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REPRESENTATIONS OF PRIMARY ARGUESIAN LATTICES

BJARNI JÓNSSON AND GEORGE S. MONK

The principal result of this paper states that every primary Arguesian lattice of geometric dimension three or more is isomorphic to the lattice of all submodules of a finitely generated module over a completely primary uniserial ring.

We shall discuss here briefly the background of this problem. Insofar as it is needed in this paper, the terminology employed here will be defined later.

In view of the correspondence between projective geometries and complemented modular lattices (cf. [4], p. 93) it follows from the classical coordinatization theorem for projective geometries (cf. [5], Chap. VI), that for $n \geqq 4$ the simple, complemented modular lattices of dimension n coincide up to isomorphism with the lattices of all subspaces of n-dimensional vector spaces over division rings or, equivalently, with the lattices of all left ideas of full rings of n by n matrices over division rings. This result has been generalized in two directions. On the one hand, von Neumann showed in [10] that every complemented modular lattice which has a homogeneous n-frame, $n \geqq 4$, is isomorphic to the lattice of all principal left ideals of a regular ring. On the other hand, Baer proved in [3] that every primary lattice of geometric dimension $n \geqq 6$ is isomorphic to the lattice of all submodules of a finitely generated module over a completely primary uniserial ring. Conversely, the lattice of all submodules of a finitely generated module over a completely primary uniserial ring is always a primary lattice. To indicate why these results are interesting, we remark that the class of rings involved contains the ring of integers modulo a prime power, and the corresponding class of modules therefore contains all finite primary Abelian groups. Baer's result was rediscovered by Inaba, who in [6] gave a different proof and, more important, replaced the condition $n \geqq 6$ by $n \geqq 4$.

Even for finite dimensional complemented modular lattices these results cannot be extended to the case $n=3$ because of the existence of projective planes in which Desargues' Law fails. In [11] Shützenberger observed that a projective geometry satisfies Desargues' Law if and only if a certain identity holds in the corresponding lattice. We adopt here a variant of Schützenberger's identity that was introduced in [8], and call a lattice Arguesian in case it satisfies this identity. Thus, for $n \geqq 3$ the simple, complemented Arguesian lattices of dimension n coincide up to isomorphism with the lattices of all subspaces of n-dimensional vector spaces over division rings. In [7] a corresponding generalization of von Neumann's result was obtained: Every complemented Arguesian lattice which possesses a homogeneous n-frame with $n \geqq 3$ is isomorphic to the lattice of all principal left ideals of a regular ring. (The condition on the frame was also relaxed in other ways that need not concern us here.) Our result provides a corresponding generalization of the Baer-Inaba theorem.

A somewhat weaker form of our principal theorem was obtained in [9]. The addition condition which the lattices considered there were assumed to satisfy, the so-called four-point property, corresponds to the geometric assumption that each line has at least four points. The basic approach in [9] is inspired by the classical construction of the coordinate ring, using the points on a line as ring elements. Our present approach is closer to Artin's method of coordinatization (cf. [2]); we single out a "hyperplane" and use the corresponding Abelian group of translations as the representation module, the scalars being the trace-preserving endomorphisms. A corresponding approach to the von Neumann theorem was first used by Amemiya in [1].

2. Preliminaries. For basic notions and results from lattice theory the reader is referred to [4]. We use $+$ and $\cdot$ to denote the binary lattice operations, Σ and Π for the corresponding operations on arbitrary families of lattice elements, $\leqq$ and $<$ for the lattice inclusion and strict inclusion, and $\ll$ for the covering relation. If $a \leqq b$, then $[a, b]$ is the interval $\{x: a \leqq x \leqq b\}$. All our lattices will be finite dimensional, and will therefore have a smallest element 0, and a largest element 1. The dimension of an element a will be denoted by $\delta(a)$.

Given a sequence of elements $a_0, a_1, \cdots, a_n$ of a finite dimensional modular lattice L, the following conditions are equivalent:

(1) $\delta(a_0 + a_1 + \cdots + a_n) = \delta(a_0) + \delta(a_1) + \cdots + \delta(a_n)$.

(2) $(a_0 + a_1 + \cdots + a_{i-1})a_i = 0$ for $i = 1, 2, \cdots, n$.

(3) For any disjoint subsets I and J of $\{0, 1, \cdots, n\}$,

$$\Sigma(a_i, i \in I) \circ \Sigma(a_i, i \in J) = 0 .$$

(4) For any subsets I and J of $\{0, 1, \cdots, n\}$,

$$\Sigma(a_i, i \in I) \circ \Sigma(a_i, i \in J) = \Sigma(a_i, i \in I \cap J) .$$

If these conditions are satisfied, then the elements a_i are said to be independent,—in symbols

$$(a_0, a_1, \cdots, a_n) \perp .$$

In order to indicate that the summands in a lattice sum are independent, we place a dot over the operation symbol, and write

$$\dot{\Sigma}(a_i, i \leqq n) \quad \text{or} \quad a_0 \dotplus a_1 \dotplus \cdots \dotplus a_n .$$

If the elements $a_0, a_1, \cdots, a_n$ of a modular lattice L generate a distributive lattice, then we write $(a_0, a_1, \cdots, a_n)\Delta$. We shall have occasions to make use of the fact that $(a_0, a_1, \cdots, a_n)\perp$ implies $(a_0, a_1, \cdots, a_n)\Delta$. Mostly, however, we will be concerned with the case of three elements a, b, c, and shall make frequent use of the fact that the one instance $a(b + c) = ab + ac$ of the distributive law (or the dual $a + bc = (a + b)(a + c)$) implies $(a, b, c)\Delta$. In particular, $(a, b, c)\Delta$ is implied by the condition $a(b + c) \leqq b$ and, a fortiori, by $a(b+c) = 0$.

An element a of a modular lattice with a zero element is said to be indecomposable if $a \neq 0$ and if a cannot be written as a sum $a = b \dotplus c$ with $b < a$ and $c < a$. We recall the fundamental theorem due to O. Ore: If an element u of a finite dimensional modular lattice L has two representations as a sum of independent indecomposable elements,

$$u = a_1 \dotplus a_2 \dotplus \cdots \dotplus a_k = b_1 \dotplus b_2 \dotplus \cdots \dotplus b_n ,$$

then $k = n$, and there exists a permutation ϕ of the indices $1, 2, \cdots, k$ such that

$$u = b_{\phi(1)} \dotplus b_{\phi(2)} \dotplus \cdots \dotplus b_{\phi(i)} \dotplus a_{i+1} \dotplus \cdots \dotplus a_k$$

for $i = 1, 2, \cdots, k$. In particular, this implies that a_i and $b_{\phi(i)}$ have the same dimension.

We finally mention the following consequence of the modular law: If the elements $a_i, b_i, i = 0, 1, \cdots, n$ of the modular lattice L are such that $a_i \leqq b_j$ whenever $i, j \leqq n$ and $i \neq j$, then

$$\Sigma(a_i b_i, i \leqq n) = \Sigma(a_i, i \leqq n) \cdot \prod (b_i, i \leqq n) .$$

3. Arguesian lattices.

We now introduce the lattice theoretic counterpart of Desargues' Law.

DEFINITION 2.1. A lattice L is said to be Arguesian if the following condition holds: For any elements $a_0, a_1, a_2, b_0, b_1, b_2 \in L$, if

$$y = (a_0 + a_1)(b_0 + b_1)[(a_0 + a_2)(b_0 + b_2) + (a_1 + a_2)(b_1 + b_2)] ,$$

then

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leqq a_0(a_1 + y) + b_0(b_1 + y) .$$

COROLLARY 3.2. *Every Arguesian lattice is modular.*

Proof. Suppose L is an Arguesian lattice. Given $u, v, w \in L$ with

$u \leqq w$, apply Definition 3.1 with

$$a_1 = a_2 = b_0 = u,\ a_0 = b_2 = v,\ b_1 = w\ ,$$

and therefore $y = (u + v)w$. This yields

$$(u + v)w \leqq u + vw\ ,$$

and therefore shows that L is modular.

The property used to define Arguesian lattices is not a very convenient one to work with. Fortunately it implies another condition, a conditional inclusion, which is much more suggestive of its geometric origin. In order to make the intuitive idea more transparent, we borrow some geometric terminology.

DEFINITION 3.3. Two ordered triples (a_0, a_1, a_2) and (b_0, b_1, b_2) of elements of a lattice L are said to be axially perspective, in symbols

$$(a_0, a_1, a_2) \barwedge (b_0, b_1, b_2)\ ,$$

if

$$(a_0 + a_1)(b_0 + b_1) \leqq (a_0 + a_2)(b_0 + b_2) + (a_1 + a_2)(b_1 + b_2)\ .$$

THEOREM 3.4. *If L is an Arguesian lattice, then for any elements $a_0, a_1, a_2, b_0, b_1, b_2 \in L$, the condition*

$$(a_0 + b_0)(a_1 + b_1) \leqq a_2 + b_2$$

implies that $(a_0, a_1, a_2) \barwedge (b_0, b_1, b_2)$.

Proof. Letting

$$c_0 = (a_1 + a_2)(b_1 + b_2)\ , \qquad c_1 = (a_2 + a_0)(b_2 + b_0)\ ,$$
$$c_2 = (a_0 + a_1)(b_0 + b_1)\ , \qquad \bar{c}_0 = (a_1 + a_2)(c_1 + c_2)\ ,$$

we first note that

$$\begin{aligned}
\bar{c}_0 + c_1 &= (a_1 + a_2 + c_1)(c_1 + c_2) \\
&= [a_1 + (a_0 + a_2)(b_0 + a_2 + b_2)](c_1 + c_2) \\
&\geqq [a_1 + (a_0 + a_2)(b_0 + (a_0 + b_0)(a_1 + b_1))](c_1 + c_2) \\
&= [a_1 + (a_0 + a_2)(a_0 + b_0)(b_0 + a_1 + b_1)](c_1 + c_2) \\
&\geqq [a_1 + a_0(b_0 + a_1 + b_1)](c_1 + c_2) \\
&= (a_1 + a_0)(b_0 + a_1 + b_1)(c_1 + c_2) \\
&\geqq (a_0 + a_1)(b_0 + b_1)(c_1 + c_2) = c_2\ .
\end{aligned}$$

Then, taking the elements $a_0, a_1, a_2, b_0, b_1, b_2$ in the definition of an

Arguesian lattice to be $c_1, b_0, a_0, \bar{c}_0, b_1, a_1$, and letting

$$y = (c_1 + b_0)(\bar{c}_0 + b_1)[(c_1 + a_0)(\bar{c}_0 + a_1) + (b_0 + a_0)(b_1 + a_1)] ,$$

we infer that

$$\begin{aligned} c_2 &= (a_0 + a_1)(b_0 + b_1)(c_1 + \bar{c}_0) \\ &\leqq c_1(b_0 + y) + \bar{c}_0(b_1 + y) \leqq c_1 + \bar{c}_0(b_1 + y) . \end{aligned}$$

Furthermore,

$$\begin{aligned} y &\leqq (b_0 + b_2)[(a_0 + a_2)(a_1 + a_2) + (a_0 + b_0)(a_1 + b_1)] \\ &\leqq (b_0 + b_2)[(a_0 + a_2)(a_1 + a_2) + b_2] \\ &= (b_0 + b_2)(a_0 + a_2)(a_1 + a_2) + b_2 = c_1(a_1 + a_2) + b_2 , \end{aligned}$$

so that

$$\begin{aligned} \bar{c}_0(b_1 + y) &\leqq (a_1 + a_2)[b_1 + b_2 + c_1(a_1 + a_2)] \\ &= (a_1 + a_2)(b_1 + b_2) + c_1(a_1 + a_2) \leqq c_0 + c_1 . \end{aligned}$$

Consequently $c_2 \leqq c_0 + c_1$.

The converse of this theorem does not hold. In fact, even in a projective geometry that satisfies Desargues' Law, axial perspectivity does not imply central perspectivity if the given triangles are degenerate. This difficulty can be avoided by weakening somewhat the condition for central perspectivity.

COROLLARY 3.5. *Suppose* $a_0, a_1, a_2, b_0, b_1, b_2$ *are elements of an Arguesian lattice* L. *Then* $(a_0, a_1, a_2) \barwedge (b_0, b_1, b_2)$ *if and only if*

(i) $(a_0 + b_0)(a_1 + b_1) \leqq (a_0 + a_2)(a_1 + a_2) + (b_0 + b_2)(b_1 + b_2)$.

Proof. Again let $c_0 = (a_1 + a_2)(b_1 + b_2)$ and cyclically. If $(a_0, a_1, a_2) \barwedge (b_0, b_1, b_2)$, then by the hypothesis

$$(a_0 + a_1)(b_0 + b_1) \leqq c_0 + c_1 ,$$

whence it follows by the preceding theorem that $(a_0, b_0, c_1) \barwedge (a_1, b_1, c_0)$. Inasmuch as

$$\begin{aligned} (a_0 + c_1)(a_1 + c_0) &\leqq (a_0 + a_2)(a_1 + a_2) , \\ (b_0 + c_1)(b_1 + c_0) &\leqq (b_0 + b_2)(b_1 + b_2) , \end{aligned}$$

this yields (i). For the converse we apply the preceding theorem with a_2 and b_2 replaced by

$$a_2' = (a_0 + a_2)(a_1 + a_2) \quad \text{and} \quad b_2' = (b_0 + b_2)(b_1 + b_2) ,$$

observing that $a_i + a_2' = a_i + a_2$ and $b_i + b_2' = b_i + b_2$ for $i = 0, 1$.

One of the most important consequences of Desargues' Law is the theorem that states that five of the six points of a quadragular six-tuple determine the sixth point. The next theorem is a counterpart to this result. For a geometric interpretation m is to be thought of as a line, a, b, c, x and y as the five given points, s, t, q and r as the vertices of one of the quadrangles used to construct the sixth point, and s', t', q' and r' as the vertices of the other quadrangle. It is quite possible that the condition

$$(s + s')m = (t + t')m = (s + s')(t + t')$$

in the hypothesis is redundant, and that the summands bd and xd can be omitted from the conclusion, but so far we have been unable to prove this. It is only after further restricting the class of lattices, and even then with considerable difficulty, that we are able to obtain a result fully analogous to the classical theorem.

THEOREM 3.6. *Suppose L is an Arguesian lattice, and assume that the elements $a, b, c, d, x, y, m, s, s', t, t' \in L$ are such that $a, b, c, d, x, y \leqq m$ and*

$$(s + t)m = (s' + t')m = a,\ sm = tm = s'm = t'm = 0\ ,$$
$$(s + s')m = (t + t')m = (s + s')(t + t') = d\ .$$

Let

$$q = (s + b)(t + c)\ , \qquad r = (s + x)(t + y)\ , \qquad z = (q + r)m\ ,$$
$$q' = (s' + b)(t' + c)\ , \qquad r' = (s' + x)(t' + y)\ , \qquad z' = (q' + r')m\ .$$

Then

$$z \leqq z' + bd + xd\ .$$

Proof. By Theorem 3.4, $(b, s, s') \barwedge (c, t, t')$, $(x, s, s') \barwedge (y, t, t')$, and therefore $q \leqq q' + d$ and $r \leqq r' + d$. Observing that

$$z \leqq (s + b + x)m = b + x\ ,$$

and therefore $z = (q + r)(b + x)$, we complete the proof by showing that

(1) $(b, x, s') \barwedge (q, r, d)$, $(b + s')(q + d) \leqq q' + bd$,

$$(x + s')(r + d) \leqq r' + xd\ .$$

Since $s(b + x) \leqq sm = 0$, we have $(s, b, x)\Delta$. Therefore

$$\begin{aligned}(b + q)(x + r) &\leqq (s + b)(t + b + c)(s + x) = (s + bx)(t + b + c) \\ &= s(t + b + c) + bx = s(s + t)(t + b + c) + bx \\ &= s[t + (s + t)(b + c)] + bx \leqq s(t + a) + bx,\ s' + d \\ &= (s + s')(s' + t + t') \geqq s(t + a)\ .\end{aligned}$$

From this the first formula in (1) follows by Corollary 3.5. Again using the fact that $q \leqq q' + d$ and $r \leqq r' + d$, we now compute

$$(b + s')(q + d) \leqq (s' + b)(q' + d) = q' + (s' + b)d = q' + bd ,$$
$$(x + s')(r + d) \leqq (s' + x)(r' + d) = r' + (s' + x)d = r' + xd ,$$

and the proof is complete.

