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COUNTABLE RETRACING FUNCTIONS AND Π_2^0 PREDICATES

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In this paper our attention centers on partial recursive retracing functions, especially *countable* ones (*as defined below*), and on their relationship with classes of number theoretic functions constituting solution sets for Π_2^0 function predicates in the Kleene hierarchy. Arithmetical function predicates which have *singleton* solution sets (i.e., so called *implicit arithmetical definitions*) have received ample attention in the recursion-theoretic literature. We shall be concerned with such predicates, at the levels Π_1^0 and Π_2^0; but we shall primarily be concerned with the wider classes of Π_1^0 and Π_2^0 predicates having *countable* solution sets. In § 5, we show (by obtaining examples which range over the whole of $\mathscr{H} \cap \{D \mid D > 0'\}$, $\mathscr{H}$ as defined in § 4) that a solution of a countable Π_1^0 predicate need not be definable by means of a "strong" Π_2^0 predicate; in fact, we establish the corresponding (slightly stronger) proposition for countable, *finite-to-one*, general recursive retracing functions. The question whether all solutions of a countable Π_2^0 predicate are Π_2^0 definable is left open but subjected to conjecture.

In § 4, we present a new and somewhat more compact proof for one of the main theorems obtained by C. E. M. Yates in [20] (indeed, we obtain a slightly stronger theorem); and we shall derive one of the other principal results of [20] as a corollary to some of our theorems. In § 4 and § 5 systematic use is made of the main content of Myhill's paper [14].

We proceed now to lay down the conventions which are to be in force throughout the rest of the paper; at the end of this section we shall indicate briefly the contents of each of the remaining sections. The symbol N always denotes the set $\{0, 1, 2, \cdots\}$ of natural numbers. We shall in general use lower case Greek letters for subsets of N and lower case Latin letters for functions (partial or total) with domain and range included in N, although this particular convention will not be adhered to with absolute rigor. Given a function $f: \alpha \rightarrow N$ where $\alpha \subseteq N$, we denote by δf the domain, α, of f, and by ρf the range of f. We fix a standard recursive enumeration ([10]) of the partial recursive functions of one variable, and denote this enumeration by $\{\varphi_e\}_{e=0}^{\infty}$; similarly, we fix a standard recursive enumeration $\{\varphi_e^2\}_{e=0}^{\infty}$ of the partial recursive functions of two variables. We further fix a recursive enumeration $\mathscr{E}_1$ of the set $\{(e, x, y) \mid \varphi_e(x) = y\}$; and we denote by φ_e^s the set $\{(x, y) \mid (\exists t)_{t \leqq s}(\mathscr{E}_1(t) = (e, x, y))\}$. Similarly, we fix a recursive enumeration $\mathscr{E}_2$ of the set $\{(e, x_1, x_2, y) \mid \varphi_e^2(x_1, x_2) = y\}$; and we denote by $\varphi_e^{2,s}$ the set $\{(x_1, x_2, y) \mid (\exists t)_{t \leqq s}(\mathscr{E}_2(t) = (e, x_1, x_2, y))\}$. We degree $\varphi_0^2 = \varnothing$. If $\mathscr{F}$ is a class of partial recursive functions, then by the *index set*, $G(\mathscr{F})$, of $\mathscr{F}$ we mean $\{e \mid \varphi_e \in \mathscr{F}\}$. We denote by W_e the set $\delta\varphi_e$; and we define $D_0 = \varnothing$ and $D_{n+1} = \{m_1, \cdots, m_r\}$, where $n + 1 = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r}$ and where $m_1 < m_2 < \cdots < m_r$ in case $r > 1$. For any set $\beta \subseteqq N$, we denote by c_β the characteristic function of β, taking value 1 on members of β and 0 on nonmembers. By a *finite initial function* we mean a function $w: \alpha \rightarrow N$ such that $(\exists n)(\alpha = \{x \mid x < n\})$. By $lh(w)$, w a finite initial function, we mean the cardinality of δw. Such standard notations as p_k, $(m)_n$, and μ (the "least number operator") are used as in [6]. If e is any natural number and w any total or finite initial function, the notation $\{e\}^w$ shall have the meaning given it on page 5 of [17]. We use the notation $\bar{f}(x)$ (for any f, partial or total, such that f is defined at least for all $y < x$) according to the convention of [17, p. 4]. As in [6], we shall say that n is a *sequence number* $\Leftrightarrow (\exists t)(\exists f)[n = \bar{f}(t)]$. We shall employ boldface notation for Turing degrees; more particularly, if $\alpha \subseteqq N$ then $\boldsymbol{\alpha}$ denotes the Turing degree of α, if f is a function from N into N then $\boldsymbol{f}$ denotes the Turing degree of f, and notations such as $\boldsymbol{D}$ and $\boldsymbol{C}$ stand simply for Turing degrees. $\leqq$ denotes the ordering relation on Turing degrees. Our notations for the jump and (finitely) iterated jump operations are those of [17]. Henceforth, we shall refer simply to *degree* when Turing degree is meant. If α is an infinite subset of N, we denote by p_α the principal function of α, i.e., the function from N into N which enumerates α in order of magnitude. We shall refer to any strictly increasing function $f: N \rightarrow N$ as a *principal function*. Let f be a principal function with range α, and suppose that h is a partial recursive function such that $\alpha \subseteqq \delta h$, $h(f(0)) = f(0)$, and $(\forall n)(h(f(n + 1)) = f(n))$. Then we say that f is *retraceable*, also that α is *retraceable*, and that h *retraces* f and also α. A partial recursive function h is a *retracing function* $\Leftrightarrow h$ retraces at least one principal function. The basic properties of such pairs (f, h) have been considered in [1] and [2]. A retracing function f is *special* $\Leftrightarrow \rho f \subseteqq \delta f$ & $(\forall n)(n \in \delta f \Rightarrow f(n) \leqq n)$. If f is a special retracing function, then $\hat{f}(n)$ is finite for all $n \in \delta f$, where $\hat{f}(n)$ denotes the set $\{n, f(n), f(f(n)), \cdots\}$. It is easily seen that if α is retraced by h then α is retraceable via some *special* retracing subfunction of h.

A finite-to-one special retracing function is called *basic*. If f is a special retracing function and $n \in \delta f$, we denote by $f^*(n)$ the number $\mu y(f^y(n) = f^{y+1}(n))$; here $f^y(n)$ is defined inductively by $f^0(n) = n$, $f^{y+1}(n) = f(f^y(n))$. Number- and function-predicate levels $\prod_n^0$, $\sum_n^0$, $\prod_n^1$ $\sum_n^1$, for arbitrary $n \geqq 0$, are defined as in [16, p. 383]. As is well known, every $\prod_1^0$ predicate of one function variable can be expressed in the form $(\forall x)R(\bar{f}(x))$ where R is a primitive recursive predicate of numbers. If a $\prod_2^0$ predicate P of one function variable can be expressed in the form $(\forall x)Q(\bar{f}(x))$ with Q a number predicate of degree $\leqq \boldsymbol{O}'$, then we say that P is a *strong* $\prod_2^0$ predicate; this is equivalent to expressibility of P in the form $(\forall x)(\exists y)R(\bar{f}(x), y)$, R recursive. For $\prod_2^0$ predicates in general, various "normal forms" are available. In this paper we find it convenient to observe that every $\prod_2^0$ predicate of one function variable can be expressed in the form $(\forall x)(\exists y)_{y>x}(\forall z)_{z\leqq x}R(\bar{f}(z), \bar{f}(y))$, R recursive; such an expression we refer to as a $\prod_2^0$ *normal form*. (To verify this equivalence the reader should proceed in easy steps, as follows: (i) a $\prod_2^0$ predicate $P(f)$ can be represented in the form $(\forall x)(\exists y)S(\bar{f}(x), \bar{f}(y))$, S recursive, as may be seen by considering $\sum_2^0$ predicates and taking into account the uniformity, in an extra number variable, of the corresponding fact about $\prod_1^0$ predicates; (ii) a predicate of the form $(\forall x)(\exists y)S(\bar{f}(x), \bar{f}(y))$, S recursive, is easily seen to be equivalent to a predicate of the form $(\forall x)(\exists y)_{y>x}Q(\bar{f}(x), \bar{f}(y)))$, Q recursive; and finally (iii) $(\forall x)(\exists y)_{y>x}Q(\bar{f}(x), \bar{f}(y))$, Q recursive, is evidently equivalent to $(\forall x)(\exists y)_{y>x}(\forall z)_{z\leqq x}R(\bar{f}(z), \bar{f}(y))$ for a suitable recursive R.) A function $f: N \to N$ is said to be $\prod_1^0$ *definable* ($\prod_2^0$ *definable*) $\Leftrightarrow f$ is the unique solution of some $\prod_1^0$ predicate of functions (some $\prod_2^0$ normal form). A predicate P of functions is said to be *countable* (*unique*) $\Leftrightarrow$ there are at most $\aleph_0$ functions f such that $P(f)$ holds (exactly one function f such that $P(f)$ holds); a retracing function is *countable* (*unique*) $\Leftrightarrow$ it retraces $\leqq \aleph_0$ sets (exactly one set).

We now turn to some preliminary remarks on solution classes for function predicates. Let $\mathscr{F}$ be a set of functions $f: N \to N$. By the *closure*, $K_{\mathscr{F}}$, of $\mathscr{F}$, we mean the set of all functions $g: N \to N$ such that $(\forall n)(\exists f)[f \in \mathscr{F}$ & $(\forall m)_{m\leqq n}(g(m) = f(m))]$. (This, of course, is exactly the topological closure of $\mathscr{F}$ in Baire Space.) To say that $\mathscr{F}$ is *closed* means, of course, that $\mathscr{F} = K_{\mathscr{F}}$. Let P be a predicate of one function variable; and let $\mathscr{F}(P)$ denote the set of "solutions" of P: $\mathscr{F}(P) = \{f \mid P(f)\}$. We shall say that a predicate Q is a *finite restriction of* $P \Leftrightarrow$ there are numbers $m_1, n_1, \cdots, m_k, n_k$, $k > 0$, such that $[Q(f) \Leftrightarrow (P(f)$ & $f(m_1) = n_1$ & $\cdots$ & $f(m_k) = n_k)]$. We note the following very simple proposition:

THEOREM 1.1. *Suppose* $\mathscr{F}(P)$ *is closed, nonempty and countable. Then* $\mathscr{F}(P)$ *contains a function* f *such that, for some finite restriction* Q *of* P, $\mathscr{F}(Q) = \{f\}$.

The proof of Theorem 1.1 consists either in appealing to the fact that a nonempty, closed, countable set in a complete metric space has an isolated point, or else in a few simple direct observations about branching in $\mathscr{F}(P)$ (as in [12, proof of Theorem 7]); we omit details.

