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ON CONTINUOUS MAPPINGS OF METACOMPACT ČECH COMPLETE SPACES

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Under what may be thought of as a guise of a description of pathology are indicated here certain ways in which Čech completeness, Arhangel'skiĭ's p-space concept, and metacompactness enlarge on the respective concepts of metric absolute G_δ's, metrizability, and paracompactness. This is done through examination of certain aspects of the theory of multivalued mappings. It is taken as a point of orientation that the topic of Tychonoff locally bicompact spaces has a substantial mathematical interest. It is assumed obvious that such spaces are locally paracompact p-spaces. An underlying point of view is that the class of regular locally paracompact p-spaces extends along natural lines the class of regular locally metrizable spaces.

Let us observe these theorems: (1) A Hausdorff space is paracompact if and only if it is fully normal [13]. (2) A space is metacompact if and only if for every collection G of open sets covering it there exists a collection H of open sets covering it such that if P is a point, the collection of all members of H containing P refines a finite subcollection K of G [22]. (3) A T_1 space is metrizable if and only if it is fully normal and has a base of countable order (cf. definitions below) [3]. (4) A T_1 space has a uniform base (cf. definitions below) if and only if it is metacompact and has a base of countable order [27]. (5) Metacompactness is invariant under the action on a topological space of a closed continuous mapping [23]. We may then see that whether or not a metacompact T_1 topological space S has a perfect mapping onto a space with a uniform base depends only on whether S has a perfect mapping onto a space having a base of countable order. Similarly, since full normalcy of a topological space is also an invariant under the action of a closed continuous mapping [10], whether a fully normal T_1 space S has a perfect mapping onto a metrizable space depends only on whether S has a perfect mapping onto a space having a base of countable order. These reductions achieve heightened interest in view of the invariance of the base of countable order property under the actions of peripherally bicompact closed continuous mappings on T_1 spaces [21], the intimacy of its relation to the topic of interior mappings [16, 19], and certain work of Frolík and Arhangel'skiĭ which will now be described.

Frolík showed that a paracompact Hausdorff space is Čech complete (cf. definition below) if and only if it has a perfect mapping onto a complete metric space [7]. This fundamental contribution was enlarged by Arhangel'skiĭ, first of all by his exercise of an extraordinary ingenuity in isolating the concept of a p-space (cf. definition below), and secondly by his equating for the Hausdorff paracompact cases the property of being a p-space with the admitting of a perfect mapping onto a metrizable space [4]. Since Čech completeness is preserved under the action of a perfect mapping between Tychonoff spaces [8], we may now see that both Frolík's theorem and Arhangel'skiĭ's theorem may be interpreted as having in common the remarkable feature of pivoting on the existence of perfect mappings of the respective (paracompact) spaces onto spaces having bases of countable order.

We now observe that the first four theorems reviewed above put one in a position to interpret that for spaces satisfying the first Trennungsaxiom and having uniform bases, the distinction metrizable-nonmetrizable corresponds precisely to the distinction unity-finitude of order >1 of certain refined collections K. Can it be the case, one may ask with this preparation, that nevertheless there exists a metacompact Čech complete space admitting of no perfect mapping onto an absolute G_δ space with a uniform base or, equivalently (in view of the above theorems), onto a space having a base of countable order? An answer in the negative might be suspected to have profound structural implications.

Let us look at this in another way. H. H. Wicke and the author proposed at the last meeting of the International Congress of Mathematicians in Moscow the thesis that the base of countable order concept, especially when enriched by an appropriate notion of completeability, express substantially much of what is topologically fundamental in the concept of metrizability [18]. If this be valid, then heuristically one might conclude that either every Tychonoff p-space has a perfect mapping onto a space having a base of countable order or else there exists a metacompact Tychonoff p-space which cannot be so transformed. If the reader but put himself in a frame of mind receptive to this line of reasoning, he may sense a heuristic justification for the either/or conclusion. For radical as it may seem in the contemporary milieu the thesis carries with it the corollary that paracompactness-like properties are not naively a part of the essential content of metrizability. (In this connection, see [14], [17], [20], [24], [25].)

Insofar as technique and exposition can be distinguished in a work of this kind, the technical portion of this mémoire will be devoted to the demonstration of the existence of a first countable, regular T_0 locally bicompact, screenable, metacompact space S of power c which admits of no Lindelöfian continuous mapping whatsoever onto a Hausdorff space having a base of countable order. It follows that

S has no perfect mapping onto a space having a base of countable order.

Let us inquire now into the significance of the requirement of first countability of the example. One may say that owing to the definitional sacrifice by any such example of the uniform first countability of the base of countable order property any further interest in first countability is eclipsed to at best a peripheral position of interest. Therefore the underlying issue likely has to do with whether first countability in itself must in certain situations go a long way toward uniformity in this sense. This is in fact the point here. There exist Čech complete spaces Σ of ordinal numbers with respect to the order topology which have no perfect mappings onto spaces with bases of countable order. But every first countable subspace of such a Σ has a base of countable order since it contains no dense subspace [20]. The significance of this is reinforced by the behavior of the base of countable order property under the action of Cartesian products [15] and its hereditary character [27].