COROLLARY 3.7. *Under the hypothesis of Theorem* 3.6, *if either*

$$bd \leqq c \quad \textit{and} \quad xd \leqq y ,$$
$$\textit{or} \quad cd \leqq b \quad \textit{and} \quad yd \leqq x ,$$

then $z = z'$.

Proof. We may assume that $bd \leqq c$ and $xd \leqq y$. Then $bd \leqq bc \leqq bq' \leqq z'$ and $xd \leqq xy \leqq xr' \leqq z'$, hence

$$z \leqq z' + bd + xd = z' .$$

The opposite inclusion follows by symmetry.

4. **Semi-primary lattices.** In this and the next two sections we present the basic properties of semi-primary and primary lattices. Some of these results can be found in [3] and [6], but are included here for the sake of completeness.

DEFINITION 4.1. An element c of a finite dimensional lattice is called a cycle if $[0, c]$ is a chain, and a dual cycle if $[c, 1]$ is a chain.

DEFINITION 4.2. A lattice L is said to be semi-primary if L is finite dimensional and modular, and every element of L is the sum of cycles and the product of dual cycles.

THEOREM 4.3. *For any finite dimensional modular lattice L, the following conditions are equivalent*:

(i) *Every element of L is the sum of cycles.*

(ii) *Every interval in L with a single dual atom is a chain.*

Proof. Assuming (i), consider an interval $[x, y]$ in L with a single dual atom v. Since y is the sum of cycles, there exists a cycle z that is contained in y but not in v. Thus the element $x + z$ belongs to $[x, y]$ and is not contained in v, whence it follows that $x + z = y$. Consequently

$$[x, y] = [x, z + x] \cong [xz, z] .$$

Since z is a cycle, this last interval is a chain, and hence so is $[x, y]$.

Conversely, if (ii) holds, then every element that covers a unique element is a cycle. But every element that covers more than one element is of course the sum of lower dimensional elements. Hence (i) follows by induction on the dimension of the element involved.

COROLLARY 4.4. *Every interval in a semi-primary lattice is semi-primary.*

Some additional terminology is needed.

DEFINITION 4.5. Suppose L is a semi-primary lattice. If $c, c' \in L$ are cycles and $c' \leqq c$, then c' is called a subcycle of c. A cycle whose dimension is k is called a k-cycle. For $a \in L$, the least upper bound of the dimensions of the cycles $c \leqq a$ is called the rank of a,—briefly rank(a)—and the number rank(1) is also called the rank of L. For $a \in L$ we let $a[k]$ be the sum of all cycles $c \leqq a$ with $\delta(c) \leqq k$.

THEOREM 4.6. *If L is a semi-primary lattice and $x_i \in L\ (i \in I)$ are cycles with $1 = \Sigma(x_i, i \in I)$, then*

$$\operatorname{rank} L = \max\{\delta(x_i) : i \in I\} .$$

Proof. The maximum n of the integers $\delta(x_i)$ is certainly not greater than the rank of L. We will show by induction on n that equality holds.

If $n = 1$, then the unit of L is the sum of atoms, and L is therefore complemented. Consequently L contains no 2-cycle, and rank$(L) = 1$.

Now suppose $n > 1$, and let

$$u = \Sigma(x_i[n-1], i \in I) .$$

Given a cycle $x \in L$, xu is a cycle in $[0, u]$ and by the inductive hypothesis $\delta(xu) \leqq n - 1$. For $i \in I$,

$$[u, u + x_i] \cong [ux_i, x_i] \subseteqq [x_i[n-1], x_i] ,$$

and x_i either covers or equals $x_i[n-1]$. Hence $u + x_i$ either covers or equals u. Thus 1 is the sum of elements that cover u, and $[u, 1]$ is therefore complemented. Consequently

$$\operatorname{rank}[xu, x] = \operatorname{rank}[u, x + u] \leqq \operatorname{rank}[u, 1] = 1 ,$$

and since $[xu, x]$ is a chain, this shows that x either equals or covers xu, and $\delta(x) \leqq n$.

COROLLARY 4.7. *For any interval* $[a, b]$ *in a semi-primary lattice* L, $\operatorname{rank}[a, b] \leqq \operatorname{rank} L$.

Proof. There exist cycles $b_i \in L\,(i \in I)$ whose sum is b. Then $a + b_i\,(i \in I)$ are cycles in $[a, b]$ whose sum is b, and since the dimension of $a + b_i$ in $[a, b]$ does not exceed the dimension of b_i in L, the conclusion follows by the preceding theorem.

THEOREM 4.8. *If* L *is a semi-primary lattice,* $\operatorname{rank} L = n$, *and* $a \in L$ *is an* n-*cycle, then* a *has a complement in* L. *In fact, every element* $x \in L$ *with* $ax = 0$ *is contained in a complement of* a.

Proof. Except in the trivial case when $a=1$, there exists a nonzero element that is disjoint from a, and we may therefore assume that $x \neq 0$. The element $a + x$ is an n-cycle in $[x, 1]$ and if we assume that the theorem holds with L replaced by $[x, 1]$, then there exists $b \in L$ such that $(a+x)+b=1$ and $(a+x)b = x$, but this clearly implies that b is a complement of a in L. The theorem now follows by induction on the dimension of L.

THEOREM 4.9. *Every element* a *of a semi-primary lattice* L *is the sum of independent cycles. Moreover, if*

$$a = \dot{\Sigma}(b_i, i < k) = \dot{\Sigma}(c_i, i < n)$$

are two such representations, with all the the summands distinct from 0, *then* $k=n$, *and there exists a permutation* ϕ *of the indices such that* $\delta(b_i) = \delta(c_{\phi(i)})$ *for* $i = 0, 1, \cdots, k-1$.

Proof. Choose a cycle x of maximal dimension in $[0, a]$. Then x has a complement a' in $[0, a]$. If $a \neq 0$, then $x \neq 0$, and therefore $\delta(a') < \delta(a)$. The first part of the theorem now follows by induction on $\delta(a)$; the second part is but a special case of Ore's Theorem.

The last theorem makes the following definition possible:

DEFINITION 4.10. Suppose L is a semi-primary lattice. By the type of an element $a \in L$ we mean the sequence $(k_1, k_2, \cdots, k_n)$ where n is the rank of $[0, a]$ and, for $i = 1, 2, \cdots, n$, k_i is the number of cycles of dimension i in a representation of a as a sum of independent cycles. The type of 1 is also called the type of L.

COROLLARY 4.11. *The type of a semi-primary lattice is equal to the type of its dual.*

Proof. Given a representation

$$1 = \dot{\Sigma}(c_i, i < n)$$

of 1 as the sum of independent cycles, the element

$$c'_j = \Sigma(c_i, i < n, i \neq j)$$

is a complement of c_j. Therefore

$$[0, c_i] = [c_i c'_i, c_i] \cong [c'_i, c_i + c'_i] = [c'_i, 1] ,$$

whence it follows that c'_i is a dual cycle whose dual dimension is equal to the dimension of a. We have

$$c'_0 c'_1 \cdots c'_i + c'_{i+1} \geqq c_{i+1} + c'_{i+1} = 1 ,$$

so that the elements c'_i are independent in the dual lattice. Finally, using the fact that independent elements generate a distributive lattice, we see that the product of the elements c'_i is 0. Thus in the dual of L, the unit 0 is the sum of the independent cycles c'_i, whence the conclusion follows.

LEMMA 4.12. *Suppose L is a semi-primary lattice, $a \in L$ and $k = \text{rank}\,(a)$. If $b_0, b_1, \cdots, b_{n-1} \in L$ are independent, and $a \leqq \Sigma(b_i, i<n)$, then $a \leqq \Sigma(b_i[k], i < n)$.*

Proof. For $j < n$ let

$$b'_j = \Sigma(b_i, i < n, i \neq j) , \qquad c_j = b_j(a + b'_j) .$$

Then $b_i \leqq a + b'_j$ for $i \neq j$, whence it follows that

$$\Sigma(c_i, i < n) = \Sigma(b_i, i < n) \prod (a + b'_i, i < n) \geqq a .$$

We observe that

$$\begin{aligned}[0, c_i] &= [b'_i c_i, c_i] \cong [b'_i, b'_i + c_i] \\ &= [b'_i, a + b'_i] \cong [ab'_i, a] .\end{aligned}$$

Consequently $\text{rank}\,(c_i) \leqq \text{rank}\,(a) = k$, $c_i \leqq b_i[k]$, and the conclusion follows.

COROLLARY 4.13. *If a, b and c are elements of a semi-primary lattice, and if $a = b \dotplus c$, then $a[k] = b[k] \dotplus c[k]$.*

THEOREM 4.14. *If L is a semi-primary lattice and $a_0, a_1, \cdots, a_{n-1} \in L$, then the conditions*

$$(a_i, i < n) \perp , (a_i[1], i < n) \perp$$

are equivalent.

Proof. If the elements a_i are not independent, then there exists a smallest index j such that the product $(a_0 + a_1 + \cdots + a_{j-1})a_j$ is not 0, and therefore contains an atom p. Clearly $p \leqq a_j[1]$, and since the sequence $(a_0, a_1, \cdots, a_{j-1})$ is independent, it follows from Lemma 4.12 that p is contained in the sum of the elements $a_i[1]$ with $i < j$. Consequently the elements $a_i[1]$ are not independent.

Conversely, if the elements a_i are independent, then so is any sequence of elements $b_i \leqq a_t$. In particular this holds for $b_i = a_i[1]$.

DEFINITION 4.15. Suppose L and L' are semi-primary lattices. By a cycle isomorphism of L onto L' we mean a one-to-one map of the cycles of L onto the cycles of L' such that, for any cycles a, b, c of L, the conditions

$$a \leqq b + c\ , \qquad f(a) \leqq f(b) + f(c)$$

are equivalent.

LEMMA 4.16. *If L and L' are semi-primary lattices and f is a cycle isomorphism of L onto L', then for any cycles $a, b_0, b_1, \cdots, b_{n-1}$ the conditions*

$$a \leqq \dot{\Sigma}(b_i, i < n), \quad f(a) \leqq \dot{\Sigma}(f(b_i), i < n)$$

are equivalent.

Proof. We first show that

$$(1) \qquad a \leqq \dot{\Sigma}(b_i, i < n) \quad \text{implies} \quad f(a) \leqq \Sigma(f(b_i), i < n)\ .$$

Inasmuch as this is true by definition for $n = 2$, we may proceed by induction on n. Letting

$$c = (a + b_0)\cdot\Sigma(b_i, 0 < i < n)\ ,$$

observe that $a + b_0 = b_0 \dotplus c$, hence $[0, c] \cong [ab_0, a]$, and c is therefore a cycle. Since $a \leqq b_0 + c$ and $c \leqq \Sigma(b_i, 0 < i < n)$, the inductive hypothesis yields

$$f(a) \leqq f(b_0) + f(c) \leqq \Sigma(f(b_i), i < n)\ .$$

Having established (1), we show next that

$$(2) \qquad (f(b_i), i < n)\perp \quad \text{implies} \quad (b_i, i < n)\perp\ .$$

In fact, if the cycles b_i are not independent, then for some $j < n$ the element $c = b_j(b_0 + b_1 + \cdots b_{j-1})$ is a nonzero cycle that is contained in both b_j and $b_0 \dotplus b_1 \dotplus \cdots \dotplus b_{j-1}$. But then $f(c)$ is a nonzero cycle that is contained in both $f(b_j)$ and $f(b_0) + f(b_1) + \cdots f(b_{j-1})$. Thus the cycles $f(b_i)$ also fail to be independent.

The proof is now completed by appling (1) and (2) both to f and to its inverse.

THEOREM 4.17. *If L and L' are semi-primary lattices, and f is a cycle isomorphism of L onto L', then there exists a unique isomorphism g of L onto L' containing f.*

Proof. Given an element $x \in L$, as a consequence of the preceding lemma we can define

$$g(x) = \dot{\Sigma}(f(b_i), i < n)$$

where $b_0, b_1, \cdots, b_{n-1}$ are cycles such that $x = \Sigma(b_i, i < n)$. We also infer from the lemma that g is one-to-one and onto, and has the property that $g(x) \leqq g(y)$ if and only if $x \leqq y$. Therefore g is the desired isomorphism. The uniqueness of g is obvious.

5. Geometric elements. The atoms in a semi-primary lattice L can of course be considered as the points of a projective geometry (possibly with degenerate lines), with three atoms being collinear if and only if they are not independent. The cycles of dimension $n =$ rank L also play a role somewhat similar to the points in a geometry. In particular, the maximum number of independent n-cycles serves to some extent as a substitute for the notion of dimension in geometry.

DEFINITION 5.1. Suppose L is a semi-primary lattice of rank n. A cycle $c \in L$ of dimension n is called a point of L. An element $a \in L$ that is the sum of independent points is said to be geometric. If the type of an element $a \in L$ is $(k_1, k_2, \cdots, k_n)$, then k_n is called the geometric dimension of a, $-gd(a) = k_n$. We let $gd(L) = gd(1)$, and call $gd(L)$ the geometric dimension of L.

THEOREM 5.2. *If L is a semi-primary lattice and $a \in L$ is geometric, then a has a complement. In fact, every element x with $ax = 0$ is contained in a complement of a.*

Proof. Write a and 1 each as the sum of independent nonzero cycles, $a = \dot{\Sigma}(y_i, i < k)$, $1 = \dot{\Sigma}(z_j, j < l)$. Except in the trivial case when $a = 1$, we must have $k < l$, because $\delta(z_j) \leqq \text{rank}\, L = \delta(y_i)$ for all $i < k$ and $j < l$. It follows that at least one of the atoms $z_j[1]$ is not contained in the sum of the atoms $y_i[1]$, and therefore, by Theorem

4.14, $az_j = 0$. This shows that we can assume without loss of generality that $x \neq 0$. From this point on we can proceed exactly as in the proof of Theorem 4.8.

THEOREM 5.3. *If L is a semi-primary lattice and $a \in L$ is geometric, then for any cycle $c \in L$ with $c \leqq a$ there exists a point $b \in L$ with $c \leqq b \leqq a$.*

Proof. Write a as the sum of independent points, $a = \dot{\Sigma}(y_i, i < k)$, and for $j < k$ let $y'_j = \Sigma(y_i, i < k, i \neq j)$. The product of the elements y'_j is 0, and therefore $\prod(cy'_j, j < k) = 0$. Since c is a cycle, this implies that $cy'_j = 0$ for some $j < k$. Applying the preceding theorem with L replaced by $[0, a]$, we see that the geometric element y'_j has a complement b in $[0, a]$ which contains c. From the fact that $a = y'_j + y_j = y'_j + b$, it follows that b must be a point.

