdots$, and that for $F \in \mathscr{F}(n)$, $\partial_* F$, $\mathscr{E}_*(F)$, $\mathscr{E}(F)$, $\mathscr{E}^*(F)$, $\partial^* F$ and $\rho(F)$ have been defined and that for each $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$, $\partial_E F$ has been defined. We will now define some special subsets of each $F \in \mathscr{F}(n)$ which we will use to define the relation δ_n.

In order for the final relation Δ to be transitive it will be necessary that the elements of $\partial F - (\partial_* F \cup \partial^* F)$ have no successors in the relation Δ. To this end we want to find for each $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$ a partitionable subset of F which contains $\partial_* F$ and $\partial_E F$ but contains no points of ∂F which are not "close" to $\partial_* F$ or $\partial_E F$. We use the following lemma.

LEMMA 2. *Let $\varepsilon > 0$ and let F be a partitionable compact subset of a metric space X such that the interior of F is connected and locally connected. Let B_0 be either a nonempty closed subset of ∂F or a point in the interior of F. Let $\{B_i\}_{i=0}^m$ be a finite set of nonempty closed subsets of ∂F, such that $\bigcup_{i=0}^m B_i \supset \partial F$. Then there exists a set $\{C_i\}_{i=0}^m$ of partitionable subsets of F such that for $i = 0, 1, \cdots, m$ C_i is closed, $(\operatorname{int} C_i) \cup B_i \cup B_0$ is connected, $B_i \subset C_i$ and if $x \in \partial F \cap C_i$ then either $d(x, B_i) < \varepsilon$ or $d(x, B_0) < \varepsilon$. Further $C_0 \subset C_i$, $i = 0, 1, \cdots, m$ and $F = \bigcup_{i=0}^m C_i$.*

Proof. By Lemma 1, F is partitionable so let $\mathscr{P}(F)$ be a partition of F such that for $P \in \mathscr{P}(F)$, $\operatorname{diam}(P) < \varepsilon/2$ and P is partitionable.

For $x \in \operatorname{int} F$ let U_x be a connected open set containing x whose closure misses ∂F. Let $\mathscr{U} = \{U_x : x \in \operatorname{int} F\}$. For each $P \in \mathscr{P}(F)$ choose $x(P) \in \operatorname{int} P \cap \operatorname{int} F$ and let

$$Q = \{x(P) : P \in \mathscr{P}(F)\} \cup \bigcup \{P \in \mathscr{P}(F) : P \cap \partial F = \emptyset\} .$$

Let $\mathscr{U}_1$ be a finite cover of the closed set Q by elements of $\mathscr{U}$. We can write $\mathscr{U}_1 = \{U_i\}_{i=1}^k$. Now fix some element $P_0 \in \mathscr{P}(F)$ such that $P_0 \cap B_0 \neq \emptyset$. The interior of F is connected by the connected open sets of $\mathscr{U}$, so that for each $U_i \in \mathscr{U}_1$ there exists $\{U_{ij}\}_{j=0}^{k(i)} \subset \mathscr{U}$ such that $x(P_0) \in U_{i0}$, $U_{ik(i)} = U_i$ and $U_{ij} \cap U_{il} \neq \emptyset$ if and only if $|j - l| \leqq 1$. That is, there is a finite chain of sets of $\mathscr{U}$ connecting each element of $\mathscr{U}_1$ with $x(P_0)$. Let

$$\mathscr{U}' = \{U_{ij} : i = 0, \cdots, k ; j = 0, \cdots, k(i)\} \\ \cup \{P \in \mathscr{P}(F) : P \cap B_0 \neq \emptyset\} .$$

Note that $\cup \mathscr{U}'$ is a connected subset of F and that if $x \in Cl(\cup \mathscr{U}')$ and $d(x, B_0) > \varepsilon/2$, then $x \notin \partial F$. This is because the boundary of each element of $\mathscr{U}$ misses the boundary of F, so that if x were in ∂F, x would be an element of P for some $P \in \mathscr{P}(F)$ such that $P \cap B_0 \neq \emptyset$ and we have that $\operatorname{diam}(P) < \varepsilon/2$. Also note that

$$F \subset (\cup \mathscr{U}') \cup \bigcup \{P \in \mathscr{P}(F) : P \cap \partial F \neq \emptyset\}$$

since $\mathscr{U}_1 \subset \mathscr{U}'$ and $\mathscr{U}_1$ is a cover of $\cup \{P \in \mathscr{P}(F) : P \cap \partial F = \emptyset\}$.