One might pursue the significance of the existence of such examples in considerable additional detail. Why the emphasis on the power of the space? Why the reference to σ-discrete refinements? Why the specific mention of screenability? Certain of these questions bear on the intimacy of the interplay between the topics of bases of countable order and paracompactness-like properties [22]. Suffice it here to say that it was felt virtually obligatory to resolve the above question prior to stating the general thesis [28, cf. also 17].

2. Definitions and notation.

Except that the null set convention is not employed, general terminology usually follows along the lines of [9]. As a technical point it is noted that *compactness* is taken in the Fréchet sense, though in the present context one might apply the theorem of [2] that T_1 compact metacompact spaces are bicompact. For *screenable space* and *development of a space*, see [5]; for *metacompact space*, see [2]. As in [11], if K is a collection of sets, K^* denotes the sum of the elements of K. If M is a point set, $\bar{M}$ denotes the (contextually implicit) closure of M. By an *arc* will be understood a bicompact connected Hausdorff point set having exactly two noncut points. By an endpoint of an arc α is meant a noncut point of α. A *perfect mapping* is a bicompact, closed continuous mapping. A *uniform base* for a space S is a base B for S such that if B' is an infinite subcollection of B and P belongs to all members of B', then B' is a base for S at P [1]. Note that a developable space has a uniform base if and only if it is metacompact [1].

A space is *Čech complete* if and only if it is a Tychonoff space S the set of all points of which is an inner limiting set in a Stone-Čech bicompactification $\beta(S)$ [6]. It follows that the set of all points of S is an inner limiting set of any Hausdorff space in which S can be densely embedded [6]. Note that all regular T_0 locally bicompact spaces are Čech complete. A T_1 space S is a *p-space* if and only if it is covered by each term of a sequence $G_1, G_2, \cdots$ of collections of open sets of a Wallman bicompactification ωS such that for each point P of S, all points common to the sets $st(G_n)_P$ belong to S [4]. For the Tychonoff cases, this requires such a sequence $G_1, G_2, \cdots$ with respect to S for any Hausdorff space in which S can be densely embedded. Note the analogy with developability [cf. 26].

A *base of countable order* for a space S is a base B for S such that if $D_1, D_2, \cdots$ is a sequence of distinct members of B each including its successor and P is a point common to all the sets D_n, then $\{D_1\} + \{D_2\} + \cdots$ is a base for S at P [3]. Particularly to be noted are the role of bases of countable order in the characterization of developability involving a paracompactness-like refinement condition [27], the close bearing of the concept on the topic of interior transformations [16, 19], and its tractability to appropriate completeness formulations [18, 19, 20].

3. **The construction.** The technique of construction utilizes in a rather straightforward manner classical theorems on transfinite cardinalities of a sort such as are developed in [12]. The proof of properties is designed to reduce the question of the existence of certain Lindelöfian mappings in effect to that of the existence of a perfect mapping onto a space having a base of countable order through utilization of restrictions to certain bicompact domains.

THEOREM. *There exists a metacompact screenable locally compact Hausdorff space S of the power of the continuum satisfying these conditions*: (1) *Any collection of open sets covering S is refined by a σ-discrete collection of point sets covering S.* (2) *No Lindelöfian continuous mapping with S as its domain has a Hausdorff space with a base of countable order as its range.* (3) *S is first countable.*

Proof. There exists a sequence $\alpha_1, \alpha_2, \cdots$ of mutually exclusive first countable arcs of cardinal number c such that for each n, there exists a collection Q_n of mutually exclusive subarcs of α_n satisfying these conditions: (1) No element of Q_n contains an endpoint *of* α_n. (2) Q_n^* is dense in α_n. (3) If q and q' are two elements of Q_n then (a) there exists a nonseparable subset Y of $\alpha_n - Q_n^*$ such that q

separates Y from one endpoint of α_n (in the sense of [11]) and q' separates Y from the other and (b) there exist c members of Q_n similarly separated from the endpoints of α_n by q and q'.

Let Γ denote the set of all sequences $J_1, J_2, \cdots$ such that each J_n is a subarc of α_n the endpoints of which belong to $\alpha_n - Q_n^*$. Let M denote a set of power c not intersecting $\alpha_1 + \alpha_2 + \cdots$. There exists a transformation θ of M onto a collection W of simple infinite sequences such that (1) the n^{th} term of each sequence in W belongs to Q_n, (2) no element of $Q_1 + Q_2 + \cdots$ is a term of two members of W, and (3) if $J_1, J_2, \cdots$ belongs to Γ, there exist c sequences $q_1, q_2, \cdots$ in W such that each J_n includes q_n. For each q in $Q_1 + Q_2 + \cdots$, let X_q denote a cut point of q. For each n, let α'_n denote the set of all points P of α_n such that either (1) $\alpha - Q_n^*$ contains P or (2) P is a noncut point of some member of Q_n or (3) P is X_q for some q in Q_n.