6. Primary lattices. The next, and final, condition which we impose on our lattices corresponds to the geometric axiom that excludes degenerate lines containing only two points.

DEFINITION 6.1. A lattice L is said to be primary if L is semi-primary and every interval in L that is not a chain has at least three atoms.

COROLLARY 6.2. *The dual of a primary lattice is primary.*

Proof. If in a semi-primary lattice L, an interval $[a, b]$ is not a chain, then, by the dual of Theorem 4.3, $[a, b]$ has at least two distinct atoms x and y, and $[a, x + y]$ is therefore a two-dimensional interval that is not a chain. From this it is clear that L is primary if and only if every two-dimensional interval in L that is not a chain has at least three atoms. Since this latter condition is selfdual, so is the property of being primary.

THEOREM 6.2. *Every primary lattice that is not a chain is simple.*

Proof. It suffices to show that if $a, b, c \in L$ and $a \ll b \ll c$, then every congruence relation θ over L that identifies a and b also identifies b and c. Since every two-dimensional lattice with more than two atoms is simple, we need only consider the case when $[a, c]$ is a chain.

If c is a cycle, then there exists an atom $p \in L$ with $p \nleqq c$. In this case $[a, b + p]$ and $[b, c + p]$ are two-dimensional intervals that are not chains, and are therefore simple. Thus, since θ identifies a and b, it also identifies b and $b + p$, and therefore the simplicity of the second interval implies that θ identifies b and c.

If c is not a cycle, then there exists an element $x \in L$ with $x \ll c$ and $x \neq b$. In this case $[bx, c]$ is a two-dimensional interval that is not a chain. Also, $bx \neq a$, because $[a, c]$ is a chain. This shows that $ax \ll a$, so that $[ax, b]$ is a two-dimensional interval that is not a chain. The conclusion now follows exactly as in the preceding case.

THEOREM 6.3. *If L is a primary lattice that has at least three independent atoms, then any two two-dimensional intervals in L that are not chains are projective.*

Proof. We use induction on the dimension of L. It suffices to show that any two-dimensional interval $[a, b]$ in L that is not a chain is projective to an interval of the form $[0, c]$.

If there exists an atom p with $p \nleqq b$, then $[a, b]$ is projective to $[a + p, b + p]$. Repeating this process, we arrive at an interval $[a', b']$ that is projective to $[a, b]$ and has the property that every atom of L is contained in b'. Therefore, if $b' \neq 1$, then the given problem reduces to the corresponding problem for the lower dimensional lattice $[0, b']$.

We are thus left with the case $b' = 1$. Let u be the product of all the dual atoms of L. Then $u \leqq a'$, and $[u, 1]$ is a complemented lattice, whence it follows that any two two-dimensional intervals of $[u, 1]$ are projective. Except in the trivial case when L is complemented, we can find a dual atom x that includes all the atoms of L. Using Corollary 4.11 we see that the dimension of $[u, 1]$ is at least three, and we can therefore find an element y such that $u \leqq y < x$ and $[y, x]$ is two-dimensional. Then $[a', b']$ is projective to $[y, x]$, and the given problem therefore reduces to the corresponding problem for the lower dimensional lattice $[0, x]$.

LEMMA 6.4. *If L is a primary lattice and $a, b \in L$ are cycles with $ab = 0$ and $\delta(a) \geqq \delta(b)$, then there exists a cycle $c \in L$ such that $a \dotplus b = a + c = b \dotplus c$.*

Proof. We may assume that $b \neq 0$. Let u be the subcycle of a such that $\delta(u) = \delta(a) - \delta(b)$. Then the interval $[u, a + b]$ is not a chain, and since a and $b + u$ are cycles in $[u, a + b]$, there exists an atom p of $[u, a + b]$ that is contained in neither a nor $b + u$. Observe that $b + u$ is a point in the lattice $[u, a + b]$ and therefore, by Theorem 4.8, has a complement c in that lattice with $p \leqq c$. Thus

$$ac = (b + u)c = u, bc = bu = 0, \delta(c) = \delta(a)$$

and $c \leqq a + b$, and the conclusion follows by a simple dimension argument.

DEFINITION 6.5. A primary lattice L is said to have the four-point property if every interval in L that is not a chain has at least four atoms.

The primary lattices in which the four-point property fails correspond to projective geometries with three points on each line. As was mentioned in the introduction, these lattices cause considerable complications, and have to some extent to be treated separately.

We now introduce the appropriate class of rings and prove the easy converse of the representation theorem.

DEFINITION 6.6. A ring R with unit is said to be completely primary and uniserial if there is a two-sided ideal P of R such that every left or right ideal of R is of the form P^k (where $P^0 = R$). The rank of such a ring is the smallest integer k such that $P^k = (0)$.

The lattice of all submodules of a module over a ring is always Arguesian (cf. [8]). We prove

THEOREM 6.7. *The lattice of all submodules of a finitely generated module over a completely primary and uniserial ring is primary.*

Proof. Let R be the given ring, U the module, and L the lattice of all submodules of U.

For any element $x \in U$, the module epimorphism $\alpha \rightarrow x\alpha$ of R onto xR induces an isomorphism of the lattice of all submodules of xR onto the lattice of all those right ideals of R that contain the kernel of the epimorphism. Consequently the submodules of xR form a chain, and xR is therefore a cycle in L. Since U is finitely generated, this shows that U is the lattice sum of finitely many cycles. Since each xR is finite dimensional, it follows that L is finite dimensional, and from this we infer that each submodule of U is finitely generated, and is therefore the sum of finitely many cycles in L.

We next want to prove that, dually, every member of L is the lattice product of dual cycles. According to the dual of Theorem 4.3 this is equivalent to the assertion that every interval in L that has only one atom is a chain. If this is not the case, then there exists an interval $[V, W]$ such that V is covered by a unique element $X \subseteqq W$, and X is covered by two distinct elements $Y, Z \subseteqq W$. We may assume that $V = (0)$, for otherwise we could replace U by the factor module U/V.

There exist $y \in Y$ and $z \in Z$ with $y, z \notin X$. Since $(0) \neq yR \subseteqq Y$, and X is the unique atom contained in Y, we must have $X \subseteqq yR$. Equality is excluded because $y \notin X$, and yR cannot be strictly between X and Y because Y covers X. Therefore $Y = yR$. Similarly $Z = zR$.

Let P be the unique maximal proper ideal of R. Since yR covers X, we have $X = yP$. Choosing $\lambda \in R$ with $P = \lambda R$ we therefore find that $X = y\lambda R$, $y\lambda \in Z$, $y\lambda = z\mu$ for some $\mu \in R$. Since $z\mu R = X = zP$ we have $\mu \in P$, $\mu = \nu\lambda$ for some $\nu \in R$. Letting $z' = z\nu$ we conclude that $(y-z')\lambda = 0$, $(y-z')P = (0)$, $(y-z')R$ at most covers (0), $(y-z')R \subseteqq X \subseteqq Z$, $y \in Z$, $Y \subseteqq Z$. We have thus arrived at a contradiction, there by proving the assertion.

Finally, we must prove that if an interval $[V, W]$ has two distinct atoms X and Y, then it has a third atom. As before, we need only consider the case when $V = (0)$. Then $X = xR$ and $Y = yR$ for some $x, y \in W$, and $Z = (x - y)R$ is easily seen to be our third atom. For, since xR and yR are atoms, we have $xP = yP = (0)$, and therefore $(x - y)P = (0)$. On the other hand $Z \neq (0)$ because $x - y \neq 0$. Also, $X + Z = Y + Z = X + Y$, and this shows that Z is distinct from both X and Y.

7. **Partial translations.** This section is devoted to the derivation of a number of technical lemmas needed in the proof of the quadrangle property. Actually these lemmas will be used only in the treatment of the special case in which the four-point property fails. The proof for lattices with the four-point property is relatively simple and direct, and if the reader is willing to restrict himself to these lattices, he may omit this section as well as Lemma 8.3 and the last part of the proof of Theorem 8.4.

We assume throughout this section that L is a primary Arguesian lattice, $m \in L$ is a complemented dual cycle and $gd(m) \geqq 2$.

DEFINITION 7.1. $C(L, m) = \{p \in L \colon 1 = p \dotplus m\}$.

DEFINITION 7.2. Given $p, p' \in C(L, m)$, we let

$$\sigma_{p,p'}(u) = [(p + p')m + u][(p + x)m + p']$$

whenever $u \in L$ and $((p + p')m, p, u)\Delta$.

For the remainder of this section we consider fixed elements $p, p' \in C(L, m)$ and let $e = (p + p')m$. For any element $u \in L$ with $(e, p, u)\Delta$, we let

$$u_p = (u + p)m,\ u' = \sigma_{p,p'}(u) = (e + u)(u_p + p')\,.$$

LEMMA 7.3. *If $(e, p, u)\Delta$, then*

$$u' + e = u + e, \quad u'm = um$$

COROLLARY 7.4. *If $u \in C(L, m)$ and $(e, p, u)\Delta$, then $u' \in C(L, m)$.*

LEMMA 7.5. *If $(e, p, u)\Delta$, then $(e, p', u')\Delta$ and $\sigma_{p',p}(u) = u$.*

LEMMA 7.6. *If $(e, p, u)\Delta$ and $(e, p, v)\Delta$, then*

$$(u' + v')m = (u + v)m \ .$$

Proof. We may assume that $u \leqq v + m$. Then

(1) $$(u, e, v) \barwedge (u_p, p', v_p) \ ,$$

because

$$\begin{aligned}(u + u_p)(e + p') &= (u + p)(e + p)(u + m) \\ &= ue + p(u + m) \leqq ue + p(v+m) \\ &\leqq ue + v + v_p \ .\end{aligned}$$

By (1), $u' \leqq (u + v)m + v'$, and using Lemma 7.3 we find that

$$(u' + v')m \leqq (u + v)m + v'm = (u + v)m \ .$$

The opposite inclusion follows by Lemma 7.5.

LEMMA 7.7. *If $q \in C(L, m)$ and $(e, p, q)\Delta$, then*

$$(q + q')m = e \quad \textit{and} \quad \sigma_{q,q'}(p) = p' \ .$$

Proof. The first equation is trivial, and routine calculations show that

$$\begin{aligned}\sigma_{q,q'}(p) &= (p + e)(q' + q_p) = (p + e)(p' + q_p) \\ &= p' + (p + e)q_p = p' + (p + e)(p + q)m \\ &= p' + pm = p'.\end{aligned}$$

LEMMA 7.8. *Suppose $q \in C(L, m)$ and $(e, p, q)\Delta$. If $(e, p, u)\Delta$ and $(e, q, u)\Delta$, then $\sigma_{q,q'}(u) = u'$.*

Proof. Letting

$$u_q = (q + u)m \ , \qquad \bar{u} = \sigma_{q,q'}(u) \ ,$$

we have $(u + u_p)(e + p') \leqq p + eu \leqq q + q_p + eu$, therefore

$$(u, e, q) \barwedge (u_p, p', q_p) \ .$$

From this it follows that

$$u' \leqq (u + q)(u_p + q_p) + q' \leqq u_q + q' ,$$
$$u' \leqq (u + e)(u_q + q') = \bar{u} .$$

In view of Lemma 7.3, strict inclusion is impossible, and we must have $u' = \bar{u}$.

LEMMA 7.9. *Suppose* $q, r \in C(L, m)$, $(p+p')q = 0$, *and* $qr \neq 0$. *For every cycle* $u \in L$, *if* $(q + e)u = (r + e)u = eu$, *then* $\sigma_{q,q'}(u) = \sigma_{r,r'}(u)$.

Proof. Observe that, since $qr \neq 0$, the condition $(p + p')q = 0$ implies that $(p + p')r = 0$, and therefore both q' and r' are defined. Also, the condition $(q + e)u = (r + e)u = eu$ implies that $(e, q, u)\Delta$ and $(e, r, u)\Delta$, and therefore u is in the domains of $\sigma_{q,q'}$ and $\sigma_{r,r'}$.

If $(e, q, r)\Delta$, then by Lemma 7.8, $\sigma_{q,q'}(r) = r'$, and replacing p and q by q and r, we again use Lemma 7.8 to infer that $\sigma_{q,q'}(u) = \sigma_{r,r'}(u)$.

Now assume $(e, q, r)\Delta$ fails, and therefore $(q + r)e \neq 0$. Let $r_q = (q + r)m$. There exist cycles $e_1, e_2 \leqq m$ such that

$$r_q = e_1 + e_2, r_q e_1 = r_q e_2 = e_1 e_2 = ee_1 = ee_2 = 0 .$$

In fact, we can take for e_1 any cycle contained in m that is disjoint from r_q and has the same dimension as r_q, and then apply Lemma 6.4 to obtain a cycle e_2 such that $r_q \dotplus e_1 = r_q + e_2 = e_1 \dotplus e_2$. Since the three cycles r_q, e_1 and e_2 have the same dimension, r_q and e_2 must be disjoint. Finally, since $r_q e \neq 0$, the cycles e_1 and e_2 must be disjoint from e also. The element $s = (q + e_1)(r + e_2)$ is clearly a complement of m, and $sq \geqq sr > 0$, therefore $(p + p')s = 0$, and s' is defined. Also $(q + s)e = (s + r)e = 0$ and $(s + e)u = eu$. To prove the last equation we observe that $(e + u)q = eq + uq = uq \leqq eq = 0$ and $sq \neq 0$, hence

$$(e + u)s = 0, (s, e, u)\Delta, (s + e)u = su + eu = eu .$$

We can now apply the first part of the proof twice to conclude that $\sigma_{q,q'}(u) = \sigma_{s,s'}(u) = \sigma_{r,r'}(u)$.

LEMMA 7.10. *Suppose* $p^* \in C(L, m)$, *and let* $e^* = (p + p^*)m$, $\bar{e} = (p' + p^*)m$. *For any* $u \in L$, *if* $(e, p, u)\Delta$, $(e^*, p, u)\Delta$ *and* $(\bar{e}, p', u')\Delta$, *then* $\sigma_{p,p^*}(u) = \sigma_{p',p^*}\sigma_{p,p'}(u)$.

Proof. Let

$$u^* = \sigma_{p,p^*}(u), \quad \bar{u} = \sigma_{p,p^*}(u') .$$

Since u, u', u^* and $\bar{u}$ all have the same dimension, it suffices to show that $\bar{u} \leqq u^*$. Observe that

$$\begin{aligned}(u + u_p)(e + p') &\leqq (p + u)(p + e) = p + ue \\ &\leqq e^* + p^* + ue ,\end{aligned}$$

so that $(u, e, e^*) \overline{\wedge} (u_p, p', p^*)$. This yields $u' \leqq u^* + \bar{e}$, and therefore

$$\begin{aligned}\bar{u} &= (\bar{e} + u')(u_p + p^*) \leqq (\bar{e} + u^*)(u_p + p^*) \\ &= u^* + \bar{e}(u_p + p^*) = u^* + \bar{e}u_p .\end{aligned}$$

But, by Lemma 7.3 and the assumption that $(\bar{e}, p', u')\Delta$,

$$\bar{e}u_p = \bar{e}(u + p) = \bar{e}(u' + p') = \bar{e}u' + \bar{e}p' = \bar{e}u \leqq u^* ,$$

and we conclude that $\bar{u} \leqq u^*$, as was to be shown.