Now consider $\mathscr{U}_2 = \{U \in \mathscr{U}' : \bar{U} \cap \partial F = \emptyset\}$. Let

$$\nu(F) = \min\{\varepsilon/2, \min\{d(\bar{U}, \partial F) : U \in \mathscr{U}_2\}\} .$$

For each $P \in \mathscr{P}(F)$ let $\mathscr{G}(F, P)$ be a partition of P such that if

$$F' \in \mathscr{G}(F, P) , \quad \text{then} \quad \operatorname{diam}(F') < \frac{\nu(F)}{4}$$

and F' is partitionable. Let

(5) $$\mathscr{G}(F) = \cup\{\mathscr{G}(F, P): P \in \mathscr{P}(F)\} .$$

We are ready now to define the sets C_i, $i = 0, 1, \cdots, m$. The set C_0 will meet ∂F only "close" to B_0 and C_i, $i = 1, \cdots, m$ will meet ∂F only "close" to B_i or B_0. Let $D = [(\cup \mathscr{U}') - \partial F] \cup B_0$. The set D is a connected subset of $(\operatorname{int} F) \cup B_0$. Let

$$C_0 = \cup\{F' \in \mathscr{G}(F): F' \cap D \neq \varnothing\} .$$

Because D is connected and covered by $\mathscr{G}(F)$, C_0 is a closed and connected subset of F. Also, if $x \in C_0 \cap \partial F$, then $d(x, B_0) < \varepsilon$, for if $x \in C_0 - B_0$ then $x \in F' \in \mathscr{G}(F)$ such that $F' \cap D \neq \varnothing$. Consequently there exists a $U \in \mathscr{U}'$ such that $F' \cap U \neq \varnothing$. It then follows that if x were in $F' \cap \partial F$ then, by definition of $\nu(F)$, $U = P$ for some $P \in \mathscr{P}(F)$ such that $P \cap B_0 \neq \varnothing$, and

$$d(x, B_0) \leqq \operatorname{diam}(F') + \operatorname{diam}(P) < \frac{\nu(F)}{4} + \varepsilon/2 \leqq \varepsilon/8 + \varepsilon/2 < \varepsilon .$$

If we let $C'_0 = [(\operatorname{int} F) \cap C_0] \cup B_0$, then C'_0 is connected because C'_0 contains D and

$$C'_0 = \cup\{[F' \cap (\operatorname{int} F)] \cup [F' \cap B_0]: F' \subset C_0\} ,$$

which is a union of connected sets which cover D and each of which has nonempty intersection with D.

Now let

$$C_i = C_0 \cup \cup \{\cup \mathscr{G}(F, P): P \in \mathscr{P}(F), P \cap B_i \neq \varnothing\} .$$

We see that C_i is a closed subset of F and it is connected because C_0 and each $P \in \mathscr{P}(F)$ is connected and $x(P) \in P \cap C_0$. Let $C'_i = [(\operatorname{int} F) \cap C_i] \cup B_i \cup B_0$. Then C'_i is a connected subset of F, for

$$C'_i = C'_0 \cup \cup \{[P \cap \operatorname{int} F] \cup [B_i \cap P]: P \in \mathscr{P}(F), P \cap B_i \neq \varnothing\} ,$$

and C'_0 and $[P \cap \operatorname{int} F] \cup [P \cap B_i]$ are connected and $x(P) \in C'_0 \cap P \cap \operatorname{int} F$ for each $P \in \mathscr{P}(F)$.

Further note that if $x \in C_i \cap \partial F$, then either $d(x, B_i) < \varepsilon$ or $d(x, B_0) < \varepsilon$. Also F is a subset of $\bigcup_{i=0}^{m} C_i$.

This completes the proof of Lemma 2.

To apply this lemma to the theorem we let $\varepsilon = \rho(F)/3$, $B_0 = \partial_* F$ and $\{B_i\}_{i=1}^{m(F)} = \{\partial_E F: E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)\}$. Thus for $F \in \mathscr{F}(n)$ we get sets C_i, $i = 0, 1, \cdots, m(F)$ satisfying the conditions of the lemma. For clarity we will sometimes write $C(F)$ for $C_0(F)$ and use $C(F, E)$ for $C_i(F)$ when $B_i = \partial_E F$ for $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$. We will also use $C'(F)$ for C'_0 and $C'(F, E)$ for C'_i.

We will now define the relation δ_n on X. First we inductively define sets $J(F)$ and $J(F, E)$ for each $F \in \mathscr{F}(n)$, $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$. The elements of $J(F)$ and $J(F, E)$ will, in a sense, be "predecessors" of the elements of $C(F)$ and $C(F, E)$ respectively.