Let τ denote the collection to which an element belongs if and only if it is the sum of some sets D satisfying one of these conditions:

(1) For some n, D is an open set of α'_n (in the relative topology).

(2) For some n and element μ of M, D is

$$\{\mu\} + \{X_{q_n}\} + \{X_{q_{n+1}}\} + \cdots ,$$

where $q_1, q_2, \cdots$ denotes $\theta(\mu)$. Let S denote τ^*.

The first countable regular T_0 space (S, τ) is screenable, locally compact, and has the σ-discrete refinement property stated in the theorem. All regular spaces with this σ-discrete refinement property are countably metacompact. Moreover, every countably metacompact screenable space is metacompact. Hence (S, τ) is metacompact. Since (S, τ) is a regular T_0 locally bicompact space, it is Čech complete. Clearly, $\bar{\bar{S}} = c$.

Suppose there exists a Lindelöfian continuous mapping f of (S, τ) onto a Hausdorff space having a base of countable order.

(I) Each f/α'_n is closed and bicompact. Since having a base of countable order is an hereditary property for a space [27], $f(\alpha'_n)$ has such a base [21]. Thus the bicompact Hausdorff space $f(\alpha'_n)$ is metrizable [3].

For some n let G denote the decomposition of α'_n induced by f. There exists a meaning for the notation $U_{i,h}$, for positive integers i and subsets h of G, such that for some development $H_1, H_2, \cdots$ of G (with respect to the quotient topology) the terms of which are finite, these conditions are satisfied: (1) For each i and element h of H_i, $U_{i,h}$ is a finite collection of sets covering h^* any nondegenerate element of which is the common part of α'_n and some connected open subset of α_n. (2) For each i and element h of H_{i+1} there exists some h' in H_i including h such that the closure of each member of $U_{i+1,h}$ is a subset of some member of $U_{i,h'}$ and is covered by h'. For each i, let K_i denote the sum of all collections $U_{i,h}$ for elements h of H_i.

With application of König's lemma it may be seen that if P belongs to $\alpha_n - Q_n^*$ and g is the member of G containing P there exist sequences $h_1, h_2, \cdots$ and $D_1, D_2, \cdots$ of sets such that (1) each h_i belongs to H_i, contains g, and includes $\bar{h}_{i+1}$, (2) each D_i is a member of U_{k,h_i} containing P, and (3) each D_i includes $\bar{D}_{i+1}$. With use of the compactness of α'_n it may be seen that if $\{P\}$ is the common part of the sets D_i then $\{D_1\} + \{D_2\} + \cdots$ is a base for α'_n at P. Since $K_1 + K_2 + \cdots$ is countable and $\alpha_n - Q_n^*$ is nonseparable, there exist some P in $\alpha_n - Q_n^*, h_1, h_2, \cdots$ and $D_1, D_2, \cdots$ as above such that the common part L of the sets $\bar{D}_i$ is nondegenerate. With use of conditions (2) and (3) of the first paragraph of this proof it may be seen that L contains two points of $\alpha_n - Q_n^*$. Since $H_1, H_2, \cdots$ is a development for G and each h_i includes $\bar{h}_{i+1}$, it may be seen that $h_1^*, h_2^*, \cdots$ converges to the member g of G containing P. This requires that L be a subset of g.

(II) Let Δ denote the decomposition of S induced by f. Suppose there exist a member δ of Δ and a sequence $n_1, n_2, \cdots$ of increasing positive integers such that for each i, δ includes the common part of S and some subarc v_i of α_{n_i} the endpoints of which belong to $\alpha_{n_i} - Q_{n_i}^*$. Then there exists a sequence $J_1, J_2, \cdots$ belonging to Γ such that for each i, J_{n_i} is v_i. With the use of condition (3) of the definition of θ and the definition of τ it may be seen that there exists an uncountable closed and isolated subset of M which is on the boundary of

$$S\cdot(v_1 + v_2 + \cdots)$$

and which therefore must be included by the closed point set δ. But this involves a contradiction, for δ is Lindelöfian. With application of (I) above it follows that there exist a sequence $\delta_1, \delta_2, \cdots$ of distinct members of Δ and a sequence $n_1, n_2, \cdots$ of increasing positive integers suchthat for each i, α_{n_i} has a subarc v_i the endpoints of which belong to $\alpha_{n_i} - Q_{n_i}^*$ such that δ_i includes $v_i \cdot \alpha'_{n_i}$. Since no δ_i contains uncountably many elements of M, there exists a countable subset T of M such that $\delta_1 + \delta_2 + \cdots$ does not intersect $M - T$. Uncountably many points of M belong to the boundary of $S\cdot(v_1 + v_2 + \cdots)$. So there exists an element δ of Δ intersecting $M - T$ and an infinite subsequence σ of $\delta_1, \delta_2, \cdots$ such that $f(\sigma)$ converges uniquely to δ. But a contradiction is involved, for $T + \delta\cdot M$ is countable, and uncountably many points of M are limit points of the sum of the terms of σ.

It follows that there exists no Lindelöfian continuous mapping of (S, τ) onto a Hausdorff space having a base of countable order.

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