8. The quadrangle property. We introduce two auxiliary definitions.

DEFINITION 8.1. Suppose L is a primary lattice, $m, a, s, t \in L$, and $a \leqq m$. We say that (s, t) is admissible for a if s, t, a are cycles and

$$(s + t)m = a, s + t = t + a = a + s, st = ta = as = 0 .$$

DEFINITION 8.2. Suppose L is a primary lattice and $m, s, t \in L$. For any cycles $b, c, x, y \leqq m$ we define

$$Q_{s,t}(b, c, x, y) = [(s + b)(t + c) + (s + x)(t + y)]m .$$

LEMMA 8.3. *Suppose L is a primary lattice, $m \in L$ is a dual cycle, and $a, b, c, x, y \in L$ are cycles contained in m. If (s, t) is admissible for a, then*

$$\delta(Q_{s,t}(b, c, x, y)) = \delta(a(b + c)(x + y)) + \delta(bc + xy) - \delta(a(bx + cy)) .$$

Proof. Let

$$\begin{aligned}q &= (s + b)(t + c) , \qquad r = (s + x)(t + y) , \\ z &= Q_{s,t}(b, c, x, y) = (q + r)m .\end{aligned}$$

Then $q + z \leqq q + r$, and therefore

$$\delta(z) \leqq \delta(r) + \delta(qz) - \delta(qr) . \tag{1}$$

We have

$$\begin{aligned}\delta(r) &= \delta(s + x) + \delta(t + y) - \delta(s + t + x + y) \\ &= \delta(s + x) + \delta(t + y) - \delta(s + a + x + y) \\ &= \delta(t) + \delta(x) + \delta(y) - \delta(a + x + y) \\ &= \delta(a) + \delta(x) + \delta(y) - \delta(a) - \delta(x + y) + \delta(a(x + y)) \\ &= \delta(a(x + y)) + \delta(xy) .\end{aligned}$$

Since $qr = (s + bx)(t + cy)$, we can replace x and y in the above calculations by bx and cy and obtain

$$\delta(qr) = \delta(a(bx + cy)) + \delta(bcxy) ,$$

Finally $qz = qm = bc$. The left hand side of (1) is thus equal to

$$\begin{aligned}&\delta(a(x + y)) + \delta(xy) + \delta(bc) - \delta(bcxy) - \delta(a(bx + cy)) \\ &= \delta(a(x + y)) + \delta(bc + xy) - \delta(a(bx + cy)) .\end{aligned}$$

Therefore

$$(2) \qquad \delta(z) \leqq \delta(a(x + y)) + \delta(bc + xy) - \delta(a(bx + cy)) ,$$

with equality holding if and only if $r \leqq q+z$ or, equivalently, $r+m \leqq q + m$. Similarly

$$(3) \qquad \delta(z) \leqq \delta(a(b + c)) + \delta(bc + xy) - \delta(a(bx + cy)) ,$$

with equivality holding just in case $q + m \leqq r + m$. Now $q + m$ and $r + m$ are members of the chain $[m, 1]$, and one must therefore be included in the other, say $r + m \leqq q + m$. Then equality holds in (2), which implies that $\delta(a(x + y)) \leqq \delta(a(b + c))$, and therefore $a(x + y) \leqq a(b + c)$, since $a(x + y)$ and $a(b + c)$ are subcycles of a. Thus

$$a(x + y) = a(b + c)(x + y) ,$$

and the proof is complete.

THEOREM 8.4. *Suppose L is a primary Arguesian lattice, $m \in L$ is a complemented dual cycle, $gd(m) \geqq 2$, $a \in L$ is a cycle and $a \leqq m$. If (s, t) and (s', t') are admissible for a, then $Q_{s,t} = Q_{s',t'}$.*

Proof. Let $k = \delta(a)$. Suppose $b, c, x, y \leqq m$ are cycles, and assume that (s, t) and (s', t') are admissible for a. Let

$$\begin{aligned} q &= (s + b)(t + c) , & r &= (s + x)(t + y) , \\ q' &= (s' + b)(t' + c), & r' &= (s' + x)(t' + y) , \\ z &= (q + r)m = Q_{s,t}(b, c, x, y) . \end{aligned}$$

By symmetry it suffices to show that

$$(1) \qquad z \leqq q' + r' .$$

We begin by treating a very special case:

$$(2) \qquad \text{If } s = s', \text{ then } Q_{s,t} = Q_{s',t'} .$$

To prove this we observe that

$$(3) \qquad (c, y, t') \barwedge (q, r, s) \,,$$

because

$$(c + q)(y + r) \leqq (t + c)(t + y) = t + cy$$
$$\leqq a + s + cy = s + t' + cy \,.$$

From (3) it follows that

$$(4) \qquad (c + y)(q + r) \leqq (c + t')(q + s) + (y + t')(r + s) \,.$$

Now $z \leqq (t + c + y)m = c + y$, and the left hand side of (4) is therefore equal to z, while the two summands on the right are easily seen to be equal to q' and r', respectively. Thus (1) holds in the present case, and our assertion (2) is therefore established.

We next reduce the general problem to the special case in which

$$(5) \qquad (s + t)(s' + t') = a \,,$$

$$(6) \qquad (s + s')m = (t + t')m = (s + s')(t + t') \,.$$

If (5) fails, we use the hypothesis $gd(m) \geqq 2$ to secure a point $p \leqq m$ with $pa = 0$, and we then choose k-cycles $s'' \leqq s + p$ and $t'' \leqq s'' + a$ such that $ss'' = ps'' = 0$ and $s''t'' = at'' = 0$. It readily follows that (s'', t'') is admissible for a, and that $(s + t)(s'' + t'') = a$. We must also have $(s' + t')(s'' + t'') = a$, because the elements $(s + t)(s' + t')$ and $(s' + t')(s'' + t'')$ belong to the chain $[a, s' + t']$, and if they were both larger than a, then their product would also be larger than a. Our problem thus reduces to showing that $Q_{s,t} = Q_{s'',t''}$ and $Q_{s'',t''} = Q_{s',t'}$, and in each case we are in the situation described by (5).

Assuming that (5) holds, observe that any three of the four cycles s, t, s', t' are independent. Let $d = (s + s')m$ and $t'' = (s' + a)(t + d)$. Observe that $a + t = s + t$ and $s + d = s' + d$, therefore $a + t + d \geqq s'$. From this it readily follows that (s', t'') is admissible for a, and that both (5) and (6) hold with t' replaced by t''. By (2), $Q_{s',t'} = Q_{s',t''}$, and the problem thus reduces to showing that $Q_{s',t''} = Q_{s,t}$, i.e., it reduces to a situation where both (5) and (6) are satisfied.

We henceforth assume that (5) and (6) hold, and we let

$$d = (s + s')(t + t') \,.$$

The case when L has the four-point property is now readily disposed of. In this case there exists by Theorem 5.3, a k-cycle $e \leqq a+d$ that is disjoint from the cycles b, x and a. Letting $s'' = (s + e)(s' + t')$ and $t'' = (t + e)(s' + t')$ we easily see that (s'', t'') is admissible for a, and that (5) and (6) hold with s'' and t'' in place of s' and t'. Since $e = (s + s'')(t + t'')$ is disjoint from b and x, we infer by Theorem 3.6 that

$$Q_{s,t}(b, c, x, y) = Q_{s'',t''}(b, c, x, y) .$$

If $s''t' = 0$, then it follows from (2) that $Q_{s'',t''} = Q_{s'',t'} = Q_{s',t'}$, and if $t''s' = 0$, then we similarly obtain $Q_{s'',t''} = Q_{s',t''} = Q_{s',t'}$. Finally, if $s''t' \neq 0$ and $t''s' \neq 0$, then we can choose a k-cycle $u \leqq s' + t'$ that is disjoint from s', t' and a, and hence also from s'' and t'', and use (2) three times to infer that

$$Q_{s'',t''} = Q_{s'',u} = Q_{s',u} = Q_{s',t'} .$$

We assume from now on that the four-point property fails in L. We may also assume that

$$(7) \qquad a + b = b + c = c + a , \qquad a + x = x + y = y + a .$$

This is justified by observing that if we replace the element $a, b, c, x, y, s, t, s', t'$ by

$$\begin{aligned} a_1 &= a(b + c)(x + y) , & b_1 &= b(a_1 + c) , & c_1 &= c(a_1 + b) , \\ x_1 &= x(a_1 + y) , & y_1 &= y(a_1 + x) , & s_1 &= s(a_1 + t) , \\ t_1 &= t(a_1 + s) , & s'_1 &= s'(a_1 + t') , & t'_1 &= t'(a_1 + s') , \end{aligned}$$

then the results of the operations $Q_{s,t}, Q_{s',t'}$ are unchanged,

$$\begin{aligned} Q_{s_1,t_1}(a_1, b_1, c_1, x_1, y_1) &= Q_{s,t}(a, b, c, x, y) , \\ Q_{s'_1,t'_1}(a_1, b_1, c_1, x_1, y_1) &= Q_{s',t'}(a, b, c, x, y) . \end{aligned}$$

In fact, the elements on the left are certainly contained in the elements on the right, and it follows from Lemma 8.3 that both sides have the same dimension. Since the new elements satisfy (7) this justifies the assumption. Furthermore, we may assume that

$$(8) \qquad s, t, s', t' \in C(L, m) ,$$

for the elements $s + m, t + m, s' + m, t' + m$ are in any case equal to each other, since they all belong to the chain $[m, 1]$ and all have the dimension $\delta(m) + k$, and we can therefore replace L by $[0, s + m]$.

Observe that if $\sigma_{s,s'}(q)$ exists, i.e., if $(d, s, q)\varDelta$, then $\sigma_{s,s'}(q) = q'$. In fact, by Theorem 3.4 we have $(b, s', s) \barwedge (c, t', t)$, therefore $q' \leqq d + q$. This readily implies that $q' \leqq \sigma_{s,s'}(q)$, and equality must hold because both elements have the same dimension as q.

If $(d, s, q)\varDelta$ and $(d, s, r)\varDelta$, then $q' = \sigma_{s,s'}(q)$ and $r' = \sigma_{s,s'}(r)$, and (1) holds by Lemma 7.6. The case when $(d, t, q)\varDelta$ and $(d, t, r)\varDelta$ can be treated similarly. The conditions $(d, s, q)\varDelta$ and $(d, t, q)\varDelta$ cannot both fail, for then the subcycles $d(s + q)$ and $d(t + q)$ of d would both be larger than q, which is impossible because

$$d(s + q)(t + q) = d(s + b)(t + c) = dq .$$

Similarly, only one of the conditions $(d, s, r)\Delta$ and $(d, t, r)\Delta$ can fail. In view of these observations we may assume that

$$\text{(9)} \qquad \begin{aligned} &\text{not} - (d, s, q)\Delta\ , \qquad (d, t, q)\Delta\ , \\ &\text{not} - (d, t, r)\Delta\ , \qquad (d, s, r)\Delta\ . \end{aligned}$$

It follows that

$$\text{(10)} \qquad q' = \sigma_{t,t'}(q)\ , \qquad r' = \sigma_{s,s'}(r)\ .$$

From (9) it follows that $(s + q)d \neq 0$, therefore $bd \neq 0$. If $cd = 0$, then $qm = bc = 0$, and therefore q is a cycle, but if $cd \neq 0$, then by Corollary 4.13,

$$\begin{aligned} q[1] &\leqq (s[1] + b[1])(t[1] + c[1]) \\ &= (s[1] + d[1])(t[1] + d[1]) = d[1]\ , \end{aligned}$$

and q is therefore a cycle in this case too. Since $(s + d)q \neq 0$, we have $q[1] \leqq s + d$. It follows from Theorem 6.3 that $s + d$ contains only three atoms, $s[1]$, $s'[1]$ and $d[1]$, and $q[1]$ must therefore be one of these. We therefore have $sq \neq 0$ or $s'q \neq 0$ or $dq \neq 0$. Similarly r is a cycle and one of the conditions $tr \neq 0$, $t'r \neq 0$, $dr \neq 0$ holds. The nine cases that result will be combined into four in the argument that follows.

Before dividing the argument into cases, observe that it follows from (7) and (8) that $s \leqq t + b + c$, and therefore

$$\begin{aligned} q + m &= (s + b)(t + c) + m = (s + b)(t + b + c) + m \\ &= s + b + m = 1\ , \end{aligned}$$

so that if $qm = 0$, then $q \in C(L, m)$. Similarly, if $rm = 0$, then $r \in C(L, m)$.

Case 1. $sq \neq 0$ or $tr \neq 0$.

Without loss of generality assume that $sq \neq 0$. Then $qm = 0$, so that $q \in C(L, m)$. Also $q(t + d) = 0$ because $s(t + d) = 0$. It follows that $(q + d)(t + d) = d$. Thus the product of the two sub-cycles $(q+d)r$ and $(t + d)r$ of r is dr, and since by (9), $(t + d)r > dr$, this implies that $(q + d)r = dr$, hence $(d, q, r)\Delta$. We infer by Lemma 7.9 that $\sigma_{q,q'}(r) = r'$, and since $\sigma_{q,q'}(q) = q'$, we conclude by Lemma 7.6 that (1) holds in this case.

Case 2. $dq \neq 0$ and $dr \neq 0$.

Let $e = (s + t')m$, $t'' = (s + a)(s' + e)$. It is easy to check that $t'' \in C(L, m)$. We have $(a + t)q = 0$ because $(a + t)d = 0$ and $dq \neq 0$. Similarly $(a + s)r = 0$. We can therefore form

$$\bar{q} = \sigma_{t,s}(q)\ , \qquad \bar{r} = \sigma_{s,t''}(r)\ .$$

Since $d\bar{q} = dq \neq 0$, we have $(s + t')\bar{q} = 0$. Similarly $(s' + t'')\bar{r} = 0$ because $\bar{r}d = rd \neq 0$ and $(s' + t'')d = 0$. Using Lemma 7.10 we infer that

$$q' = \sigma_{s,t'}(\bar{q})\ , \qquad r' = \sigma_{t'',s'}(\bar{r})\ .$$

We claim that

$$(11) \qquad (s + t')\bar{r} = 0\ .$$

For, $\bar{r}d = rd \neq 0$, therefore $\bar{r}r \neq 0$, and if (11) fails, this implies $(s + t')r \neq 0$, $(t + t')(s + t')r \neq 0$, $t'r \neq 0$, $t'rm \neq 0$, a contradiction. From (11) we infer that $\sigma_{s,t'}(\bar{r})$ exists. Inasmuch as $t'' = \sigma_{t',s}(s')$ and thus by Lemma 7.5, $s' = \sigma_{s,t'}(t'')$, it follows by Lemma 7.8 that $r' = \sigma_{s,t'}(\bar{r})$. Consequently, by Lemma 7.6,

$$(q' + r')m = (\bar{q} + \bar{r})m\ .$$

Furthermore, $s' = \sigma_{s,t''}(t')$, hence by Lemma 7.8,

$$\bar{r} = \sigma_{t',s'}(r) = \sigma_{t,s}(r)\ ,$$

whence it follows by Lemma 7.6 that

$$(\bar{q} + \bar{r})m = (q + r)m\ .$$

This completes the proof for the case under consideration.

Case 3. $s'q \neq 0$ and $t'r \neq 0$.