For $F \in \mathscr{F}(n, 0)$, let $J(F) = C(F)$ and for $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$ let $J(F, E) = C(F, E) \cup J(F)$. Then suppose $J(F)$ and $J(F, E)$ have been defined for all $F \in \mathscr{F}(n, t-1)$, $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$. Let F be an element of $\mathscr{F}(n, t)$. Define

$$\begin{aligned} &J(F) = C(F) \cup \{J(F_*, F): F_* \in \mathscr{E}_*(F)\} \quad \text{and let} \\ &J(F, E) = C(F, E) \cup J(F) \quad \text{for} \quad E \in \mathscr{E}(F) \cup \mathscr{E}^*(F) \,. \end{aligned} \tag{2.2}$$

Thus we can define $J(F)$ and $J(F, E)$ for all

$$F \in \mathscr{F}(n), E \in \mathscr{E}(F) \cup \mathscr{E}^*(F) \,.$$

The sets $J(F)$ and $J(F, E)$ are each closed since they are a finite union of closed sets. Also $J(F, E)$ is connected if $J(F)$ is connected since for each $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$, $C(F, E)$ contains $C(F)$. But $J(F)$ is connected since if $F_* \in \mathscr{E}_*(F)$ then for each $P \in \mathscr{P}(F_*)$ such that

$$P \cap F \neq \varnothing, P \cap \partial_* F \cap C(F_*, F) \neq \varnothing \,.$$

Thus $C(F)$ is not separated from $C(F_*, F)$ for any $F_* \in \mathscr{E}_*(F)$.

We will let $L_n(x) = \{y: (y, x) \in \delta_n\}$ and define δ_n by defining the sets $L_n(x)$ for all $x \in X$. Let $x \in X$ and $F \in \mathscr{F}(n)$. If $x \notin F$, let $K_F(x) = \varnothing$. If $x \in F$ and $x \in C(F)$, let $K_F(x) = J(F)$. If $x \in F$ and

$$x \in \cup \{C(F, E): E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)\} - C(F)$$

there exists some $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$ such that $x \in C(F, E)$, so let

$$K_F(x) = \cup \{J(F, E): x \in C(F, E), E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)\} \,.$$

Then let

$$L_n(x) = \cup \{K_F(x): F \in \mathscr{F}(n)\} \,. \tag{6}$$

Then $L_n(x)$ is closed and connected for each x, for it is a nonempty finite union of closed sets, and the nonempty sets comprising that union are each connected and contain x.

The relation δ_n is closed because

$$\begin{aligned} \delta_n = {} & \cup \{C(F) \times C(F): F \in \mathscr{F}(n)\} \\ & \cup \cup \{C(F', E') \times C(F, E): F \in \mathscr{F}(n) \,, \\ & E \in \mathscr{E}(F) \cup \mathscr{E}^*(F), C(F', E') \subset J(F, E)\} \end{aligned} \tag{7}$$

which is a finite union of products of closed sets.

To complete the induction we will assume δ_n has been defined and we will define the sets, $\mathscr{F}(n+1)$, $\mathscr{F}(n+1, t)$, $t = 0, 1, \cdots$, and for each $F \in \mathscr{F}(n+1)$ we must define $\partial_* F$, $\mathscr{E}_*(F)$, $\mathscr{E}(F)$, $\mathscr{E}^*(F)$, $\partial^* F$, $\rho(F)$ and for each $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$, $\partial_E F$.

First let $\mathscr{F}(n+1) = \cup\{\mathscr{G}(F): F \in \mathscr{F}(n)\}$ where $\mathscr{G}(F)$ is as defined in (5).

As in the initial induction step, we will assign to each $F \in \mathscr{F}(n+1)$ an order which will, in a sense, classify the sets of $\mathscr{F}(n+1)$ according to how "far away" they are from 0. But since we want to assure that $\delta_{n+1} \subset \delta_n$, or, what is the same thing, $L_{n+1}(x) \subset L_n(x)$ for all $x \in X$, we must take more care in defining the order of an element F of $\mathscr{F}(n+1)$. Because each $F \in \mathscr{F}(n+1)$ is contained in an unique element of $\mathscr{F}(n)$, the "predecessors" of the elements of F must be contained in the set of "predecessors" of that unique element of $\mathscr{F}(n)$ which contains it.

We will partition $\mathscr{F}(n+1)$ into the sets $\mathscr{F}(n+1, t), t=0, 1, \cdots$, and if $F \in \mathscr{F}(n+1, t)$ we will say F has order t. First we let

$$\mathscr{F}(n+1, 0) = \{F \in \mathscr{F}(n+1): 0 \in F\} .$$

Let F_n be an element of $\mathscr{F}(n, s-1)$ and suppose that order has been defined for the elements of some subset of $\mathscr{G}(F_n)$ which contains at least $\{F \in \mathscr{G}(F_n): F \cap \partial_* F_n \neq \emptyset\}$. Let F be an element of $\mathscr{G}(F_n)$ such that $F \cap C'(F_n) \neq \emptyset$ and such that order has not yet been defined for F. We will let F be an element of $\mathscr{F}(n+1, t)$ and say F has order t if t is the smallest positive integer such that there exists $F_* \in \mathscr{G}(F_n)$ such that $F_* \subset C(F_n)$, F_* has order $t-1$, and $F_* \cap F \cap \operatorname{int} F_n \neq \emptyset$. Let

$$\mathscr{E}_*(F) = \{F_* \in \mathscr{G}(F_n): F_* \in \mathscr{F}(n+1, t-1), F_* \subset C(F_n) \text{ and } F_* \cap F \cap \operatorname{int} F_n \neq \emptyset\} . \tag{3.2}$$