COROLLARY 1.2. *Every countable strong* $\prod_2^0$ *predicate which has at least one solution has a solution which is* the unique *solution either of the given predicate or of some finite restriction of it; and every countable retracing function extends a unique retracing function.*

Proof. Observe that the set of solutions of a strong $\prod_2^0$ predicate P is closed. This allows us to apply Theorem 1.1 to P (if P does not itself have a unique solution), and the first statement of the corollary follows. As for retracing functions, first note that the collection of principal functions retraced by a given retracing function f is the solution set of a strong $\prod_2^0$ predicate P_f of functions; and it is clear, moreover, that from a finite restriction Q of P_f we can obtain a partial recursive restriction f_Q of f such that f_Q retraces precisely those principal functions which are solutions of Q. Thus the second statement of the corollary follows from the first.

In § 2, we shall find the exact position in the Kleene hierarchy of the index set corresponding to the class of countable retracing functions. In § 3, we construct a degree $\boldsymbol{D}, \boldsymbol{O} < \boldsymbol{D} < \boldsymbol{O}''$, such that no function which is of degree $> \boldsymbol{O}$ but $\leqq \boldsymbol{D}$ satisfies a countable $\prod_2^0$ normal form. In § 4, we obtain various results relating retracing functions (countable and otherwise) to a class $\mathscr{H}$ of degrees whose representatives form a "thick skeleton" for the hyperarithmetical hierarchy. Finally, in § 5 we prove a theorem which has the following corollary: for every degree $\boldsymbol{D} \in \mathscr{H}$ such that $\boldsymbol{O} < \boldsymbol{D}$, there is a function $f \in \boldsymbol{D}$ with the properties that (i) f satisfies a countable $\prod_1^0$ predicate but is not $\prod_1^0$ definable and (ii) $\boldsymbol{D} > \boldsymbol{O}' \Rightarrow \rho f$ is retraced by a general recursive, countable, basic retracing function but is not retraced by any *unique* retracing function and indeed is not definable by any strong $\prod_2^0$ predicate.

2. In [20], Yates has shown that the index set $G(\text{Ret})$ associated with the class of all retracing functions is a complete $\sum_1^1$ set of natural numbers; i.e., every $\sum_1^1$ set of natural numbers is 1-1 reducible to $G(\text{Ret})$, and $G(\text{Ret})$ is itself expressible in $\sum_1^1$ form. In this section we shall prove, partly on the basis of a simple modification of Yates' argument, that the following two index sets are complete $\prod_1^1$ sets:

(a) $G(C\text{-Ret}) = \{e \mid \varphi_e$ is a countable retracing function$\}$;

(b) $G(U\text{-Ret}) = \{e \mid \varphi_e$ is a unique retracing function$\}$.

THEOREM 2.1. $G(C\text{-Ret})$ *and* $G(U\text{-Ret})$ *are complete* $\prod_1^1$ *sets.*

Proof. We first show that $G(C\text{-Ret}), G(U\text{-Ret})$ are $\prod_1^1$. Let us consider first the case of $G(C\text{-Ret})$. It is a well known fact that if a $\sum_1^1$ predicate of functions has only countably many solutions, then it has only hyperarithmetical solutions. But the statement that f is retraced by φ_e is easily seen to be a $\prod_2^0$ predicate of f and e, and hence a $\sum_1^1$ predicate of f and e. Thus, if $e \in G(C\text{-Ret})$ then φ_e retraces only hyperarithmetical sets. It follows that the predicate $e \in G(C\text{-Ret})$ can be expressed in the form:

$(\exists f)$ [f is hyperarithmetical & f is strictly increasing & f is retraced by φ_e] & $(\forall f)$ [f is a strictly increasing function such that φ_e retraces $f \Rightarrow f$ is hyperarithmetical.]

But "f is hyperarithmetical" is a $\prod_1^1$ predicate of f ([7], [16]); "f is strictly increasing and φ_e retraces f" is a $\prod_2^0$ predicate of f and e; and, by a well known theorem of Kleene ([7, Lemma 1]), any predicate of one number variable of the form $(\exists f)$ [f is hyperarithmetical & $A(f, x)$], where A is arithmetical, is equivalent to some $\prod_1^1$ predicate of x. Thus, we see that the above expression for $e \in G(C\text{-Ret})$ can be put into $\prod_1^1$ form as a predicate of e. To verify that $e \in G(U\text{-Ret})$ can be expressed in $\prod_1^1$ form, we merely note that

$$e \in G(U\text{-Ret}) \Leftrightarrow e \in G(C\text{-Ret}) \ \& \ (\forall f)(\forall g)\,[(f \text{ and } g \text{ are strictly increasing and } \varphi_e \text{ retraces both } f \text{ and } g) \Rightarrow (\forall x)(f(x) = g(x))];$$

since the second conjunct on the right-hand side of this last equivalence is $\prod_1^1$, we have that $G(U\text{-Ret})$ is $\prod_1^1$.

We next show that for any $\prod_1^1$ numerical predicate P there exists a recursive function h_P such that

$$(\forall x)[P(x) \Rightarrow h_P(x) \in G(U\text{-Ret})) \ \& \ (\neg P(x) \Rightarrow \varphi_{h_P(x)}$$

retraces $2^{\aleph_0}$ functions)].

Let P be given by $(\exists f)(\forall x)R(\bar{f}(x), z)$, R recursive. Let α be a set of numbers, and f a partial recursive function, such that:

(i) f is a unique retracing function which retraces α, and

(ii) $\delta f = \{2n + 1 \mid n \in N\}$.

Let a function h be defined on N-$\{0\}$ by the relation

$$h^{-1}(n) = \{2n + 1, 2n + 2\}\,.$$

We define a two-place recursive function g by cases, as follows: (a) $g(z, n) = f(n)$ if n is odd; (b) $g(z, 2^{k+1}) = 2^{k+1}$ if $R(2^{k+1}, z)$;

$$\text{(c)}\quad g(z, 2^{k_0+1} \cdots p_{\varepsilon(m)}^{k_m+1} p_{\varepsilon(m+1)}^{k_{m+1}+1}) = 2^{k_0+1} \cdots p_{\varepsilon(m)}^{k_m+1}\,,$$

provided that (ci) $\varepsilon(j) \in h^{-1}(\varepsilon(j-1))$ for $1 \leqq j \leqq m + 1$ and

$$\text{(cii)}\quad (\forall x)_{x \leqq m+1} R(\prod_{j \leqq x} p_j^{k_j+1}, z)\,;$$

and (d) $g(z, n) = 0$ in all other cases. (The idea of part (c) in our definition of g is, of course, to produce a retracing function whose graph has plenty of binary branching in case $\rightarrow P(z)$; at this point in our argument we are merely adding binary branching to Yates' proof of [20, Th. 1].) For each fixed z, let g_z denote the function $g(z, n)$. Now, if $\rightarrow P(z)$ then $(\exists f)(\forall x)R(\bar{f}(x), z)$; let f_0 be a particular function such that $(\forall x)R(\prod_{j \leqq x} p_j^{f_0(j)+1}, z)$. Let $\{r_n\}_{n=0}^{\infty}$ be any sequence such that $r_0 = 2^{f_0(0)+1}$ & $r_{n+1} = r_n p_t^{f_0(n+1)+1}$ where $t \in h^{-1}(w)$ with p_w being the largest prime dividing r_n. It is clear from the definition of g_z that g_z retraces $\{r_n\}_{n=0}^{\infty}$; moreover, since h is two-to-one with $\delta h \subseteqq \rho h$, there are $2^{\aleph_0}$ such sequences $\{r_n\}_{n=0}^{\infty}$. If, on the other hand, $P(z)$ holds, then $\rightarrow (\exists f)(\forall x)R(\bar{f}(x), z)$. But if g_z retraces a set β then, clearly, either $\beta = \alpha$ or else the exponents in the prime-power factorizations of the elements of β provide the values for a function f_0 such that $(\forall x)R(\bar{f}_0(x), z)$; hence g_z retraces only α if $P(z)$ holds. Thus, letting h_P be any one-to-one recursive function such that $(\forall z)(g_z = \varphi_{h_P(z)})$, we have that $\{z \mid P(z)\}$ is simultaneously 1-1 reduced to $G(U\text{-Ret})$ and to $G(C\text{-Ret})$ via h_P.

REMARK 2.2. $\{e \mid \varphi_e$ retraces uncountably many functions$\}$ is a complete $\sum_1^1$ set. It is $\sum_1^1$ since both $\{e \mid \varphi_e$ is a retracing function$\}$ and $\{e \mid \varphi_e$ is not a countable retracing function$\}$ are $\sum_1^1$; and it is complete by the proof of Theorem 2.1.

REMARK 2.3. It is easily seen that Theorem 2.1 continues to hold if $G(C\text{-Ret})$ and $G(U\text{-Ret})$ are replaced by the index sets corresponding to the classes of countable special retracing functions and unique special retracing functions. Furthermore, the class of retraceable functions can be replaced by the more extensive class of regressive functions as defined in [1]. This last observation is general for the present paper: those of our theorems which make universal assertions about retraceable sets and functions can easily be generalized to cover regressive sets and functions, via recursive equivalence mappings (see [1]).

3. Our principal concern in this section is to prove the existence of a nonzero degree $\boldsymbol{D}$, with $\boldsymbol{D} < \boldsymbol{O}''$, such that $\boldsymbol{O} < \boldsymbol{C} \leqq \boldsymbol{D} \Rightarrow \boldsymbol{C}$ contains no function which satisfies a countable $\prod_2^0$ normal form. We shall begin by proving a small but helpful theorem which is quite possibly known, although we are unable to supply a reference for it; apart from whatever interest it may have in its own right, this theorem has the virtue of reducing the technicalities which enter into the proof of Theorem 3.3.

THEOREM 3.1. *If P is a $\prod_2^0$ normal form, then there is a $\prod_1^0$ predicate Q, of one function variable, such that there exists a degree-preserving one-to-one correspondence between the solutions of P and the solutions of Q.*

Proof. The idea is simply to use the fact that a $\prod_2^0$ normal form has certain "Skolem functions" associated with its solutions. Let P be a $\prod_2^0$ normal form; thus $P(f) \Leftrightarrow (\forall x)(\exists y)_{y>x}(\forall z)_{z\leq x}R(\bar{f}(z), \bar{f}(y))$, for some recursive predicate R. For every function f which satisfies P we define f_* as follows:

$$f_*(2x) = f(x); f_*(2x+1) = \mu y[y > x \ \& \ (\forall z)_{z\leq x}R(\bar{f}(z), \bar{f}(y))].$$

If f satisfies P, then f_* is obviously a total function having the same degree as f. The mapping $f \to f_*$ is the desired degree-preserving one-to-one correspondence; it remains to construct the corresponding predicate Q. First, for every function $g: N \to N$ we define a function g_E by $g_E(x) = g(2x)$. Thus $P(f) \Rightarrow (f_*)_E = f$. Q is defined as follows:

$$Q(g) \Leftrightarrow (\forall x)[g(2x+1) = \mu y(y > x \ \& \ (\forall z)_{z\leq x}R(\bar{g}_E(z), \bar{g}_E(y)))] \ .$$

Clearly, Q can be expressed as a $\prod_1^0$ predicate of g (i.e., the μ-operator can be eliminated), so it remains only to see that the solutions are precisely the functions f_* such that $P(f)$ holds. But if $P(f)$ holds, then f_* satisfies Q because of the definition of f_* and the fact that $(f_*)_E = f$. And if $Q(g)$ holds, then $P(g_E)$ and so $(g_E)_* = g$.