Let $u = \sigma_{t',t}(q)$, $v = \sigma_{s',s}(r)$. Then $us \neq 0$ and $vt \neq 0$, and by Lemma 7.5

$$q = \sigma_{t,t'}(u), \quad r = \sigma_{s,s'}(v)\ .$$

Thus we have $r = \sigma_{u,q}(v)$ by Lemma 7.9, and hence $(u+v)m = (q+r)m$ by Lemma 7.6. Let

$$\begin{aligned} q_1 &= (v + q)m, \quad r_1 = (u + r)m\ , \\ u_1 &= (v + q)(u + r) = (v + q_1)(u + r_1)\ , \\ v_1 &= (r + q_1)(q + r_1)\ . \end{aligned}$$

Then $(u + v)(r_1 + q_1) \leqq (u + v)m \leqq q + r$, from which it follows that $(u, r_1, q) \barwedge (v, q_1, r)$, and therefore $u_1 \leqq d + v_1$. This in turn implies that $(v, u, d) \barwedge (q, r, v_1)$, and therefore

$$(v + u)(q + r) \leqq (v + d)(q + v_1) + (u + d)(r + v_1)\ .$$

Since the element on the left is known to contain z, it suffices to show that the two summands on the right are contained in r' and q', respectively. Indeed, by Lemma 7.9, $r' = \sigma_{u,q}(r)$, and therefore

$$r' = (r + d)(q + r_1) \geqq (v + d)(q + v_1) ,$$

and similarly $q' \geqq (u + d)(r + v_1)$.

Case 4. $dq = 0$ and $dr \neq 0$, or $dq \neq 0$ and $dr = 0$.

By symmetry it suffices to consider the first alternative. We have $bd \geqq (q + s)d > 0$ and $bcd = qd = 0$, therefore $cd = 0$. Also

$$yd \geqq (r + t)d > rd = xyd ,$$

hence $0 < xd < yd$. Let $\bar{x} = x[k], \bar{y} = y[k]$. Because of (7) we have $a \dotplus \bar{x} = a \dotplus \bar{y}$. Since $a + d$ is geometric in $[0, a + d + \bar{x}]$, it follows from Theorem 5.2 that there exists $u \in L$ such that $a + d + x = a \dotplus d \dotplus u$. Observe that

$$(\bar{x} + \bar{y})[1] \leqq a[1] + \bar{x}[1] = a[1] + d[1] ,$$

and therefore $u(\bar{x} + \bar{y}) = 0$. Let

$$\bar{d} = (u + \bar{x})(a + d) .$$

Clearly $u\bar{d} = 0$, and therefore

$$[0, \bar{d}] = [u\bar{d}, \bar{d}] \cong [u, u + \bar{d}] = [u, u + \bar{x}] \cong [u\bar{x}, \bar{x}] = [0, \bar{x}] ,$$

so that $\bar{d}$ is a k-cycle. Since $d\bar{d} \geqq d\bar{x} > 0$, this implies that $a\bar{d} = 0$ and hence $a \dotplus d = a \dotplus \bar{d}$. Letting

$$s'' = (s + \bar{d})(s' + t'), t'' = (t + \bar{d})(s' + t') ,$$

one easily verifies that $s'', t'' \in C(L, m)$. Also $s''t' = (s + \bar{d})t' = 0$, because $(s + d)t' = 0$ and the elements $s + \bar{d}$ and $s + d$ contain the same atoms. Furthermore $s''t'' = 0$, because the only atom contained in both $s + \bar{d}$ and $t + \bar{d}$ is $d[1]$, and $d(s' + t') = 0$. We thus see that

$$s'' \dotplus t'' = s' \dotplus t' = s'' + t' ,$$

and that $(s'' + t'')(s + t) = a$. It is also easy to see that

$$(s + s'')m = (t + t'')m = (s + s'')(t + t'') .$$

Finally, $c\bar{d} = 0$ and $b\bar{d} \neq 0$, because $d\bar{d} \neq 0$, and $\bar{d}y = \bar{x}y \leqq x, \bar{d}y \leqq \bar{d}x$. By Corollary 3.7 and two applications of (2) we therefore have

$$\begin{aligned} Q_{s,t}(b, c, x, y) &= Q_{s'',t''}(b, c, x, y) \\ &= Q_{s'',t'}(b, c, x, y) \\ &= Q_{s',t'}(b, c, x, y) . \end{aligned}$$

The proof is now complete.

DEFINITION 8.5. Suppose L is a primary Arguesian lattice, $m \in L$ is a complemented dual cycle and $gd(m) \geqq 2$. For any cycles $a, b, c, x, y \in L$ contained in m, if there exists (s, t) that is admissible for a, then we let

$$Q(a, b, c, x, y) = Q_{s,t}(b, c, x, y) .$$

9. Extensions of isomorphisms. A one-to-one mapping of the points on a line in a projective plane that satisfies Desargues' Law onto the points on a line in another such plane can be extended to an isomorphism between the two planes if and only if it preserves the operation Q. This section is devoted to the proof of a corresponding theorem for primary lattices.

THEOREM 9.1. *Suppose L and L' are primary Arguesian lattices of geometric dimension three or more, $m \in L$ and $m' \in L'$ are dual points, and $g\colon [0, m] \cong [0, m']$. Then the following conditions are equivalent:*

(i) *For any $u, v \in C(L, m)$ and $u', v' \in C(L', m')$, if*

$$uv = 0, u'v' = 0 \quad and \quad g((u + v)m) = (u' + v')m' ,$$

then there exists f such that

$$f\colon L \cong L', g \subseteqq f, f(u) = u', f(v) = v' .$$

(ii) *For any cycles $a, b, c, x, y \in L$ contained in m,*

$$g(Q(a, b, c, x, y)) = Q(g(a), g(b), g(c), g(x), g(y)) .$$

Proof. Assume (i). Clearly there exist

$$u, v \in C(L, m) \quad \text{and} \quad u', v' \in C(L', m')$$

that satisfy the hypothesis of (i), and this condition is therefore not vacuous. Therefore there exists as isomorphism f of L onto L' with $g \subseteqq f$. Given cycles $a, b, c, x, y \leqq m$, choose (s, t) that is admissible for a. Then $(f(s), f(t))$ is admissible for $f(a)$. We can therefore use these two ordered pairs to compute

$$z = Q(a, b, c, x, y) \quad \text{and} \quad z' = Q(g(a), g(b), g(c), g(x), g(y)) ,$$

respectively, and we infer that $f(z) = z'$, hence $g(z) = z'$.

Conversely, assume (ii), and suppose u, v, u', v' satisfy the conditions of (i). Letting $a_0 = u$ and $a_1 = v$, we can find $a_2, a_3 \in C(L, m)$ such that any three of the four cycles a_i are independent. For $i \neq j$ let

$$b_{i,j} = (a_i + a_j)m, \quad b'_{i,j} = g(b_{i,j}) .$$

Also let $a'_0 = u'$ and $a'_1 = v'$, and

$$a'_i = (u' + b'_{0,i})(v' + b'_{1,i}) \quad \text{for} \quad i = 2, 3 .$$

Then a'_0, a'_1, a'_2, a'_3 are complements of m' in L' and

$$b'_{i,j} = (a'_i + a'_j)m' .$$

For any cycle $x \in L$, and for $i = 0, 1, 2, 3$, let

$$x(i) = (a_i + x)m, \quad x'(i) = g(x(i)) ,$$

and let

$$x^* = \prod_{i=0}^{3} (a'_i + x'(i)) .$$

We claim that the map $x \longrightarrow x^*$ is a cycle isomorphism. From this it follows at once that the induced isomorphism f has the required properties, since $a_i^* = a'_i$ for $i = 0, 1, 2, 3$, and $x^* = g(x)$ for $x \leqq m$.

For any cycle $x \in L$ there exist distinct indices i and j such that $(a_i + a_j)x = 0$. For any such indices we have $(a_i, a_j, x)\perp$, and therefore

$$(a_i + x(i))(a_j + x(j)) = (a_i + x)(a_j + x) = x .$$

Under the same conditions, using (a_i, a_j) as the admissible ordered pair for $b_{i,j}$, we find that

$$Q(b_{i,j}, b_{i,k}, b_{j,k}, x(i), x(j)) = x(k) .$$

Consequently

$$Q(b'_{i,j}, b'_{i,k}, b'_{j,k}, x'(i), x'(j)) = x'(k) .$$

Using (a'_i, a'_j) as the admissible ordered pair for $b'_{i,j}$ we obtain

$$x'(k) = (a'_k + r)m , \quad \text{where} \quad r = (a'_i + x'(i))(a'_j + x'(j)) .$$

Therefore $r \leqq a'_k + x'(k)$. Since this is true for each of the indices k, we infer that $r \leqq x^*$. The opposite inclusion is obvious, and we have

$$x^* = (a'_i + x'(i))(a'_j + x'(j)) .$$

It is now easily seen that x^* is a cycle of the same dimension as x, for

$$a_i \dotplus x(i) = a_i \dotplus x \quad \text{and} \quad a'_i \dotplus x'(i) = a'_i \dotplus x^* .$$

We next observe that, for any cycles $x, y \in L$,

$$(x^* + y^*)m' = g((x + y)m) ,$$

for if we choose the distinct indices i and j so that

$$(a_i + a_j)x = (a_i + a_j)y = 0 ,$$

then

$$(x + y)m = Q(b_{i,j}, x(i), x(j), y(i), y(j)) ,$$
$$(x^* + y^*)m' = Q(b'_{i,j}, x'(i), x'(j), y'(i), y'(j)) .$$

From the special case $y = 0$ we see that $x^*m' = g(xm)$, therefore $\delta(x^*m') = \delta(xm)$, and since $\delta(x^*) = \delta(x)$, this implies that $\delta(x^* + m') = \delta(x + m)$.

Now if $x, y \in L$ are cycles, then the conditions

$$y \leqq x + m, \quad \delta(y + m) \leqq \delta(x + m) ,$$
$$\delta(y^* + m') \leqq \delta(x^* + m'), \quad y^* \leqq x^* + m'$$

are equivalent, and we thus have

$$y \leqq x + m \quad \text{if and only if} \quad y^* \leqq x^* + m .$$

We wish to show that, for any cycles $x, y, z \in L$,

$$z \leqq x + y \quad \text{if and only if} \quad z^* \leqq x^* + y^* .$$

It may be assumed without loss of generality that $y + m \leqq x + m$. Thus if $z \leqq x + y$, then $z, x, y \leqq x + m$ and $x^*, y^*, z^* \leqq x^* + m'$. Now

$$(z + x)m \leqq (x + y)m ,$$
$$g((z + x)m) \leqq g((x + y)m) ,$$
$$(z^* + x^*)m' \leqq (x^* + y^*)m' ,$$

and adding x^* to both sides, we obtain

$$z^* + x^* \leqq x^* + y^* .$$

Since the above steps can be reversed, the opposite implication also holds.

To complete the proof we need only show that the map $x \to x^*$ is one-to-one and onto. For this it suffices to observe that the corresponding map of cycles in L' onto cycles in L is an inverse of the given map.

10. Translations.

Throughout this section we assume that L is a priamary Arguesian lattice, $gd(L) \geqq 3$, and $m \in L$ is a dual point.

DEFINITION 10.1. By an m-dilation of L we mean an automorphism f of L such that $f(x) = x$ whenever $x \leqq m$. We let $D(L, m)$ be the set of all m-dilations of L.

DEFINITION 10.2. By an m-translation of L we mean an m-dilation f of L such that, for all $x, y \in C(L, m)$,

$$(x + f(x))m = (y + f(y))m .$$

We let $T(L, m)$ be the set of all m-translations of L.

DEFINITION 10.3. If f is an m-translation of L, then by the trace of f—in symbols $\mathrm{tr}(f)$,—we mean the unique element $u \in L$ such that, for all $x \in C(L, m)$,

$$u = (x + f(x))m .$$

The trace u of an m-translation f is always a cycle, for if $x \in C(L, m)$, then $x + u = x + f(x)$ and $xu = 0$, therefore $[0, u] \cong [xf(x), f(x)]$.

THEOREM 10.4. *If $f, g \in D(L, m)$, $x, y \in C(L, m)$, $xy = 0$, $f(x) = g(x)$ and $f(y) = g(y)$, then $f = g$.*

Proof. First consider a cycle z such that $(x + y)z = 0$. Then

$$z = (x + z)(y + z) = [x + (x + z)m][y + (y + z)m] ,$$

and since f and g agree on $x, y, (x + z)m$ and $(y + z)m$, we have $f(z) = g(z)$.

Now consider a cycle z with $(x + y)z \neq 0$. We may assume that $xz = 0$. We can find a member y' of $C(L, m)$ such that $(x + y)y' = 0$, and therefore $xy' = 0$ and $(x + y')z = 0$. We now apply the preceding case twice, first with z replaced by y' to infer that $f(y') = g(y')$, and then with y replaced by y' to conclude that $f(z) = g(z)$.

Thus f and g agree on all the cycles in L, and by Theorem 4.17 they are therefore equal.

LEMMA 10.5. *For all $f \in D(L, m)$ and $x \in L$,*

$$f(x) + m = x + m .$$

Proof. Since $f(m) = m$, f maps the chain $[m, 1]$ onto itself. Now, the only automorphism of a finite chain is the identity. Therefore

$$f(x) + m = f(x + m) = x + m .$$

LEMMA 10.6. *For all $f \in D(L, m)$ and $x \in L$,*

$$f(x + f(x)) = x + f(x) .$$

Proof. Let $u = (x + f(x))m$. By the preceding lemma,

$$x + f(x) = x + u = f(x) + u\ .$$

Since f maps u onto itself, the conclusion follows.

LEMMA 10.7. *For all $f \in T(L, m)$ and $x, y \in C(L, m)$, $y(x+f(x)) = xy$ if and only if $x(y + f(y)) = xy$.*

Proof. Assuming the former equality, let

$$u = \mathrm{tr}(f) = (x + f(x))m = (y + f(y))m\ .$$

Then $x + f(x) = x + u$ and $y + f(y) = y + u$. Observe that $(x, y, u)\Delta$, because $y(x + u) = y(x + f(x)) = xy$. Consequently

$$x(y + f(y)) = x(y + u) = xy + xu = xy\ .$$

LEMMA 10.8. *If $f \in T(L, m)$, $x, y \in C(L, m)$, and $y(x + f(x)) = xy$, then*

$$f(y) = [y + (x + f(x))m][f(x) + (x + y)m]\ .$$

Proof. We have $(x + f(x))m = \mathrm{tr}(f) = (y + f(y))m$, and therefore

$$\begin{aligned} &[y + (x + f(x))m][f(x) + (x + y)m] \\ &\quad = [y + f(y)][f(x) + (x + y)m] \\ &\quad = (y + f(y))f(x + (x + y)m) \\ &\quad = (y + f(y))f(x + y)\ . \end{aligned}$$

By Lemmas 10.6 and 10.7, this is equal to

$$f((y + f(y))(x + y)) = f(y + x(y + f(y)) = f(y + xy) = f(y)\ .$$

THEOREM 10.9. *For all $f, g \in T(L, m)$, if $f(x) = g(x)$ for one $x \in C(L, m)$, then $f = g$.*

Proof. Choose $y \in C(L, m)$ with $y(x + f(x)) = 0$. Then $f(y) = g(y)$ by Lemma 10.8, and therefore $f = g$ by Lemma 10.4.

LEMMA 10.10. *For any $f \in D(L, m)$, if there exist $x, y \in C(L, m)$ such that $y(x + f(x)) = 0$ and*

$$(x + f(x))m = (y + f(y))m\ ,$$

then $f \in T(L, m)$.