Notice that this is enough to define order for all $F \in \mathscr{G}(F_n)$ such that $F \cap C'(F_n) \neq \emptyset$, since $C'(F_n)$ is connected and covered by the connected sets $[F \cap \operatorname{int} F_n] \cup [F \cap \partial_* F_n]$. Now suppose $F \in \mathscr{G}(F_n)$ but $F \cap C'(F_n) = \emptyset$. Then $F \subset P$ for some unique $P \in \mathscr{P}(F_n)$, where $\mathscr{P}(F_n)$ is as defined in the proof of Lemma 2. Let F be an element of $\mathscr{F}(n+1, t)$ and say F has order t if t is the smallest positive integer such that there exists some $F_* \in \mathscr{F}(n+1, t-1)$ such that $F_* \subset P$ and $F \cap F_* \cap \operatorname{int} P \neq \emptyset$. Let

$$\mathscr{E}_*(F) = \{F_* \in \mathscr{G}(F_n): F_* \in \mathscr{F}(n+1, t-1), F_* \subset P \text{ and } F_* \cap F \cap \operatorname{int} P \neq \emptyset\} . \tag{3.3}$$

This is enough to define order for all $F \in \mathscr{G}(F_n)$ since for each $P \in \mathscr{P}(F)$, int P is connected and covered by the connected sets $F \cap \operatorname{int} P$ for $F \in \mathscr{G}(F_n, P)$ and $P \cap C'(F_n) \neq \emptyset$.

Suppose order has been defined for all $F \in \mathscr{G}(F_n)$ where $F_n \in \mathscr{F}(n, s-1)$. Let $F_{n,s}$ be an element of $\mathscr{F}(n, s)$ and let F be an element of $\mathscr{G}(F_{n,s})$ such that $F \cap \partial_* F_{n,s} \neq \varnothing$. We will let F be an element of $\mathscr{F}(n+1, t)$ and say F has order t if t is the smallest positive integer such that there exists $F_* \in \mathscr{F}(n+1, t-1)$ such that

$$F_* \subset F_{**} \in \mathscr{F}(n, s-1),\ F_{**} \in \mathscr{E}_*(F_{n,s})$$

and $F \cap F_* \cap \partial_* F_{n,s} \neq \varnothing$. Let

$$\begin{aligned}(3.4)\quad \mathscr{E}_*(F) = \{F_* \in \mathscr{F}(n+1, t-1)\colon & F_* \subset F_{**} \in \mathscr{F}(n, s-1)\ , \\ & F_{**} \in \mathscr{E}_*(F_{n,s}), \quad \text{and} \quad F \cap F_* \cap \partial_* F_{n,s} \neq \varnothing\}\ .\end{aligned}$$

With this we have defined a unique order for each $F \in \mathscr{F}(n+1)$ and we have $\mathscr{F}(n+1) = \bigcup_t \mathscr{F}(n+1, t)$.

Now let F be an element of $\mathscr{F}(n+1)$ and suppose $F \subset F_n \in \mathscr{F}(n)$. As mentioned earlier, in order to make the relation Δ a transitive order, it will be necessary that the elements of $\partial F_n - (\partial_* F_n \cup \partial^* F_n)$ have no successors. To ensure that this happens since $\partial_* F$ will have successors in the relation δ_{n+1}, we must define $\partial_* F$ for $F \in \mathscr{G}(F_n)$ so that

$$\partial_* F \cap [\partial F_n - (\partial_* F_n \cup \partial^* F_n)] = \varnothing\ ,$$

when $F \cap \partial_* F_n = \varnothing$. Also, if $F \cap C(F_n) = \varnothing$ and $F \subset P \in \mathscr{P}(F_n)$, we want $\partial_* F \cap \partial P = \varnothing$.

We do this as follows. If $F \in \mathscr{G}(F_n)$ and $F \cap \partial_* F_n \neq \varnothing$, set

$$(4.2)\qquad \partial_* F = F \cap \partial_* F_n\ .$$

If $F \cap \partial_* F_n = \varnothing$, but $F \cap C'(F_n) \neq \varnothing$, for each $E \in \mathscr{E}_*(F)$ choose $p(F, E) \in F \cap E \cap \operatorname{int} F_n$. Let

$$\begin{aligned}T(F_n) = \{p(F, E)\colon & F \in \mathscr{G}(F_n), F \cap \partial_* F_n = \varnothing, \\ & F \cap C'(F_n) \neq \varnothing \quad \text{and} \quad E \in \mathscr{E}_*(F)\}\ .\end{aligned}$$

Since $T(F_n)$ is a finite set it is a closed subset of int F_n. Because F_n is normal, we can find $S(F_n)$, an open subset of F_n such that

$$(8)\qquad Cl(S(F_n)) \cap T(F_n) = \varnothing \quad \text{and} \quad \partial F_n \subset S(F_n)\ .$$

Then for $F \in \mathscr{G}(F_n)$ such that $F \cap \partial_* F_n = \varnothing$ and $F \cap C'(F_n) \neq \varnothing$, set

$$(4.3)\qquad \partial_* F = [F \cap (\cup \mathscr{E}_*(F))] - S(F_n)\ .$$

Since $\mathscr{E}_*(F) \neq \varnothing$ and for $E \in \mathscr{E}_*(F)$, $p(F, E) \notin S(F_n)$, it follows that $\partial_* F$ is a nonempty closed subset of ∂F and $\partial_* F \cap [\partial F_n - \partial_* F_n] = \varnothing$. Similarly for $F \in \mathscr{G}(F_n)$ such that $F \cap C'(F_n) = \varnothing$, we know that $F \subset P$ for some unique $P \in \mathscr{P}(F_n)$. Now for each $F \subset P$ such that $F \cap C'(F_n) = \varnothing$ and each $E \in \mathscr{E}_*(F)$, we can choose $p(F, E)$ to be an element of $F \cap E \cap \text{int } P$. Let

$$T(F_n, p) = \{p(F, E): F \in \mathscr{G}(F_n, P), F \cap C'(F_n) = \emptyset, \text{ and } E \in \mathscr{E}_*(F)\} .$$

Since $T(F_n, P)$ is a finite set it is a closed subset of int P. Therefore we can find an open set $S(F_n, P)$ such that $\partial P \subset S(F_n, P)$ and

$$Cl(S(F_n, P)) \cap T(F_n, P) = \emptyset .$$

Now for each $F \subset P$ such that $F \cap C'(F_n) = \emptyset$, set

$$\partial_* F = [F \cap (\cup \mathscr{E}_*(F))] - S(F_n, P) . \tag{4.4}$$

It follows that $\partial_* F \cap \partial P = \emptyset$ and $\partial_* F$ is not empty.

For all $F \in \mathscr{F}(n+1)$, let

$$\mathscr{E}^*(F) = \{E \in \mathscr{F}(n+1): F \in \mathscr{E}_*(E)\}$$

and let

$$\partial^* F = \cup \{\partial_* E \cap F: E \in \mathscr{E}^*(F)\} .$$

If $E \in \mathscr{E}^*(F)$ let

$$\partial_E F = \partial_* E \cap F .$$

Let

$$\mathscr{E}(F) = \{E \in \mathscr{F}(n+1): E \neq F, (E \cap F) - (\partial_* F \cup \partial^* F) \neq \emptyset\}$$

and for $E \in \mathscr{E}(F)$ let

$$\partial_E F = Cl[(E \cap F) - (\partial_* F \cup \partial^* F)] .$$

If $\mathscr{E}^*(F) \neq \emptyset$, let $\rho_1(F) = d(\partial_* F, \partial^* F)$ and if $\mathscr{E}^*(F) = \emptyset$, let $\rho_1(F) = \text{diam } F$. If $F \cap \partial_* F_n = \emptyset$ but $F \cap C'(F_n) \neq \emptyset$, let $\rho_2(F) = d(\partial_* F, \partial F_n)$. If $F \cap C'(F_n) = \emptyset$, and $F \subset P \in \mathscr{P}(F_n)$, let $\rho_2(F) = d(\partial_* F, \partial P)$; otherwise let $\rho_2(F) = \text{diam } F$. Finally let

$$\rho(F) = \min\{\rho_1(F), \rho_2(F)\} . \tag{9}$$

This completes the definitions necessary to define δ_n for all positive integers n.

We now define a relation Δ on X by letting $\Delta = \bigcap_{i=1}^{\infty} \delta_i$. It remains to show that Δ is a partial order satisfying Theorem 2.

3. The relation Δ satisfies the hypotheses of Koch's Arc Theorem. The relation Δ is continuous on X since $\Delta = \bigcap_{n=1}^{\infty} \delta_n$ and we have shown in (7) that each δ_n is closed in $X \times X$. Also 0 is a zero for Δ since $0 \in L(x)$ for all $x \in X$. We must further show that $L(x)$, the set of predecessors of each $x \in X$ under the relation Δ, is a connected set. To do this it is enough to show that $L_{n+1}(x) \subseteqq L_n(x)$ for each $x \in X$, since then the set $\{L_n(x)\}_{n=1}^{\infty}$ will be a nest of continua and $L(x) = \bigcap_{n=1}^{\infty} L_n(x)$ will be a continuum and thus be connected.

Because $L_{n+1}(x)$, (6), is a union of sets of the forms $J(F)$ and $J(F, E)$ where $F \in \mathscr{F}(n+1)$, $E \in \mathscr{E}(F) \cup \mathscr{E}^*(F)$ and $x \in C(F)$ or $x \in C(F, E)$ to prove $L_{n+1}(x) \subset L_n(x)$, it is sufficient to show that if $x \in F \in \mathscr{F}(n+1)$ and $F \subset F' \in \mathscr{F}(n)$, then $F \cup J(F)$ is a subset of either $J(F')$ or $J(F', E')$ for some $E' \in \mathscr{E}(F') \cup \mathscr{E}^*(F')$. This proof is omitted but is a straightforward induction on t when

$$x \in F \subset F' \in \mathscr{F}(n, t)$$

using definitions (2.1–2) of $J(F)$ and (3.1–4) of $\mathscr{E}_*(F)$.

It is clear that $\varDelta$ is reflexive. That $\varDelta$ is a partial order on X will be established by the following lemmas.

LEMMA 3. *Let F_1 and F_2 be distinct elements of $\mathscr{F}(n)$ and let x be an element of $\partial F_1 - (\partial_* F_1 \cup \partial^* F_1)$. Then x is an element of $\partial F_2 - (\partial_* F_2 \cup \partial^* F_2)$.*

Proof. We will proceed by induction on n. Suppose $n = 1$, and that F_1 is an element of $\mathscr{F}(1, t)$. Then the order of F_2 is either $t-1, t$, or $t+1$, using (1) since $F_1 \cap F_2 \neq \emptyset$. If $F_2 \in \mathscr{F}(1, t-1)$ then by (4.1) $x \in \partial_* F_1$ and if $F_2 \in \mathscr{F}(1, t+1)$ by (4.1) $x \in \partial^* F_1$, and both of these situations contradict the hypothesis. Thus $F_2 \in \mathscr{F}(1, t)$. Suppose $x \in \partial_* F_2$. Then there exists a set $F_3 \in \mathscr{F}(1, t-1)$ such that $x \in F_2 \cap F_3$. But also $x \in F_1$ so $x \in F_1 \cap F_3 \subset \partial_* F_1$ which is a contradiction. Similarly, if $x \in \partial^* F_2$ there exists a set $F_3 \in \mathscr{F}(1, t+1)$ such that $x \in F_3 \cap F_2$, so $x \in F_1 \cap F_3 \subset \partial^* F_1$ and we get another contradiction.

We now suppose the lemma is true for $n = 1, 2, \cdots, k-1$. Let F_1 and F_2 be distinct elements of $\mathscr{F}(k)$ and suppose $F_1 \subset T_1 \in \mathscr{F}(k-1)$ and $F_2 \subset T_2 \in \mathscr{F}(k-1)$. By (4.1–4) we have for $i = 1, 2$

$$\partial_* F_i \cup \partial^* F_i \subset (\text{int } T_i) \cup \partial_* T_i \cup \partial^* T_i \,. \tag{10}$$

So $x \notin \partial_* F_1 \cup \partial^* F_1$ implies by (4.2) that $x \notin \partial_* T_1 \cup \partial^* T_1$.

Now if $T_1 \neq T_2$, $x \in T_1 \cap T_2$ implies $x \in \partial T_1 \cap \partial T_2$. From the induction hypothesis $x \in \partial T_2 - (\partial_* T_2 \cup \partial^* T_2)$. Therefore by (10) $x \in \partial F_2 - (\partial_* F_2 \cup \partial^* F_2)$.

If, however, both F_1 and F_2 are subsets of T_1, we will consider first the case when $x \in C(T_1)$. If $x \in S(T_1)$, where $S(T_1)$ is as defined in (8) then by (4.3) $x \notin \partial_* F \cup \partial^* F$ for any $F \in \mathscr{G}(T_1)$ such that $F \cap C(T_1) \neq \emptyset$. In particular $x \notin \partial_* F_2 \cup \partial^* F_2$. If $x \notin S(T_1)$, the argument that $x \notin \partial_* F_2 \cup \partial^* F_2$ is analogous to the situation when $n = 1$.

The final case when $x \notin C(T_1)$ follows by a similar argument using that either $F_1 \subset P_1$ and $F_2 \subset P_2$ where $P_1 \neq P_2$ and P_1 and P_2 are in

$\mathscr{P}(T_1)$; or $F_1 \cup F_2 \subset P$ for some $P \in \mathscr{P}(T_1)$ and that

$$\partial_* F_i \cup \partial^* F_i \subset [P - S(T_1, P)] \cup \partial^* T_1$$

for $i = 1, 2$.

Note. It follows from Lemma 3 that for $x \in (\text{int } F_1) \cup \partial_* F_1 \cup \partial^* F_1$ where $F_1 \in \mathscr{F}(n, t)$ and if $x \in F_2$ for some $F_2 \in \mathscr{F}(n)$, $F_2 \neq F_1$, then $x \in \partial_* F_2 \cup \partial^* F_2$. Further if $x \in \partial_* F_1$ then $F_2 \in \mathscr{F}(n, t-1) \cup \mathscr{F}(n, t)$, and if $x \in \partial^* F_1$, then $F_2 \in \mathscr{F}(n, t) \cup \mathscr{F}(n, t+1)$.

LEMMA 4. *Let $F \in \mathscr{F}(n)$ and $x \in \partial F - (\partial_* F \cup \partial^* F)$. Let m be an integer such that there exists $E \in \mathscr{F}(m)$ such that $x \in E \subset F$ and $E \cap (\partial_* F \cup \partial^* F) = \varnothing$. Then $C(E, E^*) \cap \partial F = \varnothing$ for all $E^* \in \mathscr{E}^*(E)$.*

Proof. Let $F_m = E$ and $F_n = F$. Then there exists $\{F_i\}_{i=n}^m$ such that $F_i \in \mathscr{F}(\text{i})$ and $F_{i+1} \subset F_i$. Let l be the greatest integer such that $m > l \geqq n$ and $\partial_* E - \partial_* F_l \neq \varnothing$ and let k be the greatest integer such that $\partial^* E - \partial^* F_k \neq \varnothing$. Then $m > l \geqq n$ and $m > k \geqq n$. Without loss of generality suppose $l \geqq k$. Since $\partial_* E \subset \partial_* F_{l+1}$,

$$d(\partial_* E, \partial F_l) \geqq d(\partial_* F_{l+1}, \partial F_l)$$

From (8) $\partial F_l \subset S(F_l)$ and by (4.3) and (4.4) $\partial_* F_{l+1} \subset F_l \backslash S(F_l)$. Therefore

$$d(\partial_* F_{l+1}, \partial F_l) = \rho_2(F_l) \geqq \rho(F_l) \quad \text{by} \quad (9) .$$

Thus $d(\partial_* E, \partial F_l) \geqq \rho(F_l)$. Similarly

$$d(\partial_* E, \partial F_k) \geqq d(\partial^* F_{k+1}, \partial F_k) \geqq \rho(F_k) .$$

Also

$$d(\partial^* E, \partial F) \geqq d(\partial^* E, \partial F_k) \quad \text{and} \quad d(\partial_* E, \partial F) \geqq d(\partial_* E, \partial F_l) .$$

Thus

$$\begin{aligned} d(\partial^* E \cup \partial^* E, \partial F) &\geqq \min \{d(\partial^* E, \partial F_k), d(\partial_* E, \partial F_l)\} \\ &> \min \{\rho(F_k), \rho(F_l)\} = \rho(F_l) \geqq \rho(F_{m-1}) . \end{aligned}$$

From Lemma 2 if $x \in \partial E$ and

$$d(x, \partial_* E \cup \partial^* E) > \frac{\rho(F_{m-1})}{3}$$

then $x \in C(E, E^*)$ for any $E^* \in \mathscr{E}^*(E)$. Thus $\partial F \cap C(E, E^*) = \varnothing$ for any $E^* \in \mathscr{E}^*(E)$.

LEMMA 5. *Let $x \in \partial F - (\partial^* F \cup \partial_* F)$ for $F \in \mathscr{F}(n)$. Then x has no successors other than itself in the relation Δ.*

Proof. Assume $y \geqq_\Delta x$ and $y \neq x$. Choose $m > n$ such that $d(x, y) > 2^{-m}$ and such that $d(x, \partial_* F \cup \partial^* F) > 2^{-m}$. Then since for $F' \in \mathscr{F}(m)$ we have diam $F' < 2^{-m}$, x and y are not both elements of any one $F' \in \mathscr{F}(m)$. Also, if

$$x \in F' \in \mathscr{F}(m), \quad \text{then} \quad F' \cap (\partial_* F \cup \partial^* F) = \varnothing ,$$

so that m satisfies the conditions of Lemma 5. However, since $x \in L_m(y)$ and x is in no element of $\mathscr{F}(m)$ containing y, by (6) $x \in C(F', F^*)$ for some

$$F' \in \mathscr{F}(m) \text{ and } F^* \in \mathscr{E}^*(F') .$$

But by Lemma 4, $C(F', F^*) \cap \partial F = \varnothing$. This is a contradiction and proves that such a y cannot exist.

In the next lemma we will use the following notation. If $x \in$ int F for some $F \in \mathscr{F}(n, t)$, set $q_n(x) = t$. If $x \in \partial_* F$ for some $F \in \mathscr{F}(n, t)$, set $q_n(x) = t - 1$. By the note after Lemma 3, $q_n(x)$ is well-defined and single valued for all $x \in (\text{int } F) \cup \partial_* F \cup \partial^* F$ where $F \in \mathscr{F}(n)$.

LEMMA 6. *The relation Δ is anti-symmetric.*

Proof. Assume there exist x and y in X such that $x \neq y$, $x \leqq_\Delta y$ and $y \leqq_\Delta x$. Choose n such that $d(x, y) > 2^{-n+1}$. Then, since $x \in L_n(y)$, there exists some $F_1 \in \mathscr{F}(n)$ such that $y \in F_1 \in \mathscr{F}(n, t)$ and $x \in J(F_1)$. By Lemma 5, $y \in (\text{int } F_1) \cup \partial_* F_1 \cup \partial^* F_1$, so $q_n(y)$ is defined and $q_n(y) \geqq t - 1$. Now because $d(x, y) > 2^{-n+1}$, $x \in F_2 \in \mathscr{F}(n)$ where $F_2 \in \mathscr{F}(n, s)$ and $s < t - 1$. Also by Lemma 5, $x \in \text{int } F_2 \cup \partial_* F_2 \cup \partial^* F_2$, so $q_n(x)$ is defined and $q_n(x) \leqq s < t - 1$. It follows that $q_n(y) > q_n(x)$. But by a symmetric argument since $y \in L_n(x)$, it can be shown that $q_n(x) > q_n(y)$. This contradiction proves that Δ is anti-symmetric.

LEMMA 7. *The relation Δ is transitive.*

Proof. Let x, y and z be elements of X such that $x \leqq_\Delta y$ and $y \leqq_\Delta z$. We will show $x \leqq_\Delta z$. We can assume $x < y$ and $y < z$. Choose n such that $\min\{d(x, y), d(y, z), d(x, z)\} > 2^{-n+1}$. It is enough to show $x \in L_n(z)$ since we have shown $L_{n-1}(z) \supset L_n(z)$. Since $y \in L_n(z)$, $y \in F_y$ for some $F_y \in \mathscr{F}(n, t)$ where $y \in J(F_y, E') \subset L_n(z)$ for some $E' \in \mathscr{E}^*(F_y)$. By Lemma 4, $y \in \text{int } F_y \cup \partial_* F_y \cup \partial^* F_y$. If $y \in \text{int } F_y$ then since

$$x \in L_n(y), x \in J(F_y) \subset J(F_y, E') \subset L_n(z) .$$

If $y \notin \text{int } F_y$ then either $y \in \partial_* F_y$ or $y \in \partial^* F_y$. We will consider the case when $y \in \partial_* F_y$. The argument is similar when $y \in \partial^* F_y$. By the note after Lemma 3 if $y \in F \in \mathscr{F}(n)$, then $F \in \mathscr{F}(n, t) \cup \mathscr{F}(n, t-1)$.

If $y \in F_* \in \mathscr{F}(n, t-1)$ where $x \in J(F_*)$ then $x \in J(F_*, F_y) \subset L_n(z)$. If we assume this is not the case then $x \notin J(F_*, F_y)$ for any $F_* \in \mathscr{E}_*(F_y)$. Let $\mathscr{A} = \{F_* \in \mathscr{F}(n, t-1): x \in J(F_*, F)$ for some $F \in \mathscr{F}(n, t)$ such that $y \in F\}$. The set $\mathscr{A}$ is not empty since $x \in L_n(y)$. Let

$$r = \min \{d(y, F_*): F_* \in \mathscr{A}\} .$$

Since $y \notin F_*$ for any $F_* \in \mathscr{A}, r > 0$. Choose $m > n$ such that $r > 2^{-m}$. Now because $x \in L_m(y) \subset L_n(y)$ there exists a set $T \in \mathscr{F}(m)$ such that $y \in T$ and $x \in J(T)$. Either $T \subset F$ for some $F \in \mathscr{F}(n, t)$ or $T \subset F_*$ for some $F_* \in \mathscr{F}(n, t-1)$. However if

$$T \subset F_* \in \mathscr{F}(n, t-1), x \in J(T) \subset J(F_*, F_y)$$

which contradicts our assumption. Thus there exists $F \in \mathscr{F}(n, t)$ such that $T \subset F$. Now by (2.2) $x \in J(T) \subset T \cup \bigcup \{J(T_*, T): T_* \cap T \neq \emptyset$, $T_* \in \mathscr{E}_*(T)$ and $T_* \subset F_*$ for some $F_* \in \mathscr{F}(n, t-1)\}$. By the choice of m and $r, F_* \notin \mathscr{A}$. But $x \in J(T) \subset J(F_*, F)$ implies that $F_* \in \mathscr{A}$. This contradiction says that $x \in J(F_*, F_y)$ for some $F_* \in \mathscr{E}_*(F_y)$ and thus $x \in J(F_y) \subset L_n(z)$. This completes the proof that $\varDelta$ is transitive.

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