COROLLARY 3.2. (1) *If a degree contains a $\prod_2^0$ definable function, then it contains a $\prod_1^0$ definable function.*

(2) *If a degree contains a function which satisfies some countable $\prod_2^0$ normal form, then it contains a function which satisfies some countable $\prod_1^0$ predicate of functions.*

(3) *If a degree contains a $\prod_2^0$ definable function, then it contains only $\prod_2^0$ definable functions.*

(4) *A countable $\prod_2^0$ normal form has a $\prod_2^0$ definable solution.*

Proof. Both (1) and (2) are obvious consequences of Theorem 3.1. As for (3), let P be a $\prod_2^0$ predicate of functions having f as its unique solution; and let numbers e_0, e_1 and a function h_0 be given such that $\{e_0\}^f = h_0$ and $\{e_1\}^{h_0} = f$. Let $Q(h)$ be the predicate: $\{e_1\}^h$ is total & $P(\{e_1\}^h)$ & $h = \{e_0\}^{\{e_1\}^h}$. Then it is easy to see that $Q(h)$ is a $\prod_2^0$ predicate having h_0 as its unique solution. (4) follows from (3) together with Theorem 3.1, via Corollary 1.2 (noting that $\prod_1^0$ predicates are strongly $\prod_2^0$).

THEOREM 3.3. *There exists a degree $\boldsymbol{D}$ such that*

(i) $\mathbf{O} < \mathbf{D} < \mathbf{O}''$, *and*

(ii) $[\mathbf{O} < \mathbf{C} \leqq \mathbf{D}$ & (*P is a countable* $\prod_2^0$ *normal form*) & $f \in \mathbf{C}] \Rightarrow \neg P(f)$.

Proof. By Corollary 3.2(2), it will suffice to find a $\mathbf{D}$ such that (i) and (ii) hold with "countable $\prod_2^0$ normal form" replaced by "countable $\prod_1^0$ predicate of functions" in (ii). But a function f which represents such a $\mathbf{D}$ can be defined in stages by an ordinary diagonal procedure, as we shall now show. At the end of each stage s in the definition of f, the portion $f^{(s)}$ of f which has thus far been obtained will be a finite initial function. We let $\{R_i\}_{i=0}^{\infty}$ be a recursive enumeration of all primitive recursive predicates of one number variable (we could, equally well for present our purposes, employ a $\mathbf{0}''$-enumeration of all *general* recursive predicates of one number variable); and we fix a recursive wellordering of $N \times N$.

Stage 0. Set $f^{(0)} = \varnothing$.

Stage $2s + 1$, $s \geqq 0$.

Case I. There exist a number n and a finite initial function w extending $f^{(2s)}$ such that if u is any finite initial function extending w then $\{(s)_0\}^u(n)$ is undefined.

Letting (n_0, w_0) be the first such pair (n, w), set $f^{(2s+1)} = w_0$ and proceed to Stage $2s + 2$.

(Thus if Case I holds at Stage $2s + 1$, we define $f^{(2s+1)}$ in such a way that $(s)_0$ will not be an index of a function recursive in f.)

Case II. Case I fails to hold; in addition, there exist a number n and a finite initial function w extending $f^{(2s)}$ such that $[m \geqq n$ & (w_1, w_2 are finite initial functions extending w) & ($\{(s)_0\}^{w_1}(m)$ and $\{(s)_0\}^{w_2}(m)$ are both defined)$] \Rightarrow \{(s)_0\}^{w_1}(m) = \{(s)_0\}^{w_2}(m)$.

Letting (n_0, w_0) be the first such pair (n, w), set $f^{(2s+1)} = w_0$ and proceed to Stage $2s + 2$.

(Thus if Case II holds at stage $2s + 1$, we define $f^{(2s+1)}$ in such a way that $\{(s)_0\}^f$ will, if total, be a general recursive function.)

Case III. Cases I and II both fail to hold; in addition, there exist a number n and a finite initial function w extending $f^{(2s)}$ such that (i) $\{(s)_0\}^w(k)$ is defined for all $k \leqq n$, and (ii) $\neg R_{(s)_1}(\overline{\{(s)_0\}^w}(n))$.

Letting (n_0, w_0) be the first such pair (n, w), set $f^{(2s+1)} = w_0$ and proceed to Stage $2s + 2$.

(Thus if Case III holds at stage $2s + 1$, we define $f^{(2s+1)}$ in such a way that if $\{(s)_0\}^f$ is total then it is not a solution of $(\forall x)R_{(s)_1}(\bar{g}(x))$.)

Case IV. Cases I-III all fail to hold. Then, as is easily seen, the following holds for every n: [(w is a finite initial function extending $f^{(2s)}$) & ($\{(s)_0\}^w(k)$ is defined for all $k \leqq n$)$] \Rightarrow (\forall k)_{k \leqq n} R_{(s)_1}(\overline{\{(s)_0\}^w}(k))$.

In this case, set $f^{(2s+1)} = f^{(2s)}$ and proceed to Stage $2s + 2$.

(If Case IV holds at stage $2s + 1$, then $(\forall x)R_{(s)_1}(\overline{\{(s)_0\}^g}(x))$ holds for every function g such that (a) g extends $f^{(2s)}$ and (b) $\{(s)_0\}^g$ is total. But since Cases I and II both fail to hold, there must in fact be a family $\mathscr{F}$ of $2^{\aleph_0}$ functions g, each extending $f^{(2s)}$, such that $(g_1, g_2 \in \mathscr{F}$ and $g_1 \neq g_2) \Rightarrow \{(s)_0\}^{g_1}$ and $\{(s)_0\}^{g_2}$ are total and distinct. Thus, in Case IV, $(\forall x)R_{(s)_1}(\bar{g}(x))$ has $2^{\aleph_0}$ solutions.)

Stage $2s$, $s > 0$.

Case A. φ_s is a total function.

Letting k be the least number not in $\delta f^{(2s-1)}$, set

$$f^{(2s)} = f^{(2s-1)} \cup \{(k, \varphi_s(k) + 1)\}$$

and proceed to stage $2s + 1$.

(Case A is dealt with so as to insure $f \neq \varphi_s$.)

Case B. φ_s is not total.

In this case, set $f^{(2s)} = f^{(2s-1)}$ and proceed to stage $2s + 1$.

This completes the description of the general stage in the definition of f; we of course set $f = \bigcup_s f^{(s)}$. It is easy to see that each of Cases I-IV and A, B presents us with a decision problem of degree $\leqq \boldsymbol{O}''$; hence $\boldsymbol{f} \leqq \boldsymbol{O}''$. Moreover $\boldsymbol{O} \neq \boldsymbol{f}$ because of Case A.

Let $(\forall x)R_{e_2}(\bar{g}(x))$ be any Π_1^0 predicate of one function variable and e_1 any natural number; and let $2s + 1$ be a stage such that $(s)_0 = e_1$, $(s)_1 = e_2$. Then, from the parenthetical remarks following the descriptions of actions taken under Cases I-IV, we see that if $\{e_1\}^f$ is total and satisfies $(\forall x)R_{e_2}(\bar{g}(x))$ then either $(\forall x)R_{e_2}(\bar{g}(x))$ has $2^{\aleph_0}$ solutions or $\{e_1\}^f$ is recursive. So it remains only to verify that $\boldsymbol{f} < \boldsymbol{O}''$. But as is well known, $\boldsymbol{O}''$ contains functions that are Π_1^0 definable; hence $\boldsymbol{f} \neq \boldsymbol{O}''$.

REMARK 3.4. Analogues of Theorem 3.3 for larger numbers of quantifiers can be proved; in the present paper, however, we are interested only in the Π_2^0 case.

COROLLARY 3.5. *There exists a degree* $\boldsymbol{D}$ *such that*

(i) $\boldsymbol{O} < \boldsymbol{D} < \boldsymbol{O}''$.

and

(ii) $\boldsymbol{O} < \boldsymbol{C} \leqq \boldsymbol{D} \Rightarrow$ *no function belonging to* $\boldsymbol{C}$ *can be retraced by a countable retracing function.*

Proof. For any $\alpha \subseteqq N$ and any number e, the statement that p_α is retraced by e is a Π_2^0 statement—indeed, a *strong* Π_2^0 statement—about p_α.

REMARK 3.6. Suppose α is a set of numbers such that α is *generic* (in the sense of Feferman) *for 2-quantifier prenex arithmetical statements*; and suppose $\boldsymbol{D} < \boldsymbol{O}''$ where $\boldsymbol{D} = \boldsymbol{\alpha}$. Then $\boldsymbol{D}$ meets the requirements of Theorem 3.3: given such a generic α to start with, the proof follows the pattern of Cases I-IV in our definition of f in the above proof of Theorem 3.3. But there are also degrees satisfying Theorem 3.3 that are far from having generic representatives; in particular, there are examples $\boldsymbol{D}$ with $\boldsymbol{D}$ *minimal* (constructed, of course, by mixing our argument with Spector's construction of a minimal degree.)

4. In this section we shall prove several theorems which serve variously to extend, refine, or supplement some of the contents of Yates' papers [19] and [20]. We begin by characterizing those pairs (f, α) such that f is a special retracing function and α is retraced by a basic subfunction of f. For our characterization, as well as for later theorems, we need the notion of ***D*-boundedness**:

DEFINITION 4.1. Let $\boldsymbol{D}$ be a degree, f a total function from N into N, and α an infinite subset of N. Then

(1) f is ***D*-bounded** $\Leftrightarrow$ there exists a function $h\colon N \to N$ such that h is recursive in $\boldsymbol{D}$ and $(\forall n)(h(n) > f(n))$;

(2) α is ***D*-bounded** $\Leftrightarrow p_\alpha$ is $\boldsymbol{D}$-bounded. (In the literature, infinite sets which are *not* $\boldsymbol{O}$-bounded have been called *hyperimmune*.)

THEOREM 4.2. *Let f be a special retracing function, and let α be a set retraced by f. The following three statements are equivalent*:

(i) $(\exists \tilde{f})(\tilde{f}$ *is a basic retracing function* & $\tilde{f}$ *retraces* $\alpha)$;

(ii) $(\exists \tilde{f})(\tilde{f}$ *is a basic retracing function* & $\tilde{f} \subseteq f$ & $\tilde{f}$ *retraces* $\alpha)$;

(iii) α *is* $\boldsymbol{O}'$-*bounded*.

Proof. (i) $\Leftrightarrow$ (ii) is immediate since the intersection of any two special retracing functions which retrace at least one set in common is again a special retracing function. To see that (iii) $\Rightarrow$ (ii), assume α to be $\boldsymbol{O}'$-bounded; then there exists a function h of degree $\leqq \boldsymbol{O}'$ such that $h(n) > p_\alpha(n)$ for all n. A well known convergence theorem states that if $\boldsymbol{C} \leqq \boldsymbol{D}'$ then [g a one-place function belonging to $\boldsymbol{C}$] $\Rightarrow$ [there exists a two-place function $\tilde{g}$ such that $\tilde{\boldsymbol{g}} \leqq \boldsymbol{D}$ & $(\forall n)(\lim_{s\to\infty} \tilde{g}(s, n)$ exists and is equal to $g(n))$]. Consequently there is a two-place recursive function $\tilde{h}$ such that $(\forall n)(\lim_{s\to\infty} \tilde{h}(s, n)$ exists and is equal to $h(n))$. We define a function $\tilde{f}$ as follows:

$$\tilde{f}(x) = y \Leftrightarrow f(x) = y \ \& \ (\exists s)(x \leqq \tilde{h}(s, f^*(x))) .$$

It is obvious that $\tilde{f}$ is a partial recursive subfunction of f. Moreover, it follows easily from the definition of $\tilde{f}$ that $\tilde{f}^{-1}(y)$ is finite for every $y \in \rho\tilde{f}$; for if $y \in \rho\tilde{f}$ then $[f(x) = y \ \& \ x \neq y \ \& \ f^*(y) = n] \Rightarrow f^*(x) = n + 1$, so that $f^{-1}(y)$ must be finite since $\lim_{s\to\infty} \tilde{h}(s, n + 1)$ exists. That $\tilde{f}$ retraces α is also easily verified: we have, for every n, that

$f^*(p_\alpha(n)) = n$ and also that $p_\alpha(n) < h(n) = \tilde{h}(s(n), n)$ for a suitably chosen number $s(n)$; thus $p_\alpha(n) < \tilde{h}(s(n), f^*(p_\alpha(n)))$, so that the condition for including the pair $(p_\alpha(n), f(p_\alpha(n)))$ in $\tilde{f}$ is met. It follows that if $\rho\tilde{f} \subseteqq \delta\tilde{f}$ then $\tilde{f}$ meets the requirements of (ii); otherwise, they are met by the function $\tilde{f}_0 = \{(x, y) \mid (x, y) \in \tilde{f} \;\&\; \hat{f}(x) \subseteqq \delta\tilde{f}\}$. Finally, suppose that $\tilde{f}$ is a basic retracing function which retraces α. Then

$$\{\{x \mid x \in \delta\tilde{f} \;\&\; \tilde{f}^*(x) = n\} \mid n \in N\}$$

is a sequence of finite sets; and it is easily seen that the function h defined by the identity

$$h(n) = \max\{x \mid x \in \delta\tilde{f} \;\&\; \tilde{f}^*(x) \leqq n\}$$

is recursive in $\boldsymbol{O}'$ and dominates p_α. Thus (ii) $\Rightarrow$ (iii) and the proof is complete.

COROLLARY 4.3. *Let G be the index set corresponding to $\{f \mid f$ is a retracing function which retraces at least one $\boldsymbol{O}'$-bounded set$\}$. Then G is a complete $\sum_4^0$ set of numbers.*

Proof. Theorem 4.2 and the remark following Theorem 8 in [12].

Yates observed in [20] that the Kreisel-Shoenfield basis theorem ([18, Theorems 1 and 2]) relativizes routinely to any degree $\boldsymbol{D}$ and its jump $\boldsymbol{D}'$ (further on in this section we shall explicitly state the relativized Kreisel-Shoenfield basis theorem as a part of Lemma 4.9); and he further observed that the resulting relativized basis assertion easily implies the following lemma (= Theorem 2 of [20]):

LEMMA 4.4 (*Yates*). *Every basic retracing function retraces at least one set of degree strictly less than $\boldsymbol{O}''$.*

COROLLARY 4.5. (1) *If a retracing function f retraces no set of degree $< \boldsymbol{O}''$, then f retraces only sets which fail to be $\boldsymbol{O}'$-bounded.*

(2) *If a countable retracing function f retraces no set of degree $\leqq \boldsymbol{O}'$, then f retraces only sets which fail to be $\boldsymbol{O}'$-bounded.*

Proof. (1) follows immediately from the combination of Theorem 4.2 with Lemma 4.4. Suppose now that f is countable, and that f retraces at least one $\boldsymbol{O}'$-bounded set. By Theorem 4.2, f extends a basic retracing function $\tilde{f}$. Since f is countable, $\tilde{f}$ is countable. But by [12, Theorem 7] (or by Corollary 1.2 and [20, Theorem 5.2]), a countable basic retracing function retraces at least one set of degree $\leqq \boldsymbol{O}'$. Since $\tilde{f} \subseteqq f$, (2) is proved.

REMARK 4.6. The converse of Corollary 4.5 (2) is obviously true; in fact a considerably stronger assertion than the converse of Corollary 4.5 (2) is true, namely [20, Theorem 8] (which we obtain below as Corollary 4.19). By way of contrast, the converse of Corollary 4.5 (1) is false for the class of unique retracing functions, as we shall demonstrate further along in this section.

THEOREM 4.7. (1) *To every* $\prod_2^0$ *predicate* P *of one function variable there corresponds a general recursive retracing function* g_P *such that if* $\mathscr{G}_P$ *is the collection of principal functions retraced by* g_P *then there is a one-to-one degree-preserving correspondence* F_P: $\mathscr{F}(P) \rightarrow \mathscr{G}_P$.

(2) *If* g *is a general recursive retracing function and* $\mathscr{G}$ *is the collection of principal functions retraced by* g *then* $\mathscr{G} = \mathscr{F}(P)$ *for some* $\prod_1^0$ *predicate* P*; likewise if we omit "general recursive" and replace "*$\prod_1^0$*" by "strong* $\prod_2^0$*".*

Proof. (1) Let P be a $\prod_2^0$ predicate of functions. By Theorem 3.1 there is a predicate Q of the form $(\forall x)R(\bar{h}(x))$, R recursive, whose solutions are in one-to-one degree-preserving correspondence with those of P. Let $\Omega: \mathscr{F}(P) \rightarrow \mathscr{F}(Q)$ be such a correspondence. Suppose f is a solution of Q. Let $\alpha(f) = \{\bar{f}(x) \mid x \in N\}$. Obviously $\alpha(f)$ and f have the same degree; and $p_{\alpha(f)}$ is retraced by the general recursive function g defined as follows:

$$g(x) = \begin{cases} \bar{w}(z+1), \text{ if } (\exists w)\ [w \text{ is a finite initial function } \& \\ \quad lh(w) = z+2\ \&\ x = \bar{w}(z+2)\ \&\ (\forall y)_{y \leq z+1} R(\bar{w}(y)); \\ x, \text{ otherwise.} \end{cases}$$

Moreover, if p_β is retraced by g then β must be of the form $\{\bar{h}(x) \mid x \in N\}$ where h solves Q; so the required correspondence $F_P: \mathscr{F}(P) \rightarrow \mathscr{G}_P$ is given by $F_P(f) = p_{\alpha(\Omega(f))}$, and (1) is proved. The proof of (2) is rather obvious and will be omitted.

DEFINITION 4.8. Let a degree $\boldsymbol{D}$ and a function $f: N \rightarrow N$ be given, and let $H = \{h \mid h \in N^N\ \&\ (\forall x)(h(x) > f(x))\}$. f is *uniformly* $\boldsymbol{D}$*-major-reducible* $\Leftrightarrow$ there exists an operator Φ from partial functions to partial functions such that (i) Φ is partial recursive in $\boldsymbol{D}$ (under the definition of relatively partial recursive operators given in [16]) and (ii) $h \in H \Rightarrow \Phi(h)$ is defined and $= f$.

LEMMA 4.9. *Let* $\boldsymbol{D}$ *be a degree and* D *a predicate of one number variable such that* D *has degree* $\leq \boldsymbol{D}$.

(1) (*Relativized Kreisel-Shoenfield basis theorem.*) *If* $(\forall x)D(\bar{f}(x))$

*has a **D**-bounded solution, then it has a solution of degree* $< \boldsymbol{D}'$.

(2) *(Relativized Kuznecov-Trahtenbrot-Myhill reducibility lemma.) If* $(\forall x)D(\bar{f}(x))$ *has a unique solution* f_D, *then* f_D *is uniformly* ***D**-majorreducible.*

Proof. As Yates has noted in [20], the proof of [18, Theorems 1 and 2] relativizes without essential change to become a proof of (1). We obtain (2) as an application of König's Lemma. Suppose, then, that f_D is the unique solution of $(\forall x)D(\bar{f}(x))$ and that $(\forall n)(g(n) > f_D(n))$. If w is a finite initial function, we say that w is *g-bounded* $\Leftrightarrow g(n) > w(n)$ holds for all $n \in \delta w$. For the remainder of this proof, we use u and w as variables over the set of g-bounded finite initial functions. Let $S = \{w \mid (\exists n)[n \geqq lh(w)$ & no u of length n extending w satisfies $(\forall x)_{x<n}D(\bar{u}(x))]\}$. S is recursively enumerable in D and g (under a recursive coding of all finite initial functions) because for each n there are only finitely many u's of length n. We claim that $(\forall w)[w \notin S \Leftrightarrow w \subseteqq f_D]$. The implication from right to left is obvious. Assume $w \notin S$; then for every $n \geqq lh(w)$ there is some u of length n extending w such that $(\forall x)_{x<n}D(\bar{u}(x))$. By König's Lemma, w can therefore be extended to a total function (necessarily f_D) which satisfies $(\forall x)D(\bar{f}(x))$. Thus $w \notin S \Leftrightarrow w \subseteqq f_D$. So for each n there is exactly one w such that $lh(w) = n + 1$ & $w \notin S$. We define $w_n =$ the unique w satisfying $lh(w) = n + 1$ & $w \notin S$. Since w_n can be recursively computed from D and g simply by listing S, and since $(\forall n)[f_D(n) = w_n(n)]$, we see that f_D is recursive in D and g. Moreover, the procedure which we have indicated for reducing f_D to l.u.b. $\{\boldsymbol{D}, \boldsymbol{g}\}$ is obviously uniform in g; thus the required relatively partial recursive operator exists, and (2) is proved.

LEMMA 4.9 (2), in nonrelativized form (i.e., with $\boldsymbol{D} = \boldsymbol{O}$) and phrased in term of effective closure in Baire Space, seems to have been first noticed by Kuznecov and Trahtenbrot [9]; later Myhill [14] independently proved an equivalent theorem (see [14, p. 207]). We have included our own proof because (a) [9] apparently exists only in Russian-language synopsis form and (b) the proof which can be assembled from theorems and comments in [14] is comparatively circuitous. Lemma 4.9 (2) provides us with half of the next theorem.

THEOREM 4.10 ([9]; [14]). *The* Π_1^0 *definable functions are precisely the uniformly* ***O**-majorreducible functions.*

Proof. Taking $\boldsymbol{D} = \boldsymbol{O}$ in Lemma 4.9 (2) gives uniform $\boldsymbol{O}$-majorreducibility of Π_1^0 definable functions. The reverse inclusion is easily seen to follow from [14, Theorems 4 and 8].

REMARK 4.11 The following simple consequence of Theorem 4.10 illustrates the extent to which Lemma 4.9 depends upon domination of a solution rather than domination merely of the *range* of a solution:

THEOREM. *Every degree which contains a* $\prod_1^0$ *definable function contains a* $\prod_1^0$ *definable permutation of* N.

For the proof, let a function f of degree $\tilde{f}$ be the unique solution of a $\prod_1^0$ predicate P; we may assume that $N\text{-}\rho f$ is infinite and also (see the proof of Theorem 4.7) that f is strictly increasing. Let g be the strictly increasing function such that $\rho g = N\text{-}\rho f$. If for any two functions h and k we define $[h \oplus k](2n) = h(n)$ and $[h \oplus k](2n+1) = k(n)$, then in particular we have $f \oplus g =$ a permutation of N; moreover it is obvious that $f \oplus g \in \boldsymbol{f}$. We claim that $f \oplus g$ is uniformly $\boldsymbol{O}$-majorreducible. First,it is clear that there exists a recursive operator Φ: $N^N \rightarrow N^N$ such that if h dominates $f \oplus g$ then $\Phi(h)$ dominates f. Next, by application of Theorem 4.10 to P we see that f is uniformly $\boldsymbol{O}$-majorreducible. But $\boldsymbol{f \oplus g} \leqq \boldsymbol{f}$. Hence $f \oplus g$ is uniformly $\boldsymbol{O}$-majorreducible, and so by Theorem 4.10 $f \oplus g$ is $\prod_1^0$ definable.

We now wish to define a special class $\mathscr{H}$ of degrees. In stating our definition of $\mathscr{H}$ we shall make use of the particular hyperarithmetical sets H_γ, $\gamma \in \bigcirc$, defined by Kleene in [8]; and we shall abbreviate $\boldsymbol{H}_\gamma$ by $\boldsymbol{\gamma}$.

DEFINITION 4.12. $\mathscr{H} = \{\boldsymbol{D} \mid (\exists \gamma)(\gamma \in \bigcirc \ \& \ \boldsymbol{\gamma} \leqq \boldsymbol{D} \leqq \boldsymbol{\gamma}')\}$.

THEOREM 4.13. *If* $\boldsymbol{D} \in \mathscr{H}$, *then there exists a uniformly* $\boldsymbol{O}$-*majorreducible function of degree* $\boldsymbol{D}$.

Proof. Suppose $\boldsymbol{\gamma} \leqq \boldsymbol{D} \leqq \boldsymbol{\gamma}'$, $\gamma \in \bigcirc$. We first observe that $\boldsymbol{\gamma}$ contains a uniformly $\boldsymbol{O}$-majorreducible function f; for by [3, p. 200] $\boldsymbol{\gamma}$ contains a $\prod_2^0$ definable function and hence (by Corollary 3.2 (1)) contains a $\prod_1^0$ definable function, so that Lemma 4.9 (2) applies. Let g be a function of degree $\boldsymbol{D}$. Since $\boldsymbol{\gamma} \leqq \boldsymbol{D} \leqq \boldsymbol{\gamma}'$, it follows from the convergence theorem cited in the proof of Theorem 4.2 that there exists a two-place function $\tilde{g}$ such that $\tilde{g}$ is recursive in f and $(\forall x)[g(x) = \lim_{s \to \infty} \tilde{g}(s, x)]$. As in the proof of [11, Theorem 1.2], we define a function h by the identity

$$h(n) = \mu s (\forall x)_{x \leqq n} [\tilde{g}(s, x) = g(x)] \ .$$

Since $\boldsymbol{g} = \boldsymbol{D}$ & $\tilde{\boldsymbol{g}} \leqq \boldsymbol{f} \quad \boldsymbol{\gamma}$ & $\boldsymbol{\gamma} \leqq \boldsymbol{D}$, we have $\boldsymbol{h} \leqq \boldsymbol{g}$.We now claim that there is a partial recursive operator Φ such that if k is a function which majorizes both f and h then $\Phi(k)$ is defined and $= g$. For suppose k majorizes both f and h; i.e., suppose that

$$(\forall n)[k(n) > \max\{f(n), h(n)\}] .$$

Let $\alpha = \rho h$. By the relativized form of [15, Theorem 21] there is a function p such that (i) $\boldsymbol{p} \leqq \boldsymbol{k}$ and (ii) $\{D_{p(n)}\}_{n=0}^{\infty}$ is a disjoint sequence of finite sets each term of which has nonempty intersection with α. Moreover we may assume with no loss of generality that

$$(\forall n)(\forall x)[x \in D_{p(n)} \Rightarrow x > h(n)] .$$

We shall verify the following equivalence:

$$(*) \qquad g(x) = y \Leftrightarrow (\exists n)[n \geqq x \;\&\; (\forall s)(s \in D_{p(n)} \Rightarrow \tilde{g}(s, x) = y)] .$$

Since $\boldsymbol{p} \leqq \boldsymbol{k} \;\&\; \tilde{\boldsymbol{g}} \leqq \boldsymbol{f} \;\&\; \boldsymbol{f} \leqq \boldsymbol{k}$ (recall that f is uniformly $\boldsymbol{O}$-major-reducible), (*) provides a procedure for calculating g recursively in k; furthermore, this procedure is uniform in k since (i) f is computable uniformly from k and (ii) the construction of p from k is uniform in k (as is clear from the proof of [15, Th. 21]). Thus verification of (*) is sufficient for proving the existence of the indicated operator Φ. The $\Rightarrow$ half of (*) is obvious since $g(x) = \lim_{s\to\infty} \tilde{g}(s, x)$. For the $\Leftarrow$ half, suppose that x, y and n are such that $n \geqq x \;\&\; (\forall s)[s \in D_{p(n)} \Rightarrow \tilde{g}(s, x) = y]$. Choose a number s_0 such that $s_0 \in D_{p(n)} \cap \alpha$. Then $s_0 > h(n)$; so, since h is nondecreasing, we have $s_0 = h(u)$ for some $u \geqq x$. Therefore, by the definition of h, $\tilde{g}(s_0, x) = g(x) = y$ and the verification of (*) is complete. Now define $k_0(x) = \max\{f(x), h(x)\} + 1$. We claim that k_0 is a uniformly $\boldsymbol{O}$-majorreducible function of degree $\boldsymbol{D}$. In the first place, k_0 does indeed have degree $\boldsymbol{D}$. For since k_0 majorizes both f and h, we have $\boldsymbol{D} = \boldsymbol{g} = \Phi(\boldsymbol{k}_0) \leqq \boldsymbol{k}_0$, while on the other hand $\boldsymbol{k}_0 \geqq \boldsymbol{g}$ since $\boldsymbol{f} \leqq \boldsymbol{g} \;\&\; \boldsymbol{h} \leqq \boldsymbol{g}$. Finally, if k majorizes k_0 then $\Phi(k) = g$; so, since $\boldsymbol{k}_0 \leqq \boldsymbol{g}$, there is a partial recursive operator Ψ such that $\Psi(k) = k_0$ for any k which majorizes k_0. The proof is complete.

It is easy to strengthen Theorem 4.13 so that it applies to all those relativizations of the hyperarithmetical hierarchy which arise from uniformly $\boldsymbol{O}$-majorreducible functions: suppose that f is uniformly $\boldsymbol{O}$-majorreducible and that $\gamma \in \bigcirc^f$ (where $\bigcirc^f$ = the set of Kleene notations for ordinals recursive in f; see [8]), and let $\boldsymbol{\gamma}_f$ denote the degree of the f-hyperarithmetical set H_γ^f; then $\boldsymbol{\gamma}_f \leqq \boldsymbol{D} \leqq \boldsymbol{\gamma}'_f \Rightarrow \boldsymbol{D}$ contains a uniformly $\boldsymbol{O}$-majorreducible function. It is a consequence of this extended version of Theorem 4.13 that there exist uniformly $\boldsymbol{O}$-major-reducible functions of degree incomparable with $\boldsymbol{O}'$, so that the converse of Theorem 4.13 is false.

THEOREM 4.14. (1) *If* $\boldsymbol{D} \in \mathscr{H}$ *then* $\boldsymbol{D}$ *contains a set* α *with the properties*: (1a) p_α *is retraced by a general recursive unique retracing function*; *and* (1b) $\boldsymbol{C} \ngeq \boldsymbol{D} \Rightarrow p_\alpha$ *is not* $\boldsymbol{C}$*-bounded.*

(2) *For every* $n \geqq 2$ *there exists a* $\prod_n^0$ *predicate* P *of one number variable such that if* $\beta = \{n \mid P(n)\}$ *then* β *has the properties*: (2a) $\beta \in \boldsymbol{O}^{(n)}$; (2b) p_β *is retraced by a general recursive unique retracing function*; *and* (2c) $\boldsymbol{C} \ngeqq \boldsymbol{O}^{(n)} \Rightarrow p_\beta$ *is not* $\boldsymbol{C}$*-bounded.* ((2) *provides the answer to a question raised in* [20].)

Proof. (1a) It is clear from Theorems 4.7, 4.10 and 4.13 that if $\boldsymbol{D} \in \mathscr{H}$ then $\boldsymbol{D}$ contains a set α such that p_α is retraced by a general recursive unique retracing function.

(1b) Suppose that α is a set belonging to $\boldsymbol{D}$ with the property that p_α is retraced by a general recursive unique retracing function. Suppose further that p_α is $\boldsymbol{C}$-bounded. By Theorem 4.7, p_α is $\prod_1^0$ definable; hence (by Lemma 4.9 (2)) p_α is uniformly $\boldsymbol{O}$-majorreducible. Therefore $\boldsymbol{D} = \boldsymbol{p}_\alpha \leqq \boldsymbol{C}$.

(2a—b) Myhill has shown in [14, Th. 11] that for each $n \geqq 2$ there exists a uniformly $\boldsymbol{O}$-majorreducible function f such that the set $\alpha = \{2^x 3^y \mid f(x) = y\}$ is a complete $\prod_n^0$ set of numbers. Given such a function f, define $\beta = \{\bar{f}(n) \mid n \in N\}$. The $\prod_n^0$ expressibility of β easily follows from the $\prod_n^0$ expressibility of α; moreover, it is clear that $\boldsymbol{\alpha} \leqq \boldsymbol{\beta}$ and hence $\boldsymbol{\beta} \in \boldsymbol{O}^{(n)}$. Since f is uniformly $\boldsymbol{O}$-majorreducible, Theorem 4.10 implies that f is $\prod_1^0$ definable; but from the $\prod_1^0$ definability of f it follows as in the proof of Theorem 4.7 that p_β is retraced by a general recursive unique retracing function.

(2c) The proof here, for any β satisfying (2a—b), exactly parallels the proof of (lb).

COROLLARY 4.15. *The converse of Corollary* 4.5 (1) *is false relative to the class of unique retracing functions.*

Proof. Apply Theorem 4.14 (1) to any degree $\boldsymbol{D}$ such that $\boldsymbol{O}' < \boldsymbol{D} < \boldsymbol{O}''$.

DEFINITION 4.16 Let P be a $\prod_2^0$ normal form; say,

$$P(f) \Leftrightarrow (\forall x)(\exists y)_{y>x}(\forall z)_{z \leqq x} R(\bar{f}(z), \bar{f}(y)) .$$

By a *P-sequence* we mean a sequence $\{w_n\}_{n=0}^\infty$ of nonempty finite initial functions satisfying the following two conditions:

(i) $(\forall n)(\forall j)_{j \leqq n}[lh(w_n) > n \ \& \ R(\bar{w}_n(j), \bar{w}_n(lh(w_n))]$;

(ii) $(\forall x)(\lim_{n\to\infty} w_n(x)$ exists) .

By a *pseudosolution* of P we mean a function f for which there exists a P-sequence $\{w_n\}_{n=0}^\infty$ such that $(\forall x)(f(x) = \lim_{n\to\infty} w_n(x))$. Finally, by a *strongly countable* $\prod_2^0$ predicate of functions we mean one which is equivalent to (i.e., has the same solutions as) some $\prod_2^0$ normal form having only countably many pseudosolutions.

THEOREM 4.17. (1) *If P is a $\prod_2^0$ normal form and f is a solution of P, then f is a pseudosolution of P.*

(2) *Any strong $\prod_2^0$ predicate P of one function variable can be expressed as a $\prod_2^0$ normal form Q such that the solutions of $Q =$ the pseudosolutions of Q (and hence Q is strongly countable if P is countable).*

(3) *If P is a $\prod_2^0$ normal form, then there is a strong $\prod_2^0$ predicate Q such that the solutions of $Q =$ the pseudosolutions of P.*

Proof. We omit the routine verifications of (1) and (2). Suppose that $P(g) \Leftrightarrow (\forall x)(\exists y)_{y>x}(\forall z)_{z\leqq x}R(\bar{g}(z), \bar{g}(y))$, R recursive. If u is a sequence number, we set $L(u) = \max\{n \mid (u)_n > 0\}$; and, for any two sequence numbers u_1 and u_2 we say that u_1 *extends* u_2 provided

$$L(u_1) \geqq L(u_2) \ \& \ (\forall z)_{z\leqq L(u_2)}((u_1)_z = (u_2)_z) \ .$$

Let a predicate Q be defined as follows:

$$Q(g) \Leftrightarrow (\forall x)B(\bar{g}(x)) \ ,$$

where B is defined by

$B(w) \Leftrightarrow w$ is a sequence number & $(\exists w_0)$ [w_0 is a sequence number & w_0 extends w & ($R(w_1, w_0)$ holds for every sequence number w_1 such that w_1 is extended by w)].

Clearly, $\boldsymbol{B} \leqq \boldsymbol{O}'$; so Q is a strong $\prod_2^0$ predicate. It is straightforward to verify that Q's solutions are exactly P's pseudosolutions, completing the proof of (3).

THEOREM 4.18. *Let f be a special retracing function and P a $\prod_2^0$ normal form. Denote by $\mathscr{S}_p(P)$ the collection of principal functions which are solutions of P, by $\mathscr{P}_p(P)$ the collection of principal functions which are pseudosolutions of P, and by $\mathscr{R}$ the collection of principal functions retraced by f. Then f has a partial recursive subfunction $\tilde{f}$ such that*

$$\mathscr{R} \cap \mathscr{S}_p(P) \subseteqq \tilde{\mathscr{R}} \subseteqq \mathscr{R} \cap \mathscr{P}_p(P)$$

where $\tilde{\mathscr{R}}$ is the collection of principal functions retraced by $\tilde{f}$. (Since for every e "φ_e retraces f" is a strong $\prod_2^0$ predicate of f, it follows from Theorem 4.17 (2) and Theorem 4.23 that the inclusion $\mathscr{R} \cap \mathscr{S}_p(P) \subseteqq \tilde{\mathscr{R}}$ cannot in general be replaced by equality.)

Proof. Suppose $P(g) \Leftrightarrow (\forall x)(\exists y)_{y>x}(\forall z)_{z\leqq x}R(\bar{g}(z), \bar{g}(y))$. Let w_0, w_1,

$w_2 \cdots$ be a fixed recursive enumeration of all nonempty finite initial functions. We define a partial recursive subfunction $\tilde{f}$ of f, as follows:

$$\tilde{f}(x) = u \Leftrightarrow f(x) = u \;\&\; (\exists n)[w_n(lh(w_n)) \in \delta f \;\&\; x \in \hat{f}(w_n(lh(w_n)))$$

$$\& \;(w_n \text{ extends the finite initial function which enumerates } \{y \mid y \leqq x\} \cap \hat{f}(w_n(lh(w_n))) \text{ in order of magnitude}) \;\&$$

$$(\forall z)_{z \leqq f^*(x)} R(\bar{w}_n(z), \bar{w}_n(lh(w_n)))] \,.$$

We claim that $\tilde{f}$ meets the requirements of the theorem. Suppose first that β is a set such that $P(p_\beta)$ holds and p_β is retraced by f. We wish to show that $\tilde{f}(p_\beta(n))$ is defined for all n. Now, $f^*(p_\beta(n)) = n$. Let w_t be a finite initial function such that

$$w_t(lh(w_t)) \in \beta \;\&\; p_\beta(n) \in \hat{f}(w_t(lh(w_t))) \;\&$$

$[w_t$ extends the finite initial function which enumerates

$$\hat{f}(w_t(lh(w_t))) \cap \{y \mid y \leqq p_\beta(n)\}$$

in order of magnitude] & $(\forall z)_{z \leqq n} R(\bar{w}_t(z), \bar{w}_t(lh(w_t)))$; such a w_t certainly exists since p_β is a solution of P. In view of the stipulated properties of w_t, the condition for setting $\tilde{f}(p_\beta(n)) = f(p_\beta(n))$ is met; hence $p_\beta(n) \in \delta\tilde{f}$. So we have $p_\beta \in \tilde{\mathscr{R}}$. For the remaining inclusion, suppose that p_β is a principal function retraced by $\tilde{f}$. We wish to show that $p_\beta \in \mathscr{P}_p(P)$. This means that we must define a P-sequence $\{w_{n_j}\}_{j=0}^{\infty}$ such that $(\forall x)(p_\beta(x) = \lim_{j \to \infty} w_{n_j}(x))$. As w_{n_0} we may take any w_n satisfying the defining condition for $\tilde{f}(p_\beta(0)) = p_\beta(0)$. Suppose that $w_{n_0}, \cdots, w_{n_j}$ have been defined; and assume, as part of the inductive hypothesis, that, for $0 \leqq i \leqq j$, we have

$$w_{n_i}(lh(w_{n_i})) \in \delta f \;\&\; p_\beta(i) \in \hat{f}(w_{n_i}(lh(w_{n_i}))) \,.$$

Since $\tilde{f}(p_\beta(k))$ is defined for all k, there must exist a finite initial function w_t with the following properties: $w_t(lh(w_t)) \in \delta f$; $\{y \mid y \in \beta \;\&\; y \leqq p_\beta(j+1)\} = \hat{f}(w_t(lh(w_t))) \cap \{y \mid y \leqq p_\beta(j+1)\}$; w_t extends the finite initial function which enumerates $\hat{f}(w_t(lh(w_t))) \cap \{y \mid y \leqq p_\beta(j+1)\}$ in order of magnitude; and $(\forall z)_{z \leqq j+1} R(\bar{w}_t(z), \bar{w}_t(lh(w_t)))$. Let $w_{n_{j+1}}$ be the first such w_t. Clearly, the sequence $w_{n_0}, w_{n_1}, \cdots$ defined inductively in this way has the property: $(\forall x)(\exists j)(\forall k)[k \geqq j \Rightarrow w_{n_k}(x)$ and $w_{n_j}(x)$ are defined and are both equal to $p_\beta(x)]$. Moreover it is clear that $\{w_{n_j}\}_{j=0}^{\infty}$ is a P-sequence. Thus $p_\beta \in \mathscr{P}_p(P)$, and the proof is complete.

We now exhibit [20, Th. 8] as an application of Theorems 4.2, 4.17 (2) and 4.18.

COROLLARY 4.19 (Yates). *Let f be a retracing function, and p_α*

a principal function of degree $\leqq \boldsymbol{O}'$ *such that* f *retraces* p_α. *Then* p_α *is retraced by a basic unique retracing function* $\tilde{f}$ *such that* $\tilde{f} \subseteqq f$.

Proof. It is easily seen that since p_α has degree $\leqq \boldsymbol{O}'$ it is the unique solution of some *strong* Π_2^0 predicate P. Hence, by Theorem 4.17 (2), we can conclude from Theorem 4.18 that p_α is retraced by some unique retracing function g such that $g \subseteqq f$. But by Theorem 4.2, any such function g must have a subfunction $\tilde{f}$ such that $\tilde{f}$ is a basic retracing function which retraces p_α. (The portion of Corollary 4.19 which asserts that $\tilde{f} \subseteqq f$ can, of course, be obtained simply by intersecting f with *any* basic unique retracing function h such that h retraces p_α.)

THEOREM 4.20. (1) *If a countable* Π_1^0 *predicate (countable strong* Π_2^0 *predicate) of functions has a* $\boldsymbol{O}$*-bounded (*$\boldsymbol{O}'$*-bounded) solution, then it has a recursive solution (a solution of degree* $\leqq \boldsymbol{O}'$*).*

(2) *If a basic retracing function* f *retraces* p_α *and* p_α *solves a countable strong* Π_2^0 *predicate, then* f *retraces at least one principal function of degree* $\leqq \boldsymbol{O}'$.

Proof. (1) Let P be a countable strong Π_2^0 predicate of functions; and let h be a function, recursive in $\boldsymbol{O}'$, such that h bounds some solution of P. Then the predicate

$$Q(f) \Leftrightarrow [P(f) \ \& \ (\forall n)(f(n) < h(n))]$$

has at least one solution and not more than $\aleph_0$; moreover, all solutions of Q are $\boldsymbol{O}'$-bounded. By Corollary 1.2 some solution f_0 of Q is *the unique* solution of a strong Π_2^0 predicate Q^*. Since f_0 is $\boldsymbol{O}'$-bounded, it is recursive in $\boldsymbol{O}'$ by Lemma 4.9(2) applied to Q^*. The argument for Π_1^0 predicates is similar.

(2) follows from (1), Theorem 4.7 (2), and the fact that the conjunction of two strong Π_2^0 predicates is strong Π_2^0; for if p_α is retraced by a basic retracing function then p_α is $\boldsymbol{O}'$-bounded. (Alternatively one can apply Theorem 4.17 (2), Theorem 4.18 and [12, Th. 7].)

The following lemma is implied by an elaborated version of [13, Th. 1] to be published elsewhere; we shall therefore confine ourselves to giving a brief informal account of its proof.

LEMMA 4.21. *Let* f *be a basic retracing function. There exists a basic retracing function* g_f *such that*

(i) g_f *retraces* $p_\alpha \Rightarrow (\exists \beta)$ (f *retraces* p_β & $\alpha \subseteqq \beta$)

and

(ii) g_f *retraces* $p_\alpha \Rightarrow \boldsymbol{\alpha} \geqq \boldsymbol{O}'$.

Proof (in outline). We construct g_f from f by a straightforward priority scheme. For each n, let $H_n = \{x \mid x \in \delta f \ \& \ f^*(x) = n\}$. Since f is basic, $\{H_n\}_{n=0}^{\infty}$ is a recursive sequence of disjoint finite sets. We add pairs (x, y) to g_f, adding only finitely many pairs at any given stage of the construction, in such a way that

$$(\exists m)(\exists q)[x \in H_m \ \& \ y \in H_q \ \& \ (m = q = 0 \quad \text{or} \quad m > q) \ \& \ y \in \hat{f}(x)] ;$$

moreover, if (x, y) is added to g_f at stage s, and if $x \in H_t$, then subsequently we add (z, y) to g_f provided $z \in H_t$, *unless* x is "injured" at some point after stage s. If x *is* injured after stage s, we then fix upon a new set $H_{t'}$ from which to draw g_f-preimages of y. But the construction is so arranged that only finitely many g_f-preimages of a given y are ever injured, and so g_f turns out to be a finite-to-one function satisfying (i). A number x is said to be *injured* at stage s of the construction if by stage s we have (1) $(x, y) \in g_f$ for some y and (2) $(\exists e)_{e \leqq g_f^*(x)}(\exists t)_{t \leqq g_f^*(x)}(\exists u)[(t, u) \in \varphi_e^s \ \& \ u \geqq x]$. Once a number is injured, we eventually get around to killing off (i.e., halting at a finite level) all potential solutions of the predicate "g_f retraces p_α" which pass through the injured number. Thus every surviving infinite branch in the graph of g_f must dominate (eventually) any given partial recursive function. As is well known, this implies that all surviving infinite branches have degree $\geqq \boldsymbol{O}'$, so (ii) is also satisfied.

A major part of our next theorem was established by Yates in [20], namely: there exists a basic retracing function f_0 such that f_0 retraces no function of degree $\leqq \boldsymbol{O}'$. We shall include our own proof of Yates' theorem as part of the proof of Theorem 4.22. It seems to us that our argument is a little more straightforward than the argument in [20]; however, it should be noted that in [20] Yates proved directly the (equivalent) theorem stating that there exists a basic retracing function which retraces *no* $\prod_2^0$ *set of numbers*.

THEOREM 4.22. *There exists a general recursive, basic retracing function f such that* $(\forall \alpha)$ *(f retraces $p_\alpha \Rightarrow \boldsymbol{\alpha} > \boldsymbol{O}'$).*

Proof. We first show the existence of a function f_0 as in the remarks preceding the statement of the theorem. We begin by defining a three-place partial recursive function Ψ (with recursive domain) as follows:

$$\Psi(e, x, s) \cong \begin{cases} \varphi_e^{2,s}(\max\{t \mid \varphi_e^{2,s}(t, x) \text{ is defined}\}, x), \text{ if} \\ \quad \{t \mid \varphi_e^{2,s}(t, x) \text{ is defined}\} \neq \emptyset \\ \text{undefined, otherwise .} \end{cases}$$

Now let f_1 be the function defined by $f_1^{-1}(x) = \{2x, 2x + 1\}$, and define $\tilde{f}$ from f_1 by the following equivalence:

$$\tilde{f}(x) = y \Leftrightarrow f_1(x) = y \;\; \& \;\; (\forall z)_{z \in \hat{f}_1(x)} (\exists m)_{m \leqq z} (\exists s)_{s \geqq x} \text{ [either } \Psi(f_1^*(z), m, s) \text{ is undefined or } \Psi(f_1^*(z), m, s) \neq c_{\hat{f}_1(z)}(m)].$$

It is immediately clear that $\tilde{f}$ is a partial recursive subfunction of f such that $\rho\tilde{f} \subset \delta\tilde{f}$, whence $\tilde{f}$ is special. Next, it is easy to see that $\tilde{f}$ can retrace no function of degree $\leqq \boldsymbol{O}'$. For suppose f retraces p_β and p_β has degree $\leqq \boldsymbol{O}'$. Then there must be a two-place recursive function φ_e^2 such that, for every z, $\lim_{s \to \infty} \varphi_e^2(s, z)$ exists and $= c_\beta(z)$. Let b be that element of β such that $f^*(b) = e$. Let x be an element of β such that $x > b$ & $x \geqq u \geqq w$, where u and w are numbers such that $d \leqq b \Rightarrow \varphi_e^{2,u}(w, d)$ is defined and $(\forall r)(r \geqq w$ & $d \leqq b \Rightarrow \varphi_e^2(r, d) = \lim_{s \to \infty} \varphi_e^2(s, d))$. Then, clearly, we cannot include x in $\delta\tilde{f}$; thus β is not retraced by $\tilde{f}$. It remains to prove that $\tilde{f}$ does retrace some set. We show how to define a strictly increasing sequence $\{r_i\}_{i=0}^\infty$ so that $\tilde{f}$ retraces the range of $\{r_i\}_{i=0}^\infty$. Let r_0 be any fixed point of f; since $f_1^*(r_0) = 0$, it follows from our convention that φ_0^2 is the empty function (§ 1) that there are infinitely many s for which $\Psi(f_1^*(r_0), r_0, s)$ is undefined. Now suppose $r_0, \cdots, r_l$ have been defined in such a way that $r_0 < \cdots < r_l$ (if $l > 0$), $l \geqq j > 0 \Rightarrow f(r_j) = r_{j-1}$, and, for $0 \leqq j \leqq l$, there exist $m_j \leqq r_j$ and infinitely many s such that either $\Psi(j, m_j, s)$ is undefined or $\Psi(j, m_j, s) \neq c_{\hat{f}_1(r_j)}(m_j)$. Let q_0, q_1 be the two numbers q such that $f_1(q) = r_l$. Because of the inductive hypotheses concerning $r_0, \cdots, r_l$, it suffices to show that either $(\exists m \leqq q_0)$[for infinitely many s either $\Psi(l + 1, m, s)$ is undefined or $\Psi(l + 1, m, s) \neq c_{\hat{f}_1(q_0)}(m)$] or $(\exists m \leqq q_1)$[for infinitely many s either $\Psi(l + 1, m, s)$ is undefined or $\Psi(l + 1, m, s) \neq c_{\hat{f}_1(q_1)}(m)$]. But suppose, e.g., that $q_0 > q_1$; then the only alternative to the validity of at least one of the above existential statements is to have both

$$\lim_{s \to \infty} \Psi(l + 1, q_1, s) = 0 \quad \text{and} \quad \lim_{s \to \infty} \Psi(l + 1, q_1, s) = 1 ,$$

an obvious impossibility. Similarly if $q_1 > q_0$. Thus, we can continue the induction from l to $l + 1$, and the existence of the required sequence $\{r_i\}_{i=0}^\infty$ follows. (Indeed, it is not difficult to show that—as also in Yates' proof—there is a surviving branch of every degree $\geqq \boldsymbol{O}''$.) Thus $\tilde{f}$ serves as f_0. Notice that every set retraced by $\tilde{f}$ is $\boldsymbol{O}$-bounded. (This is also a feature of Yates' construction.) To obtain the theorem as stated, we must (in view of Lemma 4.9 (1)) sacrifice the $\boldsymbol{O}$-boundedness of the solutions. Let $g_{\tilde{f}}$ be related to $\tilde{f}$ as in Lemma 4.21. By Lemma 4.21 (i) and the fact that retraceable sets are introreducible ([2]), for every set β retraced by $g_{\tilde{f}}$ there is a set β_0 retraced by $\tilde{f}$ such that $\boldsymbol{\beta} \geqq \boldsymbol{\beta}_0$; while by Lemma 4.21 (ii) every set retraced by $g_{\tilde{f}}$ has degree $\geqq \boldsymbol{O}'$. But no set retraced by $\tilde{f}$ has degree $\leqq \boldsymbol{O}'$; hence every set α retraced by $g_{\tilde{f}}$ satisfies $\boldsymbol{\alpha} > \boldsymbol{O}'$. Now by applying two successive recursive equivalences, the first one being onto N and the second being as in the proof of [1, Proposition 5 (b)], we obtain a retracing function h such that (a) the graph of h is recursively equivalent to that of $g_{\tilde{f}}$ and (b) δh is recursive. Hence there exists a basic retracing function $\tilde{h}$ such that $\delta\tilde{h} = N$ and the graph of $\tilde{h}$ is recursively equivalent to that of $g_{\tilde{f}}$. Then each set retraced by $\tilde{h}$ is recursively equivalent to one retraced by $g_{\tilde{f}}$. But any two recursively equivalent retraceable sets have the same degree, so we may take $f = \tilde{h}$.

THEOREM 4.23. *There exists a degree* $\boldsymbol{C}$ *strictly between* $\boldsymbol{O}'$ *and* $\boldsymbol{O}''$ *such that* $[\boldsymbol{D} \geqq \boldsymbol{C}$ *&* $\boldsymbol{D}$ *contains a* $\prod_1^0$ *definable function*$] \Rightarrow$ [*there exists a* $\prod_2^0$ *normal form* $P_{\boldsymbol{D}}$ *with the properties*:

(i) $P_{\boldsymbol{D}}$ *has a unique solution, call it* $f_{\boldsymbol{D}}$;

(ii) $f_{\boldsymbol{D}} \in \boldsymbol{D}$;

(iii) $f_{\boldsymbol{D}}$ *is retraced by a general recursive retracing function*;

(iv) *any* $\prod_2^0$ *normal form having* $f_{\boldsymbol{D}}$ *as a solution has* $2^{\aleph_0}$ *pseudosolutions*.]

(In particular, by Theorems 4.10 and 4.13, (i)−(iv) hold for any $\boldsymbol{D} \geqq \boldsymbol{C}$ such that $\boldsymbol{D} \in \mathscr{H}$.)

Proof. Let f be as in Theorem 4.22. Then [20, Th. 2] implies that f retraces at least one set α such that $\boldsymbol{O}' < \boldsymbol{\alpha} < \boldsymbol{O}''$. Let α_0 be one particular such set. Let g be a general recursive basic retracing function which retraces at least one set from each degree; e.g., we can take g to be the function f_1 defined by $f_1^{-1}(x) = \{2x, 2x+1\}$. Let $\boldsymbol{D}$ be a Turing degree $\geqq \boldsymbol{\alpha}_0$; and let γ_0 be a particular set of degree $\boldsymbol{D}$ such that g retraces γ_0. By [2, Proposition P4] there exists a retracing function h which retraces the range of the function $p_{\alpha_0}(p_{\gamma_0}(x))$. Moreover, a close look at the proof of [2, Proposition P4] shows that we can demand of h that it be general recursive and basic and retrace only sets which are of the form $\rho[p_\alpha[p_\gamma(x))]$ where f retraces α and g retraces γ. Since $\boldsymbol{\gamma}_0 \geqq \boldsymbol{\alpha}_0$ and α_0 is introreducible, we see that the range, β, of $p_{\alpha_0}(p_{\gamma_0}(x))$ is a set of degree $\boldsymbol{\gamma}_0(= \boldsymbol{D})$. Suppose there exists a strongly countable $\prod_2^0$ normal form P such that $P(p_\beta)$. Then, by Theorem 4.17 (3) and Theorem 4.20 (2), h retraces at least one set, say π, of degree $\leqq \boldsymbol{O}'$. But $\pi = \rho[p_\alpha(p_\gamma(x))]$ where f retraces α and g retraces γ; so, since α is introreducible, we have that $\boldsymbol{\alpha} \leqq \boldsymbol{\pi} \leqq \boldsymbol{O}'$. This, however, contradicts the properties of f. Thus p_β (i.e., $p_{\alpha_0}(p_{\gamma_0}(x))$) can satisfy no strongly countable $\prod_2^0$ normal form. Suppose $\boldsymbol{D}$ contains a function k such that k is the unique solution of a $\prod_1^0$ predicate. Then by Corollary 3.2 (3) $\boldsymbol{D}$ contains only functions which are Π_2^0 definable. So let P_0 be a Π_2^0 normal form such that p_β is the unique solution of P_0. If Q is any Π_2^0 normal form such that p_β solves Q, then Q has uncountably many pseudosolutions. But by Theorem 4.17 (3) the set of all pseudosolutions of a Π_2^0 normal form is closed in Baire Space; hence the pseudosolutions of Q are $2^{\aleph_0}$ in number and we may take $P_{\boldsymbol{D}} = P_0, f_{\boldsymbol{D}} = p_\beta$.

REMARK 4.24. The functions $\beta_{\boldsymbol{D}}$ obtained in the above proof of Theorem 4.23 are not $\boldsymbol{O}$-bounded. However, by using the analogue for strong Π_2^0 predicates of Theorem 5.1 below, we can obtain Theorem 4.23 with the functions $\beta_{\boldsymbol{D}}$ $\boldsymbol{O}$-bounded. In fact, an alternative proof of Theorem 4.23 can be given in which instead of Theorem 4.22 we use (a) the strong Π_2^0 analogue of Theorem 5.1 and (b) the fact (obtained by a minor variation on the proof of Theorem 3.3) that for any degree $\boldsymbol{D}$, there exists a degree $\boldsymbol{C}$ such that $\boldsymbol{D} < \boldsymbol{C} < \boldsymbol{D}'$ and some function belonging to $\boldsymbol{C}$ has the property of not satisfying any countable predicate of the form $(\forall x)D(\bar{f}(x))$ where D has degree $\leqq \boldsymbol{D}$. If question Q3 at the end of the paper has an affirmative answer, then the range of degrees $\boldsymbol{D}$ in Theorem 4.23 can be extended to cover precisely all $\boldsymbol{D} \nleqq \boldsymbol{O}'$ which contain Π_1^0 definable functions.

5. In this section a function f will be called *countably* Π_1^0 if f satisfies some countable Π_1^0 predicate. A set α will be called *countably* Π_1^0 (Π_1^0 *definable*) if p_α is countably Π_1^0 (Π_1^0 definable.) If α is nonrecursive and Π_1^0 definable, then it follows immediately from Theorem 4.10 and [5, Corollary 3.4] that N-α cannot be Π_1^0 definable. The countably Π_1^0 sets differ radically in this respect from the Π_1^0 definable sets, as the following theorem shows.

THEOREM 5.1. *If α is countably Π_1^0 and β is equivalent to α via (unbounded) truth tables, then β is countably Π_1^0.*

Proof. We first prove a lemma which shows that we may replace principal functions by characteristic functions.

LEMMA 5.2. *If γ is an infinite set, then p_γ is countably Π_1^0 if and only if c_γ is countably Π_1^0.*

Proof. Assume p_γ is among the countably many solutions of $(\forall x)R(\bar{f}(x))$, R recursive. In this proof we use w as a variable for strictly increasing finite initial functions. Define a new Π_1^0 predicate $Q(f)$ by

$$(\forall x)[f(x) \in \{0, 1\}] \ \& \ (\forall w)(\forall y)[\rho w = \{x \mid x \leqq y \ \& \ f(x) = 1\} \\ \Longrightarrow R(\bar{w}(lh(w)))] \ .$$

Clearly, $Q(c_\gamma)$ holds. Also, whenever $Q(f)$ holds, then f is the characteristic function of a set δ such that either δ is finite or p_γ satisfies $(\forall x)R(\bar{f}(x))$. Hence Q is countable, so c_γ is countably $\prod_1^0$. The proof of the converse is similar.

The proof of the theorem is similar to the proof of Corollary 3.2 (3). Assume $\alpha \equiv_{tt}\beta$. Then by a theorem of Nerode [16, p. 250], there exist numbers e_0, e_1 such that $\{e_0\}^{c_\alpha} = c_\beta$, $\{e_1\}^{c_\beta} = c_\alpha$, and for every total function h the functions $\{e_0\}^h$ and $\{e_1\}^h$ are total. Assume also that c_α is among the countably many solutions of $(\forall x)R(\bar{g}(x))$, R recursive. Consider the following predicate $Q(h)$:

$$(\forall x)R(\overline{\{e_1\}^h}(x)) \ \& \ \{e_0\}^{\{e_1\}^h} = h \ .$$

$Q(h)$ can be written as a $\prod_1^0$ predicate because all the functions mentioned in it are total. Clearly $Q(c_\beta)$ holds. Now any function h such that $Q(h)$ holds has the same degree as $\{e_1\}^h$, where $\{e_1\}^h$ is a solution of the countable predicate $(\forall x)R(\bar{g}(x))$. Thus Q is countable, so c_β is countably $\prod_1^0$. The theorem now follows from the lemma.

Since (by Lemma 4.9 (1)) nonrecursive $\prod_1^0$ definable sets are not $\boldsymbol{O}$-bounded, the following theorem demonstrates the existence of a variety of sets which are countably $\prod_1^0$ but not $\prod_1^0$ definable.

THEOREM 5.3. *If $\boldsymbol{D}$ contains a $\prod_1^0$ definable set then $\boldsymbol{D}$ contains a $\boldsymbol{O}$-bounded set β such that β is countably $\prod_1^0$. (Hence, in particular, $\boldsymbol{D}$ contains such a set β provided $\boldsymbol{D} \in \mathscr{H}$; a similar remark applies to Theorem 5.4 below.) If $\boldsymbol{D}$ is a recursively enumerable degree then $\boldsymbol{D}$ contains a recursively enumerable set α such that α is countably $\prod_1^0$.*

Proof. Suppose $\alpha \in \boldsymbol{D}$, $\alpha \neq \emptyset$, and α is $\prod_1^0$ definable. Let $\beta = \{2^x 3^y \mid x \in \alpha \ \& \ y \in N\}$. Then β is truth-table equivalent to α; hence, by Theorem 5.1, β is countably $\prod_1^0$. Obviously, β has an infinite recursive subset and is therefore $\boldsymbol{O}$-bounded. If $\boldsymbol{D}$ is recursively enumerable then by [19, Th. 2] $\boldsymbol{D}$ contains a recursively enumerable set α such that N-α is retraced by a general recursive unique retracing function. By Theorem 4.7 (2) N-α is $\prod_1^0$ definable. Hence α is countably $\prod_1^0$ by Theorem 5.1.

THEOREM 5.4. *Let $\boldsymbol{D}$ be a degree containing a $\prod_1^0$ definable set and such that $\boldsymbol{D} \nleqq \boldsymbol{O}'$. Then $\boldsymbol{D}$ contains a set α such that p_α is retraced by a general recursive, basic, countable retracing function but p_α does not satisfy any unique strong $\prod_2^0$ predicate (and hence,* *in particular, p_α is not retraced by any unique retracing function).*

Proof. Assume $\boldsymbol{D} \nleqq \boldsymbol{O}'$ and $\boldsymbol{D}$ contains a Π_1^0 definable set. By Theorem 5.3 there is a $\boldsymbol{O}$-bounded, countably Π_1^0 set β of degree $\boldsymbol{D}$. Let $\alpha = \{\bar{p}_\beta(n) \mid n \in N\}$. p_α is $\boldsymbol{O}$-bounded since β is $\boldsymbol{O}$-bounded. By the proof of Theorem 4.7 (1), p_α is retraced by a general recursive, countable retracing function f. Since p_α is $\boldsymbol{O}$-bounded, it follows by a trivial adjustment of the proof of Theorem 4.2 that f has a basic retracing subfunction $\tilde{f}$ such that $\tilde{f}$ retraces p_α and $\delta\tilde{f}$ is recursive. Hence there is a general recursive, basic, countable retracing function h such that h retraces p_α. Let P be a unique strong Π_2^0 predicate. Then by Lemma 4.9 (2) we have that $P(p_\alpha) \Rightarrow \boldsymbol{\alpha} \leqq \boldsymbol{O}'$; therefore, since $\boldsymbol{O}' \ngeqq \boldsymbol{D}$ and $\boldsymbol{D} = \boldsymbol{\alpha}$, we conclude that $\neg P(p_\alpha)$. (If we examine carefully the proof of Theorem 5.1 we see that Theorem 5.4 can be proved subject to the added condition that all functions *other than* p_α which are retraced by h are recursive.)

The sets which we have thus far shown to be countably Π_1^0 but not Π_1^0 definable are all $\boldsymbol{O}$-bounded; and indeed, the proof that these sets are not Π_1^0 definable is precisely that they are $\boldsymbol{O}$-bounded but not recursive. However, our last theorem provides examples which are not $\boldsymbol{O}$-bounded.

THEOREM 5.5. *$\boldsymbol{O} < \boldsymbol{D} \leqq \boldsymbol{O}' \Rightarrow \boldsymbol{D}$ contains a set α which is countably Π_1^0 but is neither Π_1^0 definable nor $\boldsymbol{O}$-bounded.*

Proof. If $\boldsymbol{O} < \boldsymbol{D} \leqq \boldsymbol{O}'$, then by [4, Theorems 4.2 and 5.2] $\boldsymbol{D}$ contains a set α such that α is semirecursive, splits every infinite recursive set, and is not $\boldsymbol{O}$-bounded. (A *semirecursive* set is a set β for which there exists a general recursive function $f(x, y)$—called a *selector function* for β—such that $(\forall x)(\forall y)[f(x, y) \in \{x, y\}$ & $((x \in \beta$ or $y \in \beta) \Rightarrow f(x, y) \in \beta)]$.) If $f(x, y)$ is a selector function for a semirecursive set β and β splits every infinite recursive set, then every set $\neq \beta$ for which $f(x, y)$ is also a selector function is either finite or cofinite. From this it follows that every such β—and hence in particular our set α—is countably Π_1^0; we omit details. It is clear from [5, Corollary 5.4] and the proof of [5, Th. 5.2] that α cannot be introreducible and hence (Th. 4.10) cannot be Π_1^0 definable.

Among the many questions relating to this paper which we have so far been unable to answer, the following three strike us as being of greatest interest:

Q1. Must a function which satisfies a countable Π_2^0 normal form be $\prod_2^0$ definable? We forcefully conjecture a negative answer, and remark that the negative answer to the corresponding question for the class of *strong* $\prod_2^0$ predicates is contained in Theorem 5.4.

Q2. Does there exist a set α, recursively enumerable in $\boldsymbol{O}'$, such that p_α satisfies no countable $\prod_2^0$ predicate of functions? (or, even, fails to be $\prod_2^0$ definable?)

Q3. Is it the case that if $\boldsymbol{D}$ and $\boldsymbol{C}$ are degrees satisfying $\boldsymbol{C} \nleq \boldsymbol{D}$ then $\boldsymbol{C}$ contains a set β such that p_β solves no countable predicate of the form $(\forall x)D(\bar{f}(x))$ where D is of degree $\leqq \boldsymbol{D}$? It seems very plausible to us that this is true.

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