Proof. Write z' for $f(z)$. Letting $u = (x + x')m = (y + y')m$, observe that $x + x' = x + u = x' + u$ and $y + y' = y + u = y' + u$. Therefore $(x + u)y = (x + x')y = 0$, so that $(x, y, u)\Delta$. Consequently $(x + x')(y + y') = (x + u)(y + u) = xy + u = u$. Letting $v = (x + y)m$, we have

$$\begin{aligned}
&x + y = x + v = y + v \ , \\
&x' + y' = (x + y)' = (x + v)' = x' + v \ , \\
&(x' + y')m = (x' + v)m = v \ , \\
&(x + x')(x + v) = (x + x')(x + y) = x + x'y = x \ .
\end{aligned}$$

This last formula shows that $(x, x', v)\Delta$. Hence

$$(x + y)(x' + y') = (x + v)(x' + v) = xx' + v \ .$$

Consider any $z \in C(L, m)$ with $(x + y)z = 0$. Let $z_x = (x + z)m$ and $z_y = (y + z)m$. Then

$$\begin{aligned}
&x + z_x = x + z = z + z_x, y + z_y = y + z = z + z_y \ , \\
&(x + z_x)(y + z_y) = (x + z)(y + z) = xy + z = z \ .
\end{aligned}$$

Consequently

$$\begin{aligned}
&z' = (x' + z_x)(y' + z_y) \ , \\
&x' + z' = (x + z_x)' = x' + z_x \ , \\
&y' + z' = (y + z_y)' = y' + z_y \ .
\end{aligned}$$

Since $(x' + y')(z_x + z_y) \leqq (x' + y')m = v \leqq x + y$, we have $(x', z_x, x) \barwedge (y', z_y, y)$, and therefore

$$(x' + z_x)(y' + z_y) \leqq (x' + x)(y' + y) + (z_x + x)(z_y + y)$$

or, in other words,

$$(1) \qquad z' \leqq u + z \ .$$

Also, $(x, y, z) \barwedge (x', y', z')$, because

$$\begin{aligned}
&(x + z)(x' + z') + (y + z)(y' + z') \\
&\quad = (x + z_x)(x' + z_x) + (y + z_y)(y' + z_y) \\
&\quad \geqq xx' + z_x + z_y = xx' + (x + z)m + (y + z)m \\
&\quad = xx' + (x + y + z)m \geqq xx' + v \\
&\quad = (x + v)(x' + v) = (x + y)(x' + y') \ .
\end{aligned}$$

Consequently,

$$(x + x')(y + y') \leqq (x + z)(y + z) + (x' + z')(y' + z') \ .$$

Thus $u \leqq z + z'$, and together with (1) this yields $z + z' = z + u$, hence

$$(z + z')m = (z + u)m = u .$$

Now consider $z \in C(L, m)$ with $(x + y)z \neq 0$. Since x, y, z are cycles and $xy = 0$, we have $xz = 0$ or $yz = 0$. Observe that

$$(y + y')x = (y + y')(x + x')x = ux = 0 ,$$

and the hypothesis of the lemma is therefore symmetric in x and y. We may therefore assume that $xz = 0$. We can find $z_1 \in C(L, m)$ that is disjoint from $x + y$ and $x + x'$, and we infer from the first part of the proof that $(z_1 + z_1')m = v$. Also, since $xz = 0$ and $(x + y)z \neq 0$, $x + y$ and $x + z$ contain the same atoms, whence it follows that $(x + z)z_1 = 0$. Thus $(x, z, z_1)\perp$, and therefore $(x + z_1)z = 0$. We again apply the first part of the proof, this time with z_1 in place of y, to conclude that $(z + z')m = u$.

Thus $(z + z')m = u$ for all $z \in C(L, m)$, and therefore $f \in T(L, m)$.

THEOREM 10.11. *For any $x, x' \in C(L, m)$, there exists a unique $f \in T(L, m)$ such that $f(x) = x'$.*

Proof. Choose $y \in C(L, m)$ with $(x + x')y = 0$, and let

$$u = (x + y)m, v = (x + x')m, y' = (x' + u)(y + v) .$$

It is easy to check that $y' \in C(L, m)$ and

$$(x' + y')m = u = (x + y)m .$$

It follows by Theorem 9.1 with g replaced by the identity automorphism of $[0, m]$ that there exists $f \in D(L, m)$ with $f(x) = x'$ and $f(y) = y'$, and we infer by the preceding lemma that $f \in T(L, m)$. The uniqueness of f follows from Theorem 10.9.

COROLLARY 10.12. *For any cycle $v \leqq m$ there exists $f \in T(L, m)$ with* $\mathrm{tr}(f) = v$.

Proof. Choosing $x \in C(L, m)$, use Lemma 6.4 to obtain $x' \in C(L, m)$ with $x + x' = x + v$. By the preceding theorem there exists $f \in T(L, m)$ with $f(x) = x'$, and $\mathrm{tr}(f) = v$.

DEFINITION 10.13. An m-translation f is said to be nonsingular if $\mathrm{tr}(f)$ is a point. In the alternative case f is said to be singular.

If $f \in T(L, m)$ and $x \in C(L, m)$, then

$$\delta(x) = \delta(\mathrm{tr}(f)) + \delta(xf(x)) .$$

Therefore, f is nonsingular if and only if $xf(x) = 0$.

THEOREM 10.14. *$T(L, m)$ is an Abelian group under composition and, for all $f, g \in T(L, m)$,*

$$\mathrm{tr}(gf) \leqq \mathrm{tr}(g) + \mathrm{tr}(f) .$$

Proof. Clearly the identity automorphism of L is an m-translation, and the inverse of an m-translation is an m-translation. Given $f, g \in T(L, m)$ and $x \in C(L, m)$,

$$\begin{aligned} \mathrm{tr}(g) + \mathrm{tr}(f) &= (x + f(x))m + (f(x) + gf(x))m \\ &= [(x + f(x))m + f(x) + gf(x)]m \\ &= (x + f(x) + gf(x))m , \end{aligned}$$

and therefore

(1) $(x + gf(x))m \leqq \mathrm{tr}(g) + \mathrm{tr}(f)$.

To complete the proof we must show that $fg = gf \in T(L, m)$.

If $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, then

$$g(x)(x + f(x)) \leqq (x + \mathrm{tr}(g))(x + \mathrm{tr}(f)) = x ,$$

whence it follows by Lemma 10.8 that

$$fg(x) = (g(x) + \mathrm{tr}(f))(f(x) + \mathrm{tr}(g)) .$$

Because of the symmetry of this formula we infer with the aid of Lemma 10.4 that $fg = gf$.

Before completing the proof of the commutativity, we establish the closure property. We consider five cases.

Case 1. $\mathrm{tr}(f)\,\mathrm{tr}(g) \neq 0$.

Choose $x \in C(L, m)$, and choose a point $p \leqq m$ that is disjoint from the cycles $\mathrm{tr}(f)$ and $(x + gf(x))m$, and then choose $y \in C(L, m)$ with $x \dotplus y = y \dotplus p = p \dotplus x$. Then

(2) $y(x + gf(x)) = 0$

(3) $(x, y, \mathrm{tr}(f))\perp, (x, y, \mathrm{tr}(g))\perp$.

We wish to show that (3) implies

(4) $(x + gf(x))m \leqq (y + gf(y))m$.

In view of (1) this is equivalent to

(5) $(x + gf(x))(\mathrm{tr}(f) + \mathrm{tr}(g)) \leqq y + gf(y)$.

We claim that

(6) $(x, \mathrm{tr}(f), y) \barwedge (gf(x), \mathrm{tr}(g), gf(y))$.

In fact, since $(x + y)m$ is invariant under both f and g, we have

$$(x + y)m = (f(x) + f(y))m = (gf(x) + gf(y))m ,$$

and therefore

$$(x + y)(gf(x) + gf(y)) \geqq (f(x) + f(y))m \;.$$

Since $(\mathrm{tr}(f) + \mathrm{tr}(g))f(x) = 0$, hence $(\mathrm{tr}(f), \mathrm{tr}(g), f(x))\Delta$, we have

$$\begin{aligned}(x + \mathrm{tr}(f))(gf(x) + \mathrm{tr}(g)) &= (x + f(x))(gf(x) + f(x))\\ &= (f(x) + \mathrm{tr}(f))(f(x) + \mathrm{tr}(g))\\ &= f(x) + \mathrm{tr}(f)\,\mathrm{tr}(g) \;.\end{aligned}$$

Similarly

$$(\mathrm{tr}(f) + y)(\mathrm{tr}(g) + gf(y)) = f(y) + \mathrm{tr}(f)\,\mathrm{tr}(g) \;.$$

Thus

$$\begin{aligned}(x + y)(gf(x) + gf(y)) &+ (\mathrm{tr}(f) + y)(\mathrm{tr}(g) + gf(y))\\ &\geqq (f(x) + f(y))m + f(y) + \mathrm{tr}(f)\,\mathrm{tr}(g)\\ &= f(x) + f(y) + \mathrm{tr}(f)\,\mathrm{tr}(g)\\ &\geqq (x + \mathrm{tr}(f))(gf(x) + \mathrm{tr}(g)) \;,\end{aligned}$$

which proves (6). We infer that

$$\begin{aligned}(x + gf(x))(\mathrm{tr}(f) + \mathrm{tr}(g)) &\leqq (x + y)(\mathrm{tr}(f) + y)\\ &\quad + (gf(x) + gf(y))(\mathrm{tr}(g) + gf(y))\\ &= y + gf(y) \;.\end{aligned}$$

The last step follows from (3); we note that because independence is preserved by automorphisms, the second formula in (3) implies that $(gf(x), gf(y), \mathrm{tr}(g))\perp$. Thus (5) holds, and hence so does (4). Interchanging x and y in the above argument, we see that the opposite inclusion also holds, and therefore

$$(x + gf(x))m = (y + gf(y))m \;.$$

Together with (2) and Lemma 10.10 this implies that $gf \in T(L, m)$.

Case 2. $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, and f and g are nonsingular.
Choose $x \in C(L, m)$. Then $xf(x) = 0$ and

$$(x + f(x))gf(x) \leqq (f(x) + \mathrm{tr}(f))(f(x) + \mathrm{tr}(g)) = f(x) \;,$$

therefore $(x + f(x))gf(x) = f(x)gf(x) = 0$. Thus

(7) $(x, f(x), gf(x))\perp$.

Observe that $(f(x) + fgf(x))m = (x + gf(x))m$ because every element contained in m is mapped onto itself by f. We now apply Lemma 10.10 with y and f replaced by $f(x)$ and $fg = gf$ to infer that $gf \in T(L, m)$.

For use later in this proof we observe that (7) implies that gf is nonsingular, and that $\mathrm{tr}(f)\,\mathrm{tr}(gf) = 0$.

Case 3. $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, f is singular and g is nonsingular.

Choose a point $p \leqq m$ disjoint from $\mathrm{tr}(g)$, and choose $x, y \in C(L, m)$ with $x \dotplus y = y \dotplus p = p \dotplus x$. By Theorem 10.11 there exist $f_1, f_2 \in T(L, m)$ such that $f_1(x) = y$ and $f_2(y) = f(x)$. Clearly f_1 and f_2 are nonsingular and

$$\mathrm{tr}(f_1)\,\mathrm{tr}(f_2) = (x + y)(y + f(x))m \geqq (y + xf(x))m \ .$$

Since $xf(x)$ is nonzero and disjoint from y, this shows that $\mathrm{tr}(f_1)\,\mathrm{tr}(f_2) \neq 0$. It follows by Case 1 that $f_2f_1 \in T(L, m)$, and since $f_2f_1(x) = f(x)$, we infer by Theorem 10.9 that $f = f_2f_1$. By Case 2, $gf_2 \in T(L, m)$. Furthermore, as observed in the treatment of Case 2, gf_2 is nonsingular and $\mathrm{tr}(gf_2)$ is disjoint from $\mathrm{tr}(f_2)$, and therefore also from $\mathrm{tr}(f_1)$. By a second application of Case 2 we therefore infer that $gf = gf_2f_1 \in T(L, m)$.

Case 4. $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, f is nonsingular and g is singular.

This case can be treated exactly like the preceding one.

Case 5. $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$ and f and g are singular.

We write f on the form $f = f_2f_1$ with f_1 and f_2 nonsingular, and then apply Case 4 twice.

It only remains to prove the commutativity for the case when $\mathrm{tr}(f)\,\mathrm{tr}(g) \neq 0$. Since every translation can be written as the composition of two nonsigular translations, we may assume that f and g are both nonsingular.

Choose a nonsingular m-translation h with $\mathrm{tr}(h)$ disjoint from $\mathrm{tr}(f)$ and therefore also from $\mathrm{tr}(g)$. Then $gf = gfhh^{-1} = ghfh^{-1}$. By the observation made in the treatment of Case 2 above, $\mathrm{tr}(hf)$ is disjoint from $\mathrm{tr}(f)$, and therefore also from $\mathrm{tr}(g)$. Consequently

$$gf = hfgh^{-1} = fhgh^{-1} = fghh^{-1} = fg \ .$$

The proof of the theorem is now complete.

A somewhat more general form of the observation made at the end of the treatment of Case 2 in the above proof will be useful in the next section.

LEMMA 10.15. *If* $f, g \in T(L, m)$, $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$ *and* g *is nonsingular, then* gf *is nonsingular and* $\mathrm{tr}(f)\,\mathrm{tr}(gf) = 0$.

Proof. Choose $x \in C(L, m)$. As we saw in the preceding proof, $(x + f(x))gf(x) \leqq f(x)$. Therefore

$$(x + f(x))gf(x) = f(x)gf(x) = 0 \ ,$$

$$\begin{aligned}\operatorname{tr}(f)\operatorname{tr}(gf) &= (x + f(x))(x + gf(x))m \\ &= [x + (x + f(x))gf(x)]m = xm = 0 ,\end{aligned}$$

and $xgf(x) \leqq f(x)gf(x) = 0$.

11. Trace-preserving endomorphisms. As in the preceding section we assume that L is a primary Arguesian lattice, $gd(L) \geqq 3$, and m is a dual point of L. We let n be the rank of L. The endomorphisms of the Abelian group $T(L, m)$ form a ring, and in accordance with the multiplicative notation used in $T(L, m)$ we write the endomorphisms as exponents. Thus, for two endomorphisms α and β, and for $f \in T(L, m)$, we have

$$f^{\alpha+\beta} = f^{\alpha}f^{\beta},\ f^{\alpha\beta} = (f^{\alpha})^{\beta} .$$

DEFINITION 11.1. An endomorphism α of $T(L, m)$ is said to be trace-preserving if $\operatorname{tr}(f^{\alpha}) \leqq \operatorname{tr}(f)$ for all $f \in T(L, m)$. We let $R(L, m)$ be the set of all trace-preserving endomorphisms of $T(L, m)$.

COROLLARY 11.2. *$R(L, m)$ is a subring of the ring of all endomorphisms of $T(L, m)$, containing the identity endomorphism.*

Proof. Given $\alpha, \beta \in R(L, m)$ and $f \in T(L, m)$, it follows from Theorem 10.14 that

$$\operatorname{tr}(f^{\alpha+\beta}) = \operatorname{tr}(f^{\alpha}f^{\beta}) \leqq \operatorname{tr}(f^{\alpha}) + \operatorname{tr}(f^{\beta}) \leqq \operatorname{tr}(f) .$$

Also,

$$\begin{aligned}\operatorname{tr}(f^{\alpha\beta}) &= \operatorname{tr}((f^{\alpha})^{\beta}) \leqq \operatorname{tr}(f^{\alpha}) \leqq \operatorname{tr}(f) , \\ \operatorname{tr}(f^{1}) &= \operatorname{tr}(f) = \operatorname{tr}(f^{-1}) .\end{aligned}$$

Therefore $R(L, m)$ is closed under the ring operations, and $1 \in R(L, m)$.

Our next objective is to prove that if f and f' are m-translations of L with $\operatorname{tr}(f') \leqq \operatorname{tr}(f)$, then $f' = f^{\beta}$ for some $\beta \in R(L, m)$.

LEMMA 11.3. *If $f, g \in T(L, m)$, f is nonsingular and* $\operatorname{tr}(f)\operatorname{tr}(g) = 0$, *then for all $\alpha \in R(L, m)$ and $x \in C(L, m)$,*

$$g^{\alpha}(x) = (x + \operatorname{tr}(g))(f^{\alpha}(x) + \operatorname{tr}(gf^{-1})) .$$

Proof. We have $g^{\alpha}(x) \leqq x + \operatorname{tr}(g^{\alpha}) \leqq x + \operatorname{tr}(g)$ and

$$g^{\alpha}(x) = (gf^{-1})^{\alpha}f^{\alpha}(x) \leqq f^{\alpha}(x) + \operatorname{tr}((gf^{-1})^{\alpha}) \leqq f^{\alpha}(x) + \operatorname{tr}(gf^{-1})) .$$

Letting

$$y = (x + \mathrm{tr}(g))(f^{\alpha}(x) + \mathrm{tr}(gf^{-1}))\ ,$$

we infer that $g^{\alpha}(x) \leqq y$. To rule out strict inclusion we need only observe that $g^{\alpha}(x) \in C(L, m)$ and by Lemma 10.15, $ym = \mathrm{tr}(g)\,\mathrm{tr}(gf^{-1}) = 0$.

LEMMA 11.4. *If $f \in T(L, m)$ is nonsingular, then for all $\alpha, \beta \in R(L, m)$, $f^{\alpha} = f^{\beta}$ implies $\alpha = \beta$.*

Proof. We want to show that $g^{\alpha} = g^{\beta}$ for all $g \in T(L, m)$, and in view of Theorem 10.9 it suffices to prove that $g^{\alpha}(x) = g^{\beta}(x)$ for $x \in C(L, m)$. If $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, then this follows from the preceding lemma. In the alternative case we choose $h \in T(L, m)$ nonsingular and with $\mathrm{tr}(f)\,\mathrm{tr}(h) = 0$, and hence also $\mathrm{tr}(g)\,\mathrm{tr}(h) = 0$. By two applications of the special case already considered we infer first that $h^{\alpha} = h^{\beta}$, and then that $g^{\alpha} = g^{\beta}$.

The following observation will be used several times in the proof of the next theorem.

LEMMA 11.5. *If $f, g \in T(L, m)$, $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, $x, y \in C(L, m)$, and $y \leqq x + \mathrm{tr}(gf^{-1})$, then*

$$(x + \mathrm{tr}(f))(y + \mathrm{tr}(g)) \in C(L, m)\ .$$

Proof. Letting $z = (x + \mathrm{tr}(f))(y + \mathrm{tr}(g))$, we easily check that $z + \mathrm{tr}(g) = y + \mathrm{tr}(g)$ and $zm = 0$, whence the conclusion follows.

THEOREM 11.6. *If $f, f' \in T(L, m)$ and $\mathrm{tr}(f') \leqq \mathrm{tr}(f)$, then $f' = f^{\beta}$ for some $\beta \in R(L, m)$.*

Proof. First assume that f is nonsingular. Choose $x \in C(L, m)$. For each $g \in T(L, m)$ with $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$ there exists, by Lemma 11.5 and Theorem 10.11 a member g^{α} of $T(L, m)$ such that

(1) $g^{\alpha}(x) = (x + \mathrm{tr}(g))(f'(x) + \mathrm{tr}(gf^{-1}))$.

Later, α will be extended to the whole group $T(L, m)$, and it will be shown that the map so obtained is the required endomorphism.

Suppose $g, h \in T(L, m)$ and $\mathrm{tr}(f)\,\mathrm{tr}(g) = \mathrm{tr}(f)\,\mathrm{tr}(h) = 0$. We shall prove that

(2) $h^{\alpha}(x) \leqq g^{\alpha}(x) + \mathrm{tr}(hg^{-1})$.

Consider the formula

(3) $(x, h(x), g(x)) \barwedge (f'(x), \mathrm{tr}(hf^{-1}), \mathrm{tr}(gf^{-1}))$.

We have

$$\begin{aligned}(x + f'(x))(h(x) + \mathrm{tr}(hf^{-1})) &\leqq (f(x) + \mathrm{tr}(f))(f(x) + \mathrm{tr}(hf^{-1})) \\ &= f(x) + \mathrm{tr}(f)\,\mathrm{tr}(hf^{-1})\ ,\end{aligned}$$

$$g(x) + \mathrm{tr}(gf^{-1}) = f(x) + \mathrm{tr}(gf^{-1}) .$$

Therefore, if

(4) $\mathrm{tr}(f)\,\mathrm{tr}(hf^{-1}) \leqq \mathrm{tr}(gf^{-1})$,

then (3) holds, and (2) readily follows. If (4) fails, then $\mathrm{tr}(f)\,\mathrm{tr}(gf^{-1}) \leqq \mathrm{tr}(hf^{-1})$, and interchanging g and h we find that

(5) $g^\alpha(x) \leqq h^\alpha(x) + \mathrm{tr}(hg^{-1})$.

However, since $g^\alpha(x)$ and $h^\alpha(x)$ are members of $C(L, m)$, the formulas (2) and (5) are equivalent, and (2) therefore holds in either case.

From (2) it follows that

(6) $h^\alpha(x) = (x + h(x))(g^\alpha(x) + \mathrm{tr}(hg^{-1}))$

whenever $g, h \in T(L, m)$, g is nonsingular, and the three cycles $\mathrm{tr}(f)$, $\mathrm{tr}(g)$ and $\mathrm{tr}(h)$ are pairwise disjoint. In fact, the left hand side is obviously contained in the right, and strict inclusion is impossible because by Lemma 11.5 the element on the right is a complement of m.

We now describe the promised extension β of α. Letting $f_0 = f$ and $\alpha_0 = \alpha$, choose nonsingular m-translations f_1 and f_2 such that the three cycles $\mathrm{tr}(f_i)$ are pairwise disjoint, let $f_1' = f_1^\alpha$ and $f_2' = f_2^\alpha$, and define the corresponding partial maps α_1 and α_2 in the same manner as $\alpha_0 = \alpha$ was defined using $f_0 = f$. It is easy to check using (6), that $f_i^{\alpha_j} = f_i'$ for $i \neq j$, and again using (6) we see that any two of the maps α_i agree wherever both are defined, and they therefore have a common extension β to $T(L, m)$. Furthermore, any two m-translations g and h belong jointly to the domain of at least one α_i, and using (2) with α replaced by α_i, we infer that

(7) $h^\beta(x) \leqq g^\beta(x) + \mathrm{tr}(hg^{-1})$.

From this it readily follows that

(8) $\mathrm{tr}(h^\beta g^{-\beta}) \leqq \mathrm{tr}(hg^{-1})$.

In fact, we have

$$\begin{aligned}\mathrm{tr}(h^\beta g^{-\beta}) &= (h^\beta(x) + g^\beta(x))m \\ &\leqq (g^\beta(x) + \mathrm{tr}(hg^{-1}))m = \mathrm{tr}(hg^{-1}) .\end{aligned}$$

Taking g in (8) to be the identity map, we obtain

(9) $\mathrm{tr}(h^\beta) \leqq \mathrm{tr}(h)$.

Replacing h by hg in (7), we find that

$$(hg)^\beta(x) \leqq g^\beta(x) + \mathrm{tr}(h) .$$

The corresponding formula with g and h interchanged also holds, and because of the commutativity of $T(L, m)$ this yields

$$(hg)^\beta(x) \leqq (g^\beta(x) + \mathrm{tr}(h))(h^\beta(x) + \mathrm{tr}(g)) .$$

If $\mathrm{tr}(g)\,\mathrm{tr}(h) = 0$, and hence $\mathrm{tr}(g^\beta)\,\mathrm{tr}(h^\beta) = 0$, then by Lemma 10.8,

$$h^\beta g^\beta(x) = (g^\beta(x) + \mathrm{tr}(h^\beta)(h^\beta(x) + \mathrm{tr}(g^\beta))$$
$$\leqq (g^\beta(x) + \mathrm{tr}(h))(h^\beta(x) + \mathrm{tr}(g)) ,$$

and since in this case the element

$$(g^\beta(x) + \mathrm{tr}(h))(h^\beta(x) + \mathrm{tr}(g))$$

is disjoint from m, it follows that

(10) $(hg)^\beta = h^\beta g^\beta$.

In order to prove (10) for the case when $\mathrm{tr}(g)\,\mathrm{tr}(h) \neq 0$, we choose $i = 0, 1$ or 2 so that $\mathrm{tr}(f_i)$ is disjoint from $\mathrm{tr}(g)$ and $\mathrm{tr}(hg)$, and therefore also from $\mathrm{tr}(h)$. By repeated applications of the case already considered we then have

$$(hg)^\beta f_i^\beta = (hgf_i)^\beta = h^\beta(gf_i)^\beta = h^\beta g^\beta f_i^\beta .$$

In the second step use has been made of the fact that, by Lemma 10.15, $\mathrm{tr}(gf_i)$ is disjoint from $\mathrm{tr}(g)$, and therefore also from $\mathrm{tr}(h)$. Canceling f_i^β, we conclude that (10) holds for all $g, h \in T(L, m)$. Thus β is an endomorphism of $T(L, m)$ and, by (9), $\beta \in R(L, m)$. Since $f^\beta = f'$, this completes the proof for case when f is nonsingular.

If f is singular, we choose $g \in T(L, m)$ nonsingular with $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$ and let

$$u = (x + \mathrm{tr}(g))(f'(x) + \mathrm{tr}(fg^{-1})) .$$

Then $u + \mathrm{tr}(g) = x + \mathrm{tr}(g)$, hence $u + m = 1$, and it follows from the dual of Theorem 4.8 that there exists $y \in C(L, m)$ with $y \leqq u$.

By Theorem 10.11, $y = g'(x)$ for some $g' \in T(L, m)$, and by the first part of this proof, $g' = g^\beta$ for some $\beta \in R(L, m)$. By Lemma 11.3,

$$\begin{aligned} f^\beta(x) &= (x + \mathrm{tr}(f))(y + \mathrm{tr}(fg^{-1})) \\ &\leqq (x + \mathrm{tr}(f))(u + \mathrm{tr}(fg^{-1})) \\ &= (f'(x) + \mathrm{tr}(f))(f'(x) + \mathrm{tr}(fg^{-1})) = f'(x) . \end{aligned}$$

Consequently $f^\beta = f'$, and the proof is complete.

The remainder of this section will be devoted to the problem of showing that $R(L, m)$ is completely primary and uniserial.

DEFINITION 11.7. For $f \in T(L, m)$ we let $\nu(f) = n - \delta(\mathrm{tr}(f))$.

Thus if $f \in T(L, m)$ and $x \in C(L, m)$, then $\nu(f) = \delta(xf(x))$.

LEMMA 11.8. *If $f, g \in T(L, m)$ and $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, then*

$$\delta(\mathrm{tr}(f)\,\mathrm{tr}(gf)) = \max\{\nu(g) - \nu(f), 0\} ,$$
$$\nu(gf) = \min\{\nu(f), \nu(g)\} .$$

Proof. First assume that $\nu(g) \leqq \nu(f)$. Choosing $x \in C(L, m)$ we have

$$(x + f(x))gf(x) \leqq (f(x) + \mathrm{tr}(f))(f(x) + \mathrm{tr}(g)) = f(x) ,$$

so that $(x, f(x), gf(x))\Delta$, and therefore

$$\mathrm{tr}(f)\,\mathrm{tr}(gf) = (x + f(x))(x + gf(x))m = (x + f(x)gf(x))m .$$

Now $xg(x) \leqq xf(x)$, and therefore both $xg(x)$ and its image under $f, f(x)fg(x)$, are contained in $f(x)$. This implies that $f(x)fg(x) = xg(x) \leqq x$, and we therefore find that

$$\mathrm{tr}(f)\,\mathrm{tr}(gf) = xm = 0 .$$

Since $\mathrm{tr}(f) + \mathrm{tr}(g) = \mathrm{tr}(f) + \mathrm{tr}(gf)$, and the two summands on each side are disjoint, $\mathrm{tr}(g)$ and $\mathrm{tr}(gf)$ must have the same dimension, and therefore $\nu(gf) = \nu(g)$, as was to be shown.

If $\nu(f) < \nu(g)$, then, by the first part of the proof, $\mathrm{tr}(g)\,\mathrm{tr}(gf) = 0$, and the cycles $\mathrm{tr}(f)$ and $\mathrm{tr}(gf)$ have the same dimension. From the equation $\mathrm{tr}(f) + \mathrm{tr}(gf) = \mathrm{tr}(g) + \mathrm{tr}(gf)$ we therefore conclude that $\delta(\mathrm{tr}(f)) = \delta(\mathrm{tr}(g)) + \delta(\mathrm{tr}(f)\,\mathrm{tr}(gf))$, and consequently

$$\delta(\mathrm{tr}(f)\,\mathrm{tr}(gf)) = \delta(\mathrm{tr}(f)) - \delta(\mathrm{tr}(g)) = \nu(g) - \nu(f) .$$

LEMMA 11.9. *For any $f, g \in T(L, m)$ and $\alpha \in R(L, m)$, if f is nonsingular, then $\nu(g^\alpha) = \min\{\nu(f^\alpha) + \nu(g), n\}$.*

Proof. First assume that $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, choose $x \in C(L, m)$, and let

$$u = \mathrm{tr}(f), v = \mathrm{tr}(g), w = \mathrm{tr}(gf^{-1}), u' = \mathrm{tr}(f^\alpha), v' = \mathrm{tr}(g^\alpha) .$$

Then by Lemma 11.3,

$$\begin{aligned} v' = (x + g^\alpha(x))m &= (x + v)(x + f^\alpha(x) + w)m \\ &= (x + v)(x + u' + w)m = v(u' + w) . \end{aligned}$$

We also have $u + v = v + w = w + u$ by Theorem 10.14, $uv = 0$ by hypothesis, and $vw = 0$ by Lemma 10.15. Using this we obtain the isomorphisms

$$[uw, u] \cong [w, u + w] = [w, v + w] \cong [vw, v] = [0, v] .$$

Here $u' + uw$ is mapped onto $(u' + w)v = v'$. Therefore

$$\delta(v') = \delta(u' + uw) - \delta(uw) = \delta(u') - \delta(u'w) ,$$

(1) $\nu(g^\alpha) = \nu(f^\alpha) + \delta(u'w)$.

If $\nu(g) + \nu(f^\alpha) \leqq n$, then $\delta(v) + \delta(u') \geqq n$. Since, by the preceding lemma, $\delta(uw) = n - \delta(v)$, this yields $\delta(u') \geqq \delta(uw)$. Inasmuch as

u' and uw are subcycles of u, we infer that $uw \leqq u'$, $uw = u'w$, $\delta(u'w) = n - \delta(v) = \nu(g)$, and therefore by (1),

$$\nu(g^\alpha) = \nu(f^\alpha) + \nu(g) .$$

If $\nu(g) + \nu(f^\alpha) \geqq n$, then $\delta(v) + \delta(u') \leqq n$, $\delta(u') \leqq n - \delta(v) = \delta(uw)$, $u' \leqq w$. In this case (1) yields

$$\nu(g^\alpha) = \nu(f^\alpha) + \delta(\mathrm{tr}(f^\alpha)) = n .$$

For the particular case when f and g are nonsingular and their traces are disjoint we see that $\nu(f^\alpha) = \nu(g^\alpha)$. To prove the lemma for the case when $\mathrm{tr}(f)\,\mathrm{tr}(g) \neq 0$, we therefore merely replace f by another m-translation h such that $\mathrm{tr}(g)\,\mathrm{tr}(h) = 0$.

DEFINITION 11.10. For any $\alpha \in R(L, m)$ we let $\nu(\alpha)$ be the unique natural number k such that $\nu(f^\alpha) = k$ for every nonsingular m-translation f.

COROLLARY 11.11. *For all $g \in T(L, m)$ and $\alpha, \beta \in R(L, m)$,*

$$\begin{aligned}\nu(g^\alpha) &= \min\{\nu(g) + \nu(\alpha), n\} ,\\ \nu(\alpha\beta) &= \min\{\nu(\alpha) + \nu(\beta), n\} .\end{aligned}$$

THEOREM 11.12. *For all $\alpha, \beta \in R(L, m)$, in order that there exist $\lambda \in R(L, m)$ with $\alpha = \lambda\beta$, it is necessary and sufficient that $\nu(\alpha) \geqq \nu(\beta)$.*

Proof. Assuming that $\nu(\alpha) \geqq \nu(\beta)$, choose nonsingular m-translations f and g with $\mathrm{tr}(f)\,\mathrm{tr}(g) = 0$, and choose $x \in C(L, m)$. Let

$$u = \mathrm{tr}(f^{-\beta}g^\alpha) , \qquad y = (f(x) + u)(x + \mathrm{tr}(g)) .$$

By Lemma 11.8, u is disjoint from $\mathrm{tr}(g^\alpha)$, and hence also from $\mathrm{tr}(g)$. Therefore $y\,\mathrm{tr}(g) \leqq ym = u\,\mathrm{tr}(g) = 0$, and since $\mathrm{tr}(g)$ is a point in the lattice $[0, x + \mathrm{tr}(g)]$, it follows by Theorem 4.8 that $\mathrm{tr}(g)$ has a complement z in that lattice with $y \leqq z$. Clearly $z \in C(L, m)$, and hence by Theorems 10.11 and 11.6 there exists $\lambda \in R(L, m)$ such that $g^\lambda(x) = z$. By Lemma 11.3,

$$g^{\lambda\beta}(x) = (x + g^\lambda(x))(f^\beta(x) + \mathrm{tr}(f^{-1}g^\lambda) .$$

Since $\mathrm{tr}(f^{-1}g^\lambda) = (f(x) + z)m \geqq (f(x) + y)m \geqq u$, this yields

$$\begin{aligned}g^{\lambda\beta}(x) &\geqq (x + y)(f^\beta(x) + u)\\ &= (x + f(x) + u)(x + \mathrm{tr}(g))(f^\beta(x) + g^\alpha(x))\\ &\geqq (x + \mathrm{tr}(f) + u)g^\alpha(x) = g^\alpha(x) .\end{aligned}$$

Thus $g^{\lambda\beta}(x) = g^{\alpha}(x)$, therefore $g^{\lambda\beta} = g^{\alpha}$ by Theorem 10.9, and hence $\lambda\beta = \alpha$ by Lemma 11.4. The opposite implication is an immediate consequence of Corollary 11.11.

THEOREM 11.13. *For all $\alpha, \beta \in R(L, m)$, in order that $\alpha = \beta\mu$ for some $\mu \in R(L, m)$, it is necessary and sufficient that $\nu(\alpha) \geqq \nu(\beta)$.*

Proof. Assuming that $\nu(\alpha) \geqq \nu(\beta)$, choose a nonsingular m-translation f. Then $\mathrm{tr}(f^{\alpha}) \leqq \mathrm{tr}(f^{\beta})$, so that by Theorem 11.6 there exists $\mu \in R(L, m)$ with $f^{\alpha} = (f^{\beta})^{\mu} = f^{\beta\mu}$. By Lemma 11.4, $\alpha = \beta\mu$. The converse is an immediate consequence of Corollary 11.11.

THEOREM 11.14. *$R(L, m)$ is a completely primary uniserial ring of rank n. The set of all elements $\alpha \in R(L, m)$ with $\nu(\alpha) \neq 0$ is the unique maximal proper ideal P of $R(L, m)$, and it coincides with the set of all nonunits of $R(L, m)$. For $k = 0, 1, \cdots, n$, P^k is the set of all $\alpha \in R(L, m)$ with $\nu(\alpha) \geqq k$.*

Proof. For $k = 0, 1, \cdots, n$ let A_k be the set of all $\alpha \in R(L, m)$ with $\nu(\alpha) \geqq k$. From the two preceding theorems we see that if $\nu(\alpha) = k$, then $\alpha R(L, m) = R(L, m)\alpha = A_k$. Thus the sets A_k are two-sided ideals of $R(L, m)$, and there are no other right or left ideals. In particular, $P = A_1$ is the unique maximal proper ideal of $R(L, m)$. Finally, choosing a ring element α with $\nu(\alpha) = 1$, we use Corollary 11.11 to infer that $\nu(\alpha^k) = k$, and therefore $A_k = \alpha^k R(L, m) = P^k$.

12. The representation theorem.

We still assume that L is a primary Arguesian lattice of rank n, $gd(L) \geqq 3$, and m is a dual point of L.

DEFINITION 12.1. For $x \leqq m$ we let

$$F_m(x) = \{f \in T(L, m) \colon \mathrm{tr}(f) \leqq x\} .$$

LEMMA 12.2. *F_m is an isomorphism of $[0, m]$ onto the lattice of all $R(L, m)$ submodules of $T(L, m)$.*

Proof. From the fact that

$$\mathrm{tr}(fg) \leqq \mathrm{tr}(f) + \mathrm{tr}(g) , \qquad \mathrm{tr}(f^{\alpha}) \leqq \mathrm{tr}(f) ,$$

we see that F_m maps $[0, m]$ into the lattice L of all $R(L, m)$ submodules of $T(L, m)$. It is also obvious that if $x \leqq y \leqq m$, then $F_m(x) \subseteqq F_m(y)$. Conversely, suppose $x, y \in [0, m]$ and $x \nleqq y$. Then there exists a cycle z such that $z \leqq x$ and $z \nleqq y$. By Corollary 10.12 there exists

$f \in T(L, m)$ such that $\mathrm{tr}(f) = z$, hence $f \in F_m(x)$ and $f \notin F_m(y)$. Thus $F_m(x) \nsubseteq F_m(y)$.

To complete the proof it suffices to show that F_m is onto. However, before doing this, we show that F_m preserves least upper bounds, i.e., that

$$F_m(u + v) = F_m(u)F_m(v)$$

for all $u, v \leqq m$. This will follow if we show that for any m-translation f with $\mathrm{tr}(f) \leqq u + v$ there exist m-translations g and h such that $f = gh$, $\mathrm{tr}(g) \leqq u$, $\mathrm{tr}(h) \leqq v$.

Letting $w = \mathrm{tr}(f)$ we may assume that $u + v = v + w = w + u$, for otherwise u and v can be replaced by the elements $u(v + w)$ and $v(u + w)$, and this condition will be satisfied. Choose $x \in C(L, m)$ and let $y = (x + u)(f(x) + v)$. Then $y + u = x + u$, hence $y + m = x + m$, and it follows from the dual of Theorem 4.8 that there exists $z \in C(L, m)$ with $z \leqq y$. Let g be the m-translation with $g(x) = z$, and let $h = g^{-1}f$. Then $\mathrm{tr}(g) = (x + z)m \leqq u$ and

$$\begin{aligned}\mathrm{tr}(h) &= (f(x) + g(x))m = (f(x) + z)m \\ &\leqq (f(x) + y)m \leqq (f(x) + v)m = v\ ,\end{aligned}$$

as was to be shown.

Now suppose $U \in L$, and let u be the sum of all the cycles $x \leqq m$ with $F_m(x) \subseteqq U$. It follows from Theorem 11.6 that for any m-translation f, $F_m(\mathrm{tr}(f))$ is the cyclic submodule $f^{R(L,m)}$ of $T(L, m)$. Therefore, if $f \in U$, then $F_m(\mathrm{tr}(f)) \subseteqq U$, and consequently $\mathrm{tr}(f) \leqq u$, $f \in F_m(u)$. Thus $U \subseteqq F_m(u)$. That this inclusion must actually be an equality follows from the fact that u is the sum of finitely many cycles x with $F_m(x) \subseteqq U$, and that $F_m(u)$ is the least upper bound of the corresponding modules $F_m(x)$.

THEOREM 12.3. *L is isomorphic to the lattice of all submodules of a finitely generated module over $R(L, m)$.*

Proof. Choose a module U that is a direct sum of a free cyclic module $\kappa R(L, m)$ and a module M isomorphic to $T(L, m)$. Let $\boldsymbol{L}(U)$ be the lattice of all submodules of U, and $\boldsymbol{L}(M)$ the lattice of all submodules of M. Obviously $\kappa R(L, m)$ is a complement of M in $\boldsymbol{L}(U)$, and since the right ideals of $R(L, m)$ form a chain with $n + 1$ elements, $\kappa R(L, m)$ is an n-cycle. In view of the preceding theorem, $\boldsymbol{L}(M)$ is of rank n and geometric dimension two or more. Therefore $\boldsymbol{L}(U)$ is of rank n, and its geometric dimension is at least three.

The given isomorphism $\phi\colon M \cong T(L, m)$, together with the isomorphism F_m, gives rise to an isomorphism $\Phi\colon [0, m] \cong \boldsymbol{L}(M)$ with

$$\Phi(x) = \{\xi \in M : \phi(\xi) \in F_m(x)\}$$

for all $x \leqq m$. By Theorem 9.1, in order to show that Φ can be extended to an isomorphism of L onto $L(U)$ it suffices to show that it preserves the operation Q. We therefore consider five cycles $a, b, c, x, y \leqq m$ and their images $A, B, C, X, Y \subseteqq M$ under Φ, and let

$$z = Q(a, b, c, x, y)\ , \qquad Z = Q(A, B, C, X, Y)\ .$$

Also let Z' be the set of all $\xi \in U$ such that for some $\eta, \zeta \in U$,

(1) $\eta \in B, \xi - \eta \in X, \zeta \in C, \xi - \zeta \in Y, \eta - \zeta \in A$.

We claim that

(2) $Z \subseteqq Z' \subseteqq \Phi(z)$.

This will complete the proof, for by Lemma 8.3, z and Z have the same dimension, and strict inclusions are therefore not possible.

Choosing submodules S and T of U such that (S, T) is admissible for A, and letting

$$V = (S + B) \cap (T + C)\ , \qquad W = (S + X) \cap (T + Y)\ ,$$

we have $Z = (V + W) \cap M$. Therefore, if $\xi \in Z$, then there exists λ such that $\lambda \in V$ and $\xi - \lambda \in W$. Since $\lambda \in V$, there exist η, ζ such that $\eta \in B$, $\lambda - \eta \in S$, $\zeta \in C$, $\lambda - \zeta \in T$. We now infer that

$$\begin{aligned} \xi - \eta &\varepsilon (W + S) \cap (Z + B) \subseteqq X\ , \\ \xi - \zeta &\varepsilon (W + T) \cap (Z + C) \subseteqq Y\ , \\ \eta - \zeta &\varepsilon (S + T) \cap (B + C) \subseteqq A\ . \end{aligned}$$

Consequently (1) is satisfied, and we have $\xi \in Z'$.

The second inclusion in (2) is equivalent to the assertion that if $f, g, h \in T(L, m)$ are such that

$$g \in F_m(b), fg^{-1} \in F_m(x), h \in F_m(c), fh^{-1} \in F_m(y), gh^{-1} \in F_m(a)\ ,$$

then $f \in F_m(z)$. In other words, it is equivalent to the assertion that if

(3) $\operatorname{tr}(g) \leqq b, \operatorname{tr}(fg^{-1}) \leqq x, \operatorname{tr}(h) \leqq c, \operatorname{tr}(fh^{-1}) \leqq y, \operatorname{tr}(gh^{-1}) \leqq a$,

then $\operatorname{tr}(f) \leqq z$. Assuming that (3) holds, choose $s \in C(L, m)$ and let $t = gh^{-1}(s)$ and $a' = \operatorname{tr}(gh^{-1}) = (s + t)m$. Then $a' \leqq a$ and (s, t) is admissible for a'. Letting

$$z' = Q(a', b, c, x, y) = Q_{s,t}(b, c, x, y)\ ,$$

we therefore have $z' \leqq z$. Letting

$$v = (s + b)(t + c)\ , \qquad w = (s + x)(t + y)\ ,$$

we have $(v + w)m = z'$. Since $\operatorname{tr}(h) = (g(s) + gh^{-1}(s))m$, we have

$$g(s) \leqq (s + \operatorname{tr}(g))(gh^{-1}(s) + \operatorname{tr}(h)) \leqq v\ .$$

Similarly

$$gf^{-1}(s) \leqq (s + \mathrm{tr}(gf^{-1}))(gh^{-1}(s) + \mathrm{tr}(hf^{-1})) \leqq w \ .$$

Consequently

$$\mathrm{tr}\,(f) = (g(s) + gf^{-1}(s))m \leqq (v + w)m \leqq z \ ,$$

as was to be shown.

BIBLIOGRAPHY

1. I. Amemiya, *On the representation of complemented modular lattices,* J. Math. Soc. Japan **9** (1957), 263-279.
2. E. Artin, *Coordinates in affine geometry,* Reports of a Math. Colloq., Notre Dame, 1940, 15-20.
3. R. Baer, *A unified theory of projective spaces and finite Abelian groups,* Trans. Amer. Math. Soc. **52** (1942), 283-343.
4. G. Birkhoff, *Lattice theory,* Amer. Math. Soc. Colloq. Publ. 25, third ed., Providence, 1967.
5. W. W. D. Hodge and D. Pedoe, *Methods of algebraic geometry,* Cambridge University Press, Cambridge, 1947.
6. E. Inaba, *On primary lattices,* Journal of the Faculty of Science, Hokkaido University **11** (1948), 39-107.
7. B. Jónsson, *Representations of complemented modular lattices,* Trans. Amer. Math. Soc. **97** (1960), 64-94.
8. ———, *On the representation of lattices,* Math. Scand. **1** (1953), 193-206.
9. G. S. Monk, *The representation of primary lattices,* Thesis, University of Minnesota, Minneapolis, 1966.
10. J. von Neumann, *Continuous geometry,* New York, 1960 (originally published in mimeographed form as Lectures on continuous geometries, Princeton, 1936-1937).
11. M. Scuhützenberger, *Sur certains axiomes de la theorie des structures,* C. R. Acad. Sci. Paris **221** (1945), 218-220.

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VANDERBILT UNIVERSITY
NASHVILLE, TENNESSEE
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON