PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 2, 1969

TWISTED COHOMOLOGY AND ENUMERATION OF VECTOR BUNDLES

LAWRENCE L. LARMORE

In the present paper we give a technique for completely enumerating real 4-plane bundles over a 4-dimensional space, real 5-plane bundles over a 5-dimensional space, and real 6-plane bundles over a 6-dimensional space. We give a complete table of real and complex vector bundles over real projective space P_k, for $k \leqq 5$. Some interesting results are:

(0.1.1.) Over P_5, there are four oriented 4-plane bundles which could be the normal bundle to an immersion of P^5 in R^9, i.e., have stable class $2h + 2$, where h is the canonical line bundle. Of these, two have a unique complex structure.

(0.1.2.) Over P_5 there is an oriented 4-plane bundle which we call C, which has stable class $6h - 2$, which has two distinct complex structures. D, the conjugate of C, i.e., reversed orientation, has no complex structure.

(0.1.3) Over P_5, there are no 4-plane bundles of stable class $5h - 1$ or $7h - 3$.

0.2. In reading the tables (4.5.2) and (4.6), remember that if ξ: $P_k \rightarrow BO(n)$ or $\xi: P_k \rightarrow BU(n)$ is a locally oriented (i.e., oriented over base-point) real or complex vector bundle, and if

$$a \in H^k(P_k; \pi_k(BO(n), \xi))$$

(local coefficients if ξ unoriented) or $a \in H^k(P_k; \pi_k(BU(n))$, then $\xi + a$ is a vector bundle obtained by cutting out a disk in the top cell of P_k and joining a sphere with some vector bundle on it.

0.3. Since some of the homotopy groups of $BO(n)$ are acted upon nontrivially by $Z_2 \cong \pi_1(BO(n))$ for n even, we study cohomology with local coefficients in § 3.

1.2. From here on, we assume that all spaces are connected C. W.-complexes with base-point, all maps are b.p.p. (base-point-preserving) and that all homotopies are b.p.p.

For any space Y, we choose a Postnikov system for Y, that is: for each integer $n \geqq 0$, a space $(Y)_n$ and a map $P_n: Y \rightarrow (Y)_n$ which induces an isomorphism in homotopy through dimension n, where all homotopy groups of $(Y)_n$ are zero above n; for each $n \geqq 1$ a fibration $p_n: (Y)_n \rightarrow (Y)_{n-1}$ such that $p_n P_n = P_{n-1}$. The fiber of each p_n is then an Eilenberg-MacLane space of type $(\pi_n(Y), n)$. If X is a space of finite dimension m, then $[X; Y]$, the set of homotopy classes of maps from X to Y, is in one-to-one correspondence with $[X; (Y)_m]$.

DEFINITION (1.2.1). For any integer $n \geqq 1$, let $G_n(Y)$ be the sheaf over $(Y)_1$ whose stalk over every y is defined to be $\pi_n(p^{-1}y)$, which is isomorphic to $\pi_n(Y)$ (where $p = p_2 \cdots p_n: (Y)_n \to (Y)_1$) if $n \geqq 2$; $\pi_1((Y)_1, y)$ if $n = 1$. If X is any space and $f: X \to (Y)_1$ is a map, let $\pi_n(Y, f)$ be the sheaf $f^{-1}G_n(Y)$ over X. This sheaf depends only on the homotopy class of f. If $g: X \to (Y)_m$ is a map for any integer $m \geqq 1$, or if $h: X \to Y$ is a map, let $\pi_n(Y, g)$ denote $\pi_n(Y, p_2 \cdots p_m g)$ and let $\pi_n(Y, n)$ denote $\pi_n(Y, P_1 h)$.

DEFINITION (1.2.2). If f and g are maps from X to $(Y)_n$ for any $n \geqq 2$, which agree on A, and if $F: X \times I \to (Y)_{n-1}$ is a homotopy of $p_n f$ with $p_n g$ which holds A fixed, let $\delta^n(f, g; F) \in H^n(X, A; \pi_n(Y, f))$ be the obstruction to lifting F to a homotopy of f with g which holds A fixed.

REMARK (1.2.3). If $g: X \to (Y)_n$ is another map which agrees with f on A, and if G is a homotopy of $p_n g$ with $p_n h$ which holds A fixed, then $\delta^n(f, g; F) + \delta^n(g, h; G) = \delta^n(f, h; F + G)$, where, for each $(x, t) \in X \times I$,

$$(F + G)(x, t) = \begin{cases} F(x, 2t) & \text{if} \quad 0 \leqq t \leqq \frac{1}{2} \\ G(x, 2t - 1) & \text{if} \quad \frac{1}{2} \leqq t \leqq 1 \, . \end{cases}$$

DEFINITION (1.2.4). Let X be a space, let $A \subset X$ be any subcomplex (possible empty), let $f: X \to (Y)_n$ be a map for some integer $n \geqq 2$, and let a be an element of $H^n(X, A; \pi_n(Y, f))$. We define $f + a$ to be that map from X to $(Y)_n$, unique up to fiber homotopy with A held fixed, such that $p_n(f + a) = p_n f$ and $\delta^n(f, f + a) = a$, where C is the constant homotopy.

REMARK (1.2.5). If b is any other element of $H^n(X, A; \pi_n(Y, f))$, then $f + (a + b) = (f + a) + b$.

REMARK (1.2.6). If $g: (X', A') \to (X, A)$ is a map, where $(X'A')$ is any other C. W. pair, then $(f + a)g = gf + g^*a$.

MAIN THEOREM (1.2.7). *For any* $a \in H^n(X, A; \pi_n(Y, f)), f + a$ *is homotophic to* f, rel A, *if and only if* $\delta^n(f, f; F) = a$ *for some homotopy* F *of* $p_n f$ *with itself which holds* A *fixed.*

Proof. Let C be the constant homotopy of $p_n f$ with itself. On the one hand, if F is any homotopy of $p_n f$ with itself which holds

A fixed, let $a = \delta^n(f, f; F)$. Then $\delta^n(f + a, f; F) = \delta^n(f + a, f; C) + \delta^n(f, f; F) = - a + a = 0$. Thus F may be lifted to a homotopy of $f + a$ with f. On the other hand, if G is a homotopy of $f + a$ with f, then $\delta^n(f, f; p_nG) = \delta^n(f, f + a; C) + \delta^n(f + a, f; p_nG) = a + 0 = a$.

DEFINITION (1.2.8). Let L_f be the subgroup of $H^n(X, A; \pi_n(Y, f))$ consisting of all a such that $f + a$ is homotopic to $f \operatorname{rel} A$. Then the set of all homotopy ($\operatorname{rel} A$) classes of liftings of p_nf to $(Y)_n$ which agree with f on A is in a one-to-one correspondence with the quotient group $H^n(X, A; \pi_n(Y, f))/L_f$; each coset $a + L_f$ corresponds to $f + a$. If $g: X \rightarrow Y$ is a map such that $p_ng = f$, let $L_g^n = L_f$. If $h: X \rightarrow (Y)_m$ is a map such that $p_{n+1} \cdots p_mh = f$, for $m \geqq n$, let $L_h^n = L_f$.

REMARK (1.2.9). If $a \in H^n(X, A; \pi_n(Y, f))$, then $L_{f+a} = L_f$.

Proof. Let F be any homotopy of $p_nf = p_n(f + a)$ with itself, and let C be the constant homotopy. Then $\delta^n(f + a, f + a; F) = \delta^n(f + a, f; C) + \delta^n(f, f; F) + \delta^n(f, f + a; C) = - a + \delta^n(f, f; F) + a = \delta^n(f, f; F)$.

1.3. In order to calculate L_f in specific cases, such as X a projective space, $A =$ base-point, and $Y = BO(m)$ for some m, we use a spectral sequence which has the following properties:

(1.3.1) ${}^fE_2^{p,q} = E_2^{p,q} = H^p(X, A; \pi_q(Y, f))$ if $2 \leqq q \leqq n, 1 \leqq p \leqq q + 1$.

(1.3.2) $E_2^{p,q} = 0$ for all other values of p and q.

(1.3.3) $d_r: E_r^{p,q} \rightarrow E_r^{p+r,q+r-1}$ for all $r \geqq 2$.

(1.3.4) $E_\infty^{n,n} = H^n(X, A; \pi_n(Y, f))/L_f$, which, by (1.2.7) and (1.2.8) can be put into one-to-one correspondence with the set of $\operatorname{rel} A$ homotopy classes of maps $X \rightarrow (Y)_n$ whose projection to $(Y)_{n-1}$ is $\operatorname{rel} A$ homotopic to p_nf.

Basically, what is happening is as follows (where, for any space Z and any map $g: A \rightarrow Z$, the set of $\operatorname{rel} A$ homotopy classes of maps $X \rightarrow Z$ which agree with g on A is denoted "$[X; Z: g]$"); consider the function:

$$[X; (Y)_n: f \mid A] \xrightarrow{(p_n)_\#} [X; (Y)_{n-1}: p_nf \mid A] .$$

Now $(p_n)_\#$ is just a function of sets, but $(p_n)_\#^{-1}(p_nf)$ is an Abelian group with 0 the homotopy class of f itself. This group, $E_\infty^{n,n}$ of our spectral sequence, depends on the choice of f.

We define our spectral sequence via an exact couple:

$$\begin{array}{ccc} D_2^{*,*} & \xrightarrow{\quad i_2 \quad} & D_2^{*,*} \\ & {\scriptstyle k_2}\nwarrow \quad \nearrow{\scriptstyle j_2} & \\ & E_2^{*,*} & \end{array}$$

where $E_2^{p,q}$ is as defined in (1.3.1) and (1.3.2), where i_2, j_2, and k_2 have bi-degrees $(-1, -1), (2, 1)$, and $(0, 0)$ respectively; and where (for all $t \leqq n$, M_t = space of maps from X to $(Y)_t$ which agree with $p_t^n f$ on A, compact-open topology):

(1.3.5) $D_2^{p,q} = \pi_{q-p}(M_q, p_q^n f)$ if $0 \leqq q \leqq n$, and $p \leqq q$.

(1.3.6) $D_2^{p,q} = 0$ if $q < p$ or $q < 0$.

(1.3.7) $D_2^{p,q} = D_2^{p-1,q-1}$ if $q > n$.

Note that $D_2^{p,q}$ is only a group if $q = p + 1$ and only a set if $q = p$. This will not affect our computation, however.

We proceed to define the homomorphisms i_2, j_2 and k_2.

(1.3.8) If $q > n$, let i_2 be the identity. If $q \leqq n$, let $i_2 = (p_q)_\#$.

(1.3.9) If $p \leqq q$ and $0 \leqq q < n$, any $x \in D_2^{p,q}$ represents a map $g: X \times I^{q-p} \to (Y)_q$, where $g(x, v) = p_q^n f(x)$ for all $(x, v) \in X \times \partial I^{q-p} \cup A \times I^{q-p}$. Let $j_2(x) = (s^{q-p})^{-1}\gamma^{q+2}(g)$, where $s^{q-p}: H^{p+2}(X, A; \pi_{q+1}(Y, f)) \to H^{q+2}(X \times I^{q-p}, X \times \partial I^{q-p} \cup A \times I^{q-p}; \pi_{q+1}(Y, g))$ is the $(q - p)$-fold suspension and $\gamma^{q+2}(g)$ is the obstruction to finding a lifting $h: X \times I^{q-p} \to (Y)_{q+1}$ of g such that $h(x, v) = p_{q+1}^n f(x)$ for all $(x, v) \in X \times \partial I^{q+p} \cup A \times I^{q-p}$. (If $p > q$ or $q < 0$ or $q \geqq n$, $j_2: D_2^{p,q} \to E_2^{p+2,q+1}$ is obviously the zero map, since $E_2^{p+2,q+1} = 0$.) This obstruction is zero if and only if g can be lifted; it follows immediately that:

(1.3.10) The sequence $D_2^{p+1,q+1} \xrightarrow{i_2} D_2^{p,q} \xrightarrow{j_2} E^{p+2,q+1}$ is exact.

Furthermore, since every homotopy, rel A, of $p_n f$ with itself represents a loop in M_{n-1}:

(1.3.11) L_f is the image of $j_2: D_2^{n-2,n-1} \to E_2^{n,n}$. For any $2 \leqq q \leqq n$, $1 \leqq p \leqq q$, and any $a \in E_2^{p,q}$, let

$$b = s^{q-p}a \in H^q(X \times I^{q-p}, X \times \partial I^{q-p} \cup A \times I^{q-p}; \pi_q(Y, C)) ,$$

where $C(x, v) = p_q^n f(x)$ for every $(x, v) \in X \times I^{q-p}$. Let $k_2(a) \in D_2^{p,q}$ be that element represented by the map $C + b$ (cf. 1.2.2). It follows from (1.2.3) that k_2 is a homomorphism if $p < q$; if $p = q$ then $D_2^{p,q}$ is only a set anyway. (For other values of p and q, $k_2 = 0$.) Since $p_q(C+b) = p_q C$, and C represents $0 \in D_2^{p,q}$:

(1.3.12) $\operatorname{Im} k_2 \subset \operatorname{Ker} i_2$.

If, on the other hand, a map g: $X \times I^{q-p} \to (Y)_q$ such that $g = C$ on $X \times \partial I^{q-p} \cup A \times I^{q-p}$ is a representative of a given $a \in \operatorname{Ker} i_2$, then $p_q g$ is homotopic, rel $X \times \partial I^{q-p} \cup A \times I$, to $p_q C$ via a homotopy F, then $a = k_2((s^{q-p})^{-1}\delta^q(C, g; F))$. Thus:

(1.3.13) $\operatorname{Ker} i_2 \subset \operatorname{Im} k_2$.

Somewhat more difficult to show is:

(1.3.14) $\operatorname{Ker} k_2 = \operatorname{Im} j_2$ if $p \leqq q$.

Proof. Let $2 \leqq q \leqq n, 1 \leqq p \leqq q$. Let $g(x, v) = p_q^n f(x) \in (Y)_q$ for all $(x, v) \in X \times I^{q-p}$; g represents $0 \in D_2^{p,q}$. Let $b \in E_2^{p,q}$. Then $b \in \operatorname{Ker} k_2$

if and only if $s^{q-p}b \in L_g$ (cf. 1.2.7). If $b = j_2 a$, then a represents F, a homotopy, $\operatorname{rel} X \times \partial I^{q-p} \cup A \times I^{q-p}$ of $p_q q$ with itself, and $s^{q-p}b = \delta^q(g, g; F) \in L_g$. If, on the other hand, $s^{q-p}b \in L_g$, then $s^{q-p}b = \delta^q(g, g; F)$ for some homotopy F, $\operatorname{rel} X \times \partial I^{q-p} \cup A \times I^{q-p}$, of $p_q g$ with itself; let $a = [F] \in D^{p-2,q-1}$, and $j_2 a = b$.

1.4. Since only finitely many of the E_2 terms are nonzero, we obtain E_∞ after a finite number of steps. We also have, by straightforward algebra, an exact sequence

$$0 \longrightarrow E_\infty \xrightarrow{k_\infty} D_\infty \xrightarrow{i_\infty} D_\infty \longrightarrow 0 .$$

Consider now the commutative diagram with exact columns:

$$\begin{array}{ccccll}
 & & D_2^{n-2,n-1} & = \pi_1(M_{n-1}, p_n f) & & [F] \\
 & & \big\downarrow j_2 & & & \big\downarrow \\
E_\infty^{n,n} & \xleftarrow{\text{epi}} & E_2^{n,n} & = H^n(X, A; \pi_n(Y, f)) & & \delta^n(f, f; F) \\
\text{mono}\big\downarrow k_\infty & & \big\downarrow k_2 & & & x \\
D_\infty^{n,n} & = & D_2^{n,n} & = [X; (Y)_n : f \mid A] & & \big\downarrow \\
\text{epi}\big\downarrow i_\infty & & \big\downarrow i_2 & & & f + x . \\
D_\infty^{n-1,n-1} & \xrightarrow[\text{mono}]{} & D_2^{n-1,n-1} & = [X; (Y)_{n-1} : p_n f \mid A] & &
\end{array}$$

A typical element of $D_2^{n-2,n-1}$ is a $\operatorname{rel} X \times \partial I \cup A \times I$ homotopy class of homotopies of $p_n f$ with itself; if F is such a homotopy, $j_2[F] = \delta^n(f, f; F)$, by (1.3.9). If $x \in H^n(X, A; \pi_n(Y, f))$, $k_2 x = f + x$, by (1.3.11). Thus $\operatorname{Im} j_2 = L_f$, and $E_\infty^{n,n} = H^n(X, A; \pi_n(Y, f))/L_f$, the set of $\operatorname{rel} A$ homotopy classes of liftings of $p_n f$.

1.5. If $g: (X', A') \to (X, A)$ is a map, g induces a map of spectral sequences.

(1.5.1) $g^*: {}^f E_r^{p,q} \to {}^{fg}E_r^{p,q}$ for all p, q, r. If $h: Y \to Z$ is a map, where Z is any other space, h determines a map $h_m: (Y)_m \to (Z)_m$ for each $m \geqq 0$ [1]. Then $h_\#: \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ induces a sheaf homomorphism from $G_n(Y)$ to $(h_1)^{-1}G_n(Z)$ which in turn induces a homomorphism.

(1.5.2) $h_*: H^*(X, A; \pi_m(Y, f)) \to H^*(X, A; \pi_m(Z, hf))$ for all $m \geqq 0$ and a map of spectral sequences

(1.5.3) $h_*: {}^f E_r^{p,q} \to {}^{hf}E_r^{p,q}$ for all p, q, r.

2. Nonbase-point-preserving homotopies.

2.1. Using the techniques of § 1, we can compute all b.p.p.

homotopy classes of maps from a finite-dimensional space X to a space Y. What if we want to know, instead, all free homotopy classes of maps?

2.2. Let $f: X \to Y$ be any b.p.p. map, and let $a \in \pi_1(Y, y_0)$. By the homotopy extension property, we can find a free homotopy F: $X \times I \to Y$ of f such that $F \mid \{x_0\} \times I$ represents a. Let $f^a(x) = F(x, 1)$ for any $x \in X$; f^a is unique up to b.p.p. homotopy, and $f^{ab}(f^a)^b$ for any other $b \in \pi_1(Y, y_0)$.

THEOREM (2.2.1). *If f and g are any* b.p.p. *maps from X to Y, then f is freely homotopic to g if and only if f^a is* b.p.p. *homotopic to g for some $a \in \pi_1(Y, y_0)$.*

Proof. If f^a is b.p.p. homotopic to g, then f is obviously freely homotopic to g since f is freely homotopic to f^a. If, on the other hand, $F: X \times I \to Y$ is a free homotopy of f with g, let a be that element of $\pi_1(Y, y_0)$ represented by the loop $F \mid \{x_0\} \times I$. Then $f^a = g$ (up to b.p.p. homotopy).

THEOREM (2.2.2). *If $n \geqq 2, f: X \to (Y)_n$ is a map,*

$$a \in H^n(X, x_0; \pi_n(Y, f)) ,$$

and $b \in \pi_1(Y, y_0)$, then $(f + a)^b = f^b + 1_*^b(a)$, where 1_*^b is the homomorphism induced by the map 1^b (cf. 1.5.2), *where 1 is the identity map on $(Y)_n$.*

Proof. The theorem follows from naturality of obstruction theory.

3. Sheaves of local coefficients.

3.1. The homotopy groups of $BO(n)$ are sometimes acted on nontrivially by π_1. We must therefore study twisted sheaves.

DEFINITION (3.1.1). A twisted group is an ordered pair (G, T), G an Abelian group, $T: G \to G$ an automorphism of order 2. If X is a space, a (G, T)-sheaf over X is a fiber bundle over X with fiber G and structural group Z_2, action determined by T. Let $G^T[u]$ be the (G, T)-sheaf over P_∞ obtained by identifying (x, g) with (Tx, Tg) for all $(x, g) \in S^\infty \times G$, where $T: S^\infty \to S^\infty$ is the antipodal map.

DEFINITION (3.1.2). If $a \in H^1(X, x_0; Z_2)$ and $f: (X, x_0) \to (P_\infty, *)$ is a map where $f^*u = a$ ($u =$ fundamental class of P_∞), let $G^T[a] = f^{-1}G^T[u]$. We call a the twisting class of $G^T[a]$.

PROPOSITION (3.1.3). *$G^T[u]$ is universal in the sense of Steenrod [6], that is, if G is a (G, T)-sheaf over a space $X, G \cong G^T[a]$ for some unique $a \in H^1(X, x_0; Z_2)$.*

Proof. $P_\infty = BZ_2$.

REMARK (3.1.4). If $F: X \times I \to P_\infty$ is a *free* homotopy of f with itself, where $f^*u = a$, then F induces an automorphism of $G^T[a]$; 1 or T depending on whether $F \mid \{x_0\} \times I$ is a trivial loop in P_∞ or not.

3.2. If X is a space, $B \subset A \subset X$ are closed, and S is a sheaf over X, we have a long exact sequence:

$$\cdots \longrightarrow H^n(X, A; S) \longrightarrow H^n(X, B; S) \longrightarrow H^n(A, B; S) \xrightarrow{\delta} H^{n+1}(X, A; S) \longrightarrow \cdots$$

PROPOSITION (3.2.1). *If S is a sheaf over a space X, and $A \subset X$ is closed, we may find an isomorphism*

$$s: H^*(X, A; S) \longrightarrow H^*(X \times I, X \times \partial I \cup A \times I; S \times I),$$

called the suspension, of degree 1, *where $S \times I = p^{-1}S$; $p: X \times I \to X$ being the projection.*

Proof. Let S' be that subsheaf of S such that $S' \mid A = 0$ and $S' \mid (X - A) = S \mid (X - A)$. According to Bredon [1],

$$H^*(X, A; S) = H^*(X; S')$$

and

$$H^*(X \times I, X \times \partial I \cup A \times I; S \times I) = H^*(X \times I, X \times \partial I; S' \times I).$$

Now $H^*(X \times I, X \times \{t\}; S') = 0$ for any $t \in I$ [1], and by the long exact sequence of $(X \times I, X \times \partial I, X \times \{1\})$ and excision we have an isomorphism $H^*(X \times \{0\}; S' \times I) \xrightarrow{\cong} H^*(X \times I, X \times \partial I; S' \times I)$ of degree 1; the left group is isomorphic to $H^*(X; S')$.

3.3. Let X be a space, $A \subset X$ closed. If $\alpha: S \to S'$ is a homomorphism of sheaves over X, we get a homomorphism $\alpha_*: H^*(X, A; S) \to H^*(X, A; S')$. If S and S' are sheaves over X and

$$E: 0 \longrightarrow S \xrightarrow{i} S'' \xrightarrow{p} S' \longrightarrow 0$$

is an extension of S' by S, then E determines a long exact sequence

$$\cdots \longrightarrow H^n(X, A; S) \xrightarrow{i_*} H^n(X, A; S'') \xrightarrow{p_*} H^n(X, A; S')$$
$$\xrightarrow{\delta^E} H^{n+1}(X, A; S) \longrightarrow \cdots$$

where δ^E is called the Bockstein of E.

PROPOSITION (3.3.1). *If S and S' are sheaves over X and if*

$$E\colon 0 \longrightarrow S \xrightarrow{i} S'' \xrightarrow{p} S' \longrightarrow 0$$

and

$$F\colon 0 \longrightarrow S \xrightarrow{j} U \xrightarrow{q} S' \longrightarrow 0$$

are elements of Ext (S', S), *then* $\delta^{E+F} = \delta^E + \delta^F$.

Proof. We use the Baer sum construction to find

$$E + F\colon 0 \longrightarrow S \longrightarrow V \longrightarrow S' \longrightarrow 0;$$

our result follows from the commutative diagram, where each row is exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S \times S & \longrightarrow & S'' \times U & \longrightarrow & S' \times S' & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 1} & & \uparrow & & \uparrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & S \times S & \longrightarrow & W & \longrightarrow & S' & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & S & \longrightarrow & V & \longrightarrow & S' & \longrightarrow & 0 .
\end{array}$$

3.4. As Abelian groups Ext $(Z_2, Z_2) \cong Z_2$; the nonzero extension is Z_4. Fix a space X; we study Ext of sheaves over X.

PROPOSITION 3.4.1. *As sheaves over X,*

$$\text{Ext}\,(Z_2, Z_2) \cong Z_2 + H^1(X, x_0; Z_2)\ .$$

For any $a \in H^1(X, x_0; Z_2)$, $(0, a)$ corresponds to the extension

$$E_a^0\colon 0 \longrightarrow Z_2 \xrightarrow{i_1} (Z_2 + Z_2)^T[a] \xrightarrow{p_2} Z_2 \longrightarrow 0\ ,$$

where $T(x, y) = (x + y, y)$, $i_1(x) = (x, 0)$, and $p_2(x, y) = y$; $(1, a)$ corresponds to

$$E_a^1\colon 0 \longrightarrow Z_2 \xrightarrow{m} Z_4^T[a] \xrightarrow{e} Z_2 \longrightarrow 0\ ,$$

where $T(x) = -x$ for all $x \in Z_4$, $m(1) = 2$, and $e(1) = 1$.

Proof. Routine computation shows that $E_a^x + E_b^y = E_{a+b}^{x+y}$ for any $x, y \in Z_2$ and $a, b \in H^1(X, x_0; Z_2)$. On the other hand, suppose that

$$E\colon 0 \longrightarrow Z_2 \xrightarrow{i} G \xrightarrow{p} Z_2 \longrightarrow 0$$

is some extension. Then the stalk of G at x_0 is Z_4, in which case $G = Z_4^T[a]$ for some $a \in H^1(X, x_0; Z_2)$, or it is $Z_2 + Z_2$. In that case, we have an exact sequence of stalks at x_0:

$$0 \longrightarrow Z_2 \xrightarrow{i_1} Z_2 + Z_2 \xrightarrow{p_2} Z_2 \longrightarrow 0 .$$

Since G is locally isomorphic to $Z_2 + Z_2$, it is a fiber bundle with fiber $Z_2 + Z_2$ and structural group $\operatorname{Aut}(Z_2 + Z_2)$. But the only nontrivial automorphism which commutes with $i_1\colon Z_2 \to Z_2 + Z_2$ and $p_2\colon Z_2 + Z_2 \to Z_2$ is T given above. So the structural group of G may be reduced to Z_2; $G = (Z_2 + Z_2)^T[a]$ for some $a \in H^1(X, x_0; Z_2)$. This gives us the isomorphism.

We have the following commutative diagram with both rows exact, for any $a \in H^1(X, x_0; Z_2)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^T[a] & \xrightarrow[2]{} & Z^T[a] & \xrightarrow[\Pi]{} & Z_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} & & {\scriptstyle 1}\downarrow & & \\
0 & \longrightarrow & Z_2 & \xrightarrow[m]{} & Z_4^T[a] & \xrightarrow[e]{} & Z_2 & \longrightarrow & 0 .
\end{array}$$

DEFINITION (3.4.2). Let $\beta^T[a]$ (or simply β^T, when a is understood) denote the Bockstein of the top row of the above diagram, and let $(S_q^1)^T[a]$ (or $(S_q^1)^T$) denote the Bockstein of the bottom row.

REMARK (3.4.3). $\Pi_* \beta^T = (S_q^1)^T$.

PROPOSITION (3.4.4). *For any $n \geqq 0$ and any $x \in H^n(X, A\colon Z_2)$, $(S_q^1)^T x = S_q^1 x + x \cup a$.*

Proof. Samelson [5].

PROPOSITION (3.4.5). *For any $n \geqq 0$ and any $x \in H^n(X, A; Z_2)$ $\delta(x) = x \cup a$, where δ is the Bockstein of $E_a^0\colon 0 \to Z_2 \to (Z_2 + Z_2)^T[a] \to Z_2 \to 0$.*

Proof. The result follows immediately from (3.3.1), (3.4.1), and (3.4.4).

3.5. Let $T(n, m) = (m - n, m)$ for any $(n, m) \in Z + Z$. If S and S' are sheaves over a space X, and if $\mu\colon S \otimes S' \to S''$ is a sheaf homomorphism, then we have a cup product defined from

$$H^*(X, A; S) \otimes H^*(X, B; S')$$

to $H^*(X, A \cup B; S'')$ for any closed $A \subset X$ and $B \subset X$. We have thus cup products generated by the following relations:

$$Z^T[a] \otimes Z^T[b] = Z^T[a+b],\ Z_2 \otimes (Z_2 + Z_2)^T[a]$$
$$= (Z_2 + Z_2)^T[a],\ Z \otimes (Z + Z)^T[a]$$
$$= (Z + Z)^T[a],\ Z^T[a] \otimes (Z + Z)^T[a] = (Z + Z)^T[a]$$

(where $n \otimes (p, q) = (np, 2np - nq)$), $Z_4^T[a] \otimes Z_4^T[b] = Z_4^T[a + b]$,

and many others.

Let (X, A) be a C. W.-pair. Let $a \in H^1(X, x_0; Z_2)$ and

$$\alpha = \beta^T[a](1) \in H^1(X; Z^T[a]) .$$

We have the following commutative diagram; where

$$i_1 x = (x, 0),\ T(x, y) = (y - x, y),\ j_1 x = (x, 2x),$$

and $q_2(x, y) = y - 2x$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^T[a] & \xrightarrow{i_1} & (Z + Z)^T[a] & \xrightarrow{p_2} & Z & \longrightarrow & 0 \\
 & & \downarrow \Pi & & \downarrow \Pi & & \downarrow \Pi & & \\
0 & \longrightarrow & Z_2 & \xrightarrow{i_1} & (Z_2 + Z_2)^T[a] & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0 \\
 & & \uparrow \Pi & & \uparrow \Pi & & \uparrow \Pi & & \\
0 & \longrightarrow & Z & \xrightarrow{j_1} & (Z + Z)^T[a] & \xrightarrow{p_2} & Z^T[a] & \longrightarrow & 0 .
\end{array}$$

PROPOSITION (3.5.1). *The Bockstein homomorphisms δ_1 and δ_2 are both cup products with α.*

Proof. By (3.4.3) and (3.4.4) we may compute that

$$H^1(P_\infty; Z^T[u]) \cong Z_2$$

and is generated by $\bar{u} = \beta^T(1)$.

Let $x \in H^n(X, A; Z)$. If $n = 0$, then the universal example is $X = P_\infty, A = \emptyset, x = 1$. Then $\alpha = \bar{u}$. Now $H^0(P_\infty; Z^T) = 0$, so $(j_1)_*$: $H^0(P_\infty; Z) \leftarrow H^0(P_\infty; (Z + Z)^T)$ is an isomorphism, and $p_2 j_1 = 2$. Thus $1 \notin \operatorname{Im}(p_2)_*$, so $\delta_1(1) = \bar{u}$. If $n \geqq 1$, the universal example is $X = K(Z, n) \times P_\infty, A = * \times P_\infty, x = v_n \times 1$. Then $\alpha = p^*\bar{u}$, where $p: X \to P_\infty$ is projection onto the second factor. Now routine computations using (3.4.3) and (3.4.4) show that $H^{n+1}(X, A; Z^T) \cong Z_2$ and is generated by $(v_n \times 1) \cup p^*\bar{u}$, which is mapped onto $\Pi_* v_n \times u$ under $\Pi_*: H^*(; Z^T) \to H^*(; Z_2)$. The result follows from (3.4.5).

Let $x \in H^n(X, A; Z^T)$. If $n = 0, x = 0$. If $n = 1$, the universal example is $X = K(Z^T, n), A = P_\infty$, and $x = v_n^T$, where $K(Z^T, n)$ is obtained as follows:[1] Let $K(Z, n)$ be a topogical group, let $T(g, y) = (g^{-1}, Ty)$ for all $g \in K(Z,n)$ and $y \in S^\infty$. Let

[1] Personal communication from C. T. C. Wall.

$$K(Z^T, n) = K(Z, n) \times S^\infty/T .$$

We have inclusion and projection

$$P_\infty \xrightarrow{i} K(Z^T, n) \xrightarrow{p} P_\infty$$

where $i[y] = [*, y]$ and $p[g, y] = [y]$; P_∞ may thus be considered to be a subset of $K(Z^T, n)$, and its cohomology group is a direct summand[1]. Then $v_n^T \in H^n(K(Z^T, n), P_\infty; Z^T[u])$ is the fundamental class.

$$H^n(X, A; Z_2) \cong Z_2$$

is generated by $\Pi_* v_n^T$; $H^{n+1}(X, A; Z_2) \cong Z_2$ generated by $\Pi_* v_n^T \cup u$. Thus, by (3.4.3) and (3.4.4), $H^{n+1}(X, A; Z) \cong Z_2$ generated by $v_n^T \cup \bar{u}$, and the result follows from (3.4.5).

(3.5.2). We summarize the results of (3.4.5) and (3.5.1) in the following commutative diagram with all rows exact:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^n(X, A; Z^T) & \xrightarrow{(i_1)_*} & H^n(X, A; (Z+Z)^T) & \xrightarrow{(p_2)_*} & H^n(X, A; Z) & \xrightarrow{\cup \alpha} & H^{n+1}(X, A; Z^T) & \longrightarrow \cdots \\
& \downarrow \Pi_* & & \downarrow \Pi_* & & \downarrow \Pi_* & & \downarrow \Pi_* & \\
\cdots \longrightarrow & H^n(X, A; Z_2) & \xrightarrow{(i_1)_*} & H^n(X, A; (Z_2+Z_2)^T) & \xrightarrow{(p_2)^*} & H^n(X, A; Z_2) & \xrightarrow{\cup \alpha} & H^{n+1}(X, A; Z_2) & \longrightarrow \cdots \\
& \uparrow \Pi_* & & \uparrow \Pi_* & & \uparrow \Pi_* & & \uparrow \Pi_* & \\
\cdots \longrightarrow & H^n(X, A; Z) & \xrightarrow{(j_1)_*} & H^n(X, A; (Z+Z)^T) & \xrightarrow{(q_2)^*} & H^n(X, A; Z^T) & \xrightarrow{\cup \alpha} & H^{n+1}(X, A; Z) & \longrightarrow \cdots
\end{array}$$

3.6. Applying the results of 3.4 and 3.5, we compute the cohomology of real projective space P_k, for $k \geqq 1$:

(3.6.1)
$$H^n(P_k; Z_2) \cong \begin{cases} Z_2, \text{ generated by } u^n, \text{ if } n \leqq k \\ 0 \quad \text{if } n > k . \end{cases}$$

(3.6.2)
$$H^n(P_k; Z) \cong \begin{cases} Z_2, \text{ generated by } \bar{u}^n, \text{ if } n \text{ even, } 0 < n \leqq k \\ Z, \text{ generated by } 1, \text{ if } n = 0 \\ 0, \text{ if } n \text{ odd, } 0 < n < k \\ Z, \text{ generated by } t(P_k), \text{ the top class, if } n = k \text{ odd} \\ 0 \quad \text{if } n > k . \end{cases}$$

(3.6.3)
$$H^n(P_k; Z^T[u]) \cong \begin{cases} Z_2, \text{ generated by } \bar{u}^n, \text{ if } n \text{ odd, } 0 < n \leqq k \\ 0, \text{ if } n \text{ even, } 0 < n < k \\ Z, \text{ generated by } t(P_k), \text{ the top class, if } n = k \text{ even} \\ 0, \text{ if } n > k . \end{cases}$$

(3.6.4) $$H^n P_k, *; Z^T[u]) \cong \begin{cases} 0, & \text{if } n = 0 \\ Z, & \text{generated by } \bar{u}, \text{ if } n = 1 . \\ H^n(P_k; Z^T[u]) & \text{if } n > 1 \end{cases}$$

(3.6.5) $$H^n(P_k; Z_2 + Z_2) \cong H^n(P_k; Z_2) \oplus H^n(P_k; Z_2) .$$

(3.6.6) $$H^n(P_k; Z + Z) \cong H^n(P_k; Z) \oplus H^n(P_k; Z) .$$

(3.6.7) $$H^n(P_k; (Z + Z)^T[u]) \cong \begin{cases} Z, & \text{generated by } (j_1)_* 1, \\ & \text{if } n = 0 \\ 0, & \text{if } 0 < n < k \\ Z, & \text{generated by } \tfrac{1}{2}(i_1)_* t(P_k) = \\ & (q_2)_*^{-1} t(P_k) \text{ if } n = k \text{ is even} \\ Z, & \text{generated by } \tfrac{1}{2}(j_1)_* t(P_k) = \\ & (p_2)_*^{-1} t(P_k) \text{ if } n = k \text{ is odd} \\ 0, & \text{if } n > k \end{cases}$$

(3.6.8) $$H^n(P_k; (Z_2 + Z_2)^T[u]) \cong \begin{cases} Z_2, & \text{generated by } (i_1)_* 1 \\ & \text{if } n = 0 \\ 0, & \text{if } 0 < n < k \\ Z_2, & \text{generated by } (p_2)_*^{-1} u^k \\ & (= \Pi_* \tfrac{1}{2}(i_1)_* t(P_k)) \text{ if } k \\ & \text{even}, = \Pi_* \tfrac{1}{2}(j_1)_* t(P_k) \text{ if } k \\ & \text{odd}) \text{ if } n = k \\ 0, & \text{if } n > k . \end{cases}$$

4. Evaluation of the differentials.

4.1. We need two remarks.

(4.1.1) If Y_1 and Y_2 are spaces, and $h: Y_1 \to Y_2$ is a map, h induces a map $(Y_1)_{n-1} \to (Y_2)_{n-1}$ and a sheaf homomorphism $\tilde{h}: \pi_n(Y_1, 1) \to \pi_n(Y_2, h)$. If k_1^{n+1} and k_2^{n+1} are the n^{th} k-invariants of Y_1 and Y_2 respectively, $\tilde{h}_* k_1^{n+1} = h^* k_1^{n+2} \in H^{n+1}((Y_1)_{n-1}; \pi_n(Y_2, h))$.

(4.1.2) Let X and Y be spaces, $2 \leqq m < n$ integers such that $\pi_k(Y) = 0$ for all $m < k < n$, and $f: X \to (Y)_n$ a map. If the k-invariant k^{n+1} of Y is based on the relation $\theta(1, k^{m+1}) = 0$, where θ is a map cohomology operation and $1: (Y)_{m-1} \to (Y)_{m-1}$ is the identity map, then; for any

$$x \in H^{m-1}(X; \pi_m(Y, f)), d_r(x) = s^{-2}\theta(p^n_{m-1} f P, s^2 x), r = n - m + 1 ,$$

where $P: X \times S^2 \to X$ is projection,

$$s^2: H^*(X, x_0) \to H^{*+2}(X \times S^2, X \times * \cup x_0 \times S^2)$$

is suspension and $p^n_{m-1} = p_m \cdots p_n \colon (Y)_n \to (Y)_{m-1}$.

Proof. Let $(S^1, *)$ be a circle, which we think of as the unit interval with end-points identified. Let $C \colon X \times S^1 \to (Y)_m$ be the constant homotopy of $p^n_m f$ with itself. Now $p_m(C + sx) = p_m C$, where $C + sx$ is as defined in (1.2.2) and $d_r(x) = \delta^n(f, f; C + sx)$ by (1.3). Finally, $s\delta^n(f, f; C + sx) = (C + sx)^* k^{n+1} = s^{-1}\theta(p^n_{m-1} fP, s^2 x)$.

4.2. Kervaire [3, p. 162] gives us the following table of homotopy groups:

	$BO(1)$	$BO(2)$	$BO(3)$	$BO(4)$	$BO(5)$	$BO(6)$	$BO(n)$ for $7 \leqq n \leqq \infty$
π_1	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2
π_2	0	Z	Z_2	Z_2	Z_2	Z_2	Z_2
π_3	0	0	0	0	0	0	0
π_4	0	0	Z	$Z + Z$	Z	Z	Z
π_5	0	0	Z_2	$Z_2 + Z_2$	Z_2	0	0
π_6	0	0	Z_2	$Z_2 + Z_2$	Z_2	Z	0 .

Now $\pi_1(BO(n)) = Z_2$ acts on $\pi_k(BO(n))$ for all $n \geqq 1, k \geqq 1$; this action is trivial if $\pi_k(BO(n))$ is stable, that is, $k < n$; because BO is simple. For n even, Z_2 acts nontrivially on $\pi_n(BO(n))$, because the first relative k-invariant of $BO(n) \to BO$ is

$$k^{n+1} = \beta^T[w_1]w_n \in H^{n+1}(BO; Z^T[w_1]) \ .$$

(Because $\Pi_* k^{n+1}$, the reduction mod 2, must be w_{n+1}). Z_2 acts trivially on $\pi_4(BO(3))$ because if acts trivially on $\pi_4(BO)$ and the map $Z \cong \pi_4(BO(3)) \to \pi_4(BO) \cong Z$ is just multiplication by 2. Since Z_2 can only act trivially on Z_2, we need only now examine the action on $\pi_4(BO(4))$ for $k = 4, 5, 6$.

PROPOSITION (4.2.1). *We may choose generators x and y of $\pi_4(BO(4))$ such that $T(x) = -x$, $T(y) = x + y$, and the maps*

$$i^3_4 \colon \pi_4(BO(3)) \longrightarrow \pi_4(BO(4)) \quad and \quad i^4_4 \colon \pi_4(BO(4)) \longrightarrow \pi_4(BO(5))$$

have the properties $i^3_4(1) = x + 2y$, $i^4_4(x) = 0$ and $i^4_4(y) = 1$.

Proof. We know that i^4_4 is onto. Choose x to be a generator of $\mathrm{Ker}\, i^4_4$, and pick a such that $i^4_4 a = 1$. Now $2a - i^3_4(1) \in \mathrm{Ker}\, i^4_4$, since $i^4_4 i^3_4 = 2$. So $2a - i^3_4(1)$ is a multiple of x. It can't be an even multiple, because then $i^3_4(1)$ would be divisible by 2, and $i^3_4\pi_4(BO(3))$ is a direct summand of $\pi_4(BO(4))$. So for some k, $2a - i^3_4(1) = (2k-1)x$. Let $y = a - kx$; then $i^3_4(1) = x + 2y$, $i^4_4(x) = 0$, and $i^4_4(y) = 1$. Now $T(x) \in \mathrm{Ker}\, i^4_4$, so $T(x)$ must be $-x$. $T(x + 2y) = x + 2y$ so $T(y) = \frac{1}{2}(x + 2y - Tx) = x + y$. We are done.

We represent $\pi_4(BO(4))$ as ordered pairs of integers, where (p, q) represents $px + qy$.

PROPOSITION (4.2.2). *$\pi_5(BO(4))$ and $\pi_6(BO(4))$ may be represented as ordered pairs of elements of Z_2, such that $i_5^3(x) = i_6^3(x) = (x, 0)$, $i_5^4(x, y) = i_6^4(x, y) = y$, and $T(x, y) = (x + y, y)$ for all $x, y \in Z_2$.*

Proof. $\pi_5(BO(n))$ and $\pi_6(BO(n))$ are the images, under η and η^2 respectively, of $\pi_4(BO(n))$, for $n = 3$, 4, or 5. Apply (4.2.1).

REMARK (4.2.3). There are two possible choices of x in (4.2.1) we retroactively make that choice such that the image of $\pi_5(BU(2)) \cong Z_2$, under the classifying map of the reallification $BU(2) \rightarrow BO(4)$, is generated by $(0, 1) \in \pi_5(BO(4))$.

4.3. We need to describe k-invariants for $BO(n)$.

(4.3.1) For all n, k^3 of $BO(n)$ is zero, since the projection

$$P_1: BO(n) \longrightarrow (BO(n))_1 = K(Z_2, 1) = BO(1)$$

has a lifting, namely, the map induced by the inclusion of $O(1)$ in $O(n)$. Also $k^4 = 0$, since $\pi_3(BO(n)) = 0$.

(4.3.2) For $BO(3), k^5 = \pm \beta_4 \mathfrak{P} w_2$, where β_4 is the Bockstein of $Z \rightarrow Z \rightarrow Z_4$ and $\mathfrak{P}: H^2(; Z_2) \rightarrow H^4(; Z_4)$ is the Pontrjagin square [2], and k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$.

(4.3.3) For $BO(5)$, $k^5 = 2\beta_4 \mathfrak{P} w_2 = \beta w_2^2$ (see [4]), and $k^6 = w_6$, based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$.

(4.3.4) Using (4.3.2), (4.3.3), we get that for $BO(4)$, $k^5 = \iota \beta_4 \mathfrak{P} w_2$, where $\iota: H^*(; Z) \rightarrow H^*(; (Z + Z)^T)$ is $(j_1)_*$ as described in (3.5.2), and k^5 is of order 4 and generates $H^5((BO(4))_4; (Z + Z)^T[w_1])$. Also, k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5$, where

$$S_q^2: H^*(; (Z_2 + Z_2)^T[a]) \longrightarrow H^{*+2}(; (Z_2 + Z_2)^T[a])$$

is that unique operation which is ordinary S_q^2 on each factor when $a = 0$, and $w_2 \cup$ is as described in (3.5).

(4.3.5) For $BO(6)$, $k^5 = 2\beta_4 \mathfrak{P} w_2 = \beta w_2^2$, and $k^7 = \beta^T[w_1] w_6$, based on the relation $\beta^T(S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5) = 0$.

4.4. Using (4.1.1) and (4.1.2) we can now evaluate some differentials $d_r = d_r^f$ for a map $f: X \rightarrow (Y)_k$.

(4.4.1) If $Y = BO(1)$ or $BO(2)$, $d_r = 0$.

(4.4.2) If $Y = BO(3)$ and $k < 4, d_r = 0$. If $k = 4, d_2 = 0$: by (4.1.2), $d_3(x) = \beta(x^3 + x \cup f^* w_2) \in H^4(X; Z)$ for all $x \in H^1(X; Z_2)$. This was also known to Dold and Whitney [2]. If

$$k = 5,\ d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; Z_2)\ ,$$

for all $x \in H^3(X; Z)$ by (4.1.2); $d_3 = 0$, and d_4 requires special computation.

(4.4.3) If $Y = BO(4)$ and $k < 4$, $d_r = 0$. If $k = 4$, $d_2 = 0$; and by (4.1.2),

$$d_3(x) = \iota\beta(x^3 + x \cup f^* w_2) \in H^4(X; (Z + Z)^T[f^* w_1])$$

for all $x \in H^1(X; Z_2)$; if

$$k = 5,\ d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; (Z_2 + Z_2)^T[f^* w_1])$$

for all $x \in H^3(X; (Z + Z)^T[f^* w_1])$ by (4.1.2), $d_3 = 0$, and d_4 must be computed specially.

(4.4.4) If $Y = BO(5)$ and $k < 5$, $d_r = 0$. If

$$k = 5,\ d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; Z_2)$$

for all $x \in H^3(X; Z)$, $d_3 = 0$, and

$$\begin{aligned} d_4(x) = x^5 + f^* w_1 \cup x^4 + f^* w_2 \cup x^3 + f^* w_3 \cup x^2 \\ + f^* w_4 \cup x + \operatorname{Im} d_2 \in E_4^{5,5} = H^5(X; Z_2)/\operatorname{Im} d_2 \end{aligned}$$

for all $x \in H^1(X; Z_2)$.

Proof. We have a map $S: \Sigma K(Z, 1)$—BSO, such that $S^* w_{i+1} = su^i$ for all $i \geqq 1$, where u is the fundamental class. Now $(BO(5))_4 = (BO)_4$ has the same homotopy as BO up through dimension 7, so we identify $H^k((BO(5))_4$ with $H^k(BO)$ for $0 \leqq k \leqq 7$. Let $h: \Sigma K(Z_2, 1)$—$(BO(5))_4$ be given by the commutative diagram:

$$\begin{array}{ccc} \Sigma K(Z_2, 1) & \xrightarrow{h} & (BO(5))_4 = (BO)_4 \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle P_4} \\ BSO & \longrightarrow & BO\ . \end{array}$$

$(BO(5))_4$ has an H-space structure $\mu: (BO(5))_4 \times (BO(5))_4 \to (BO(5))_4$ and $\mu^* w_6 = \sum_{i=0}^{6} w_i \times w_{6-i}$. Let QX be the space obtained from $X \times S^1$ by collapsing $x_0 \times S^1$; let $J: QX \to \Sigma X$ be the map which collapses $X \times *$, and let $p_1: QX \to X$ be projection onto the first factor. For any $x \in (H^* X)$, let $qx = p_1^* x$ and let $Qx = J^* sx$, both in $H^*(QX)$. We showed in [4, 5.1] that $qa \cup qb = q(a \cup b)$, $qa \cup Qb = Q(a \cup b)$, and $Qa \cup Qb = 0$ for all $a, b \in H^*(X)$. Let $C: X \to K(Z_2, 1)$ be a classifying map for a given $x \in H^1(X; Z_2)$, and let $F: QX \to (BO(5))_4$ be a map, which represents a homotopy of $p_5 f$ with itself, defined by composing the following maps:

$$QX \xrightarrow{\Delta} QX \times QX \xrightarrow{J \times p_1} \Sigma X \times X \xrightarrow{\Sigma C \times p_5 f} \Sigma K(Z_2, 1) \times (BO(5))_4$$
$$\xrightarrow{h \times 1} (BO(5))_4 \times (BO(5))_4 \xrightarrow{\mu} (BO(5))_4 \,.$$

By (1.3), $d_4(x)$ contains $\delta^5(f, f; F)$. Now routine computation shows that $f^*w_6 = Q(x^5 + x^4 f^* w_1 + x^3 f^* w_2 + x^2 f^* w_3 + x f^* w_4)$, and the result follows from [4, 5.2].

(4.4.5) If $Y = BO(6)$ and $k < 6$, $d_r = 0$. If $k = 6$, $d_2 = 0$ and $d_3(x) = \beta^T(S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x) \in H^6(X; Z^T[f^* w_1])$ for all $x \in H^3(X; Z)$; $d_4 = 0$ and

$$d_5(x) = \beta^T(x^5 + x^4 f^* w_1 + x^3 f^* w_2 + x^2 f^* w_3 + x f^* w_4) + \operatorname{Im} d_2 \in E_5^{6,6} = H^6(X; Z^T[f^* w_1])/\operatorname{Im} d_3$$

for all $x \in H^1(X; Z_2)$.

Proof. same as (4.4.4).

4.5. We are now ready to classify real vector bundles over P_k, for $k \leqq 5$.

DEFINITION (4.5.1). A locally oriented real n-dimensional vector bundle over a space X shall be a b.p.p. homotopy class of maps from X to $BO(n)$. If $f: X \to BO(n)$ represents a locally oriented v.b. ξ, let $\sim \xi$, or ξ conjugate, be that locally oriented v.b. given by a map $g: X \to BO(n)$ which is connected to f via a free homotopy which sends the base-point of X around a nontrivial loop of $BO(n)$. Obviously $\sim \xi \cong \xi$, and conjugate classes of locally oriented vector bundles correspond to equivalence classes of vector bundles.

TABLE (4.5.2). For $k \geqq 1$, let $h: P_k \to BO(1)$ be the canonical line bundle. Let "$\oplus$" denote Whitney sum. We give a complete list of all locally oriented real n-dimensional vector bundles over P_k, each n and k; all bundles are self-conjugate unless otherwise specified.

Let G denote $(q_1)_*^{-1} t(P_4) = \frac{1}{2}(i_1)_* t(P_4)$ which generates

$$H^4(P_4; (Z + Z)^T[u]) \,.$$

Also $(p_2^*)^{-1} u^5$ generates $H^5(P_5; (Z_2 + Z_2)^T[u])$. Locally oriented real n-dimensional vector bundles over P_k, for $n - 1 \leqq k \leqq 5$:

Over P_1		Over P_2		
1	2	1	2	3
h	$h \oplus 1$	h	$T_p = (h \oplus 1) + pt(P_2)$, for all $p \in Z$; stable class $h + 1$ if p even, $3h - 1$ if p odd; $\sim T_p = T_{-p}$.	$h \oplus 2$
			$2h = 2 + \bar{u}^2$	$2h \oplus 1 = 3 + u^2$
				$3h = (h \oplus 2) + u^2$

Over P_3				Over P_4				
1	2	3	4	1	2	$3=3+\bar{u}^4$	$4=4+(\bar{u}^4, 0)$	5
h	$h\oplus 1$	$h\oplus 2$	$h\oplus 3$	h	$h\oplus 1$	$h\oplus 2$	$2h\oplus 2$	$h\oplus 4$
	$2h$	$2h\oplus 1$	$2h\oplus 2$		$2h$	$(h\oplus 2)+\bar{u}^4$	$2h\oplus 2+(\bar{u}^4, 0)$	$2h\oplus 3$
		$3h$	$3h\oplus 1$			$2h\oplus 1$	$4h=4+(0,\bar{u}^4)=4h+(\bar{u}^4,0)$	$3h\oplus 2$
						$(2h\oplus 1)+\bar{u}^4$	$2h\oplus 2+(0, \bar{u}^4)$; stable class $6h-2$	$4h\oplus 1$
						$3h=3h+\bar{u}^4$	$2h\oplus 2+(\bar{u}^4, \bar{u}^4)=$ $\sim(2h\oplus 2+(0, \bar{u}^4))$	$5h$
							$E_p=h\oplus 3+pG$ for all $p\in Z$; stable class $h+3$ if p even, $5h-1$ if p odd; $\sim E_p=E_{-p}$	$((2h\oplus 2)+(0, \bar{u}^4))\oplus 1$; stably $6h-1$
							$F_p=3h\oplus 1+pG$ for all $p\in Z$; stable class $3h+1$ if p even, $7h-3$ if p odd; $\sim F_p=F_{-p}$	$F_1\oplus 1$; stable class $7h-2$

Over P_5					
1	2	3	4	$5=5+u^5$	6
h	$h\oplus 1$	$3+u^5$	$4+(u^5, 0)$	$h\oplus 4$	$h\oplus 5$
	$2h$	$h\oplus 2$	$4+(0, u^5)$	$h\oplus 4+u^5$	$2h\oplus 4$
		$h\oplus 2+u^5$	$4+(u^5, u^5)=\sim(4+(0, u^5))$	$2h\oplus 3$	$3h\oplus 3$
		$A=A+u^5$;	$h\oplus 3$	$2h\oplus 3+u^5$	$4h\oplus 2$
		$A\|P_4=h\oplus 2+\bar{u}^4$	$h\oplus 3+(p_2^*)^{-1}u_5$	$3h\oplus 2$	
		$2h\oplus 1$	$2h\oplus 2$	$3h\oplus 2+u^5$	$5h\oplus 1$
		$2h\oplus 1+u^5$	$2h\oplus 2+(u^5, 0)$	$4h\oplus 1$	$6h$
		$B=B+u^5$;	$2h\oplus 2+(0, u^5)$	$4h\oplus 1+u^5$	$C\oplus h\oplus 1$
		$B\|P_4=2h\oplus 1+\bar{u}^4$	$2h\oplus 2+(u^5, u^5)=\sim(2h\oplus 2+(0,u^5))$	$5h=5h+u^5$	
		$3h$	$B\oplus 1=B\oplus 1+(u^5, 0)$	$C\oplus 1=C\oplus 1+u^5$	
		$3h+u^5$	$B\oplus 1+(0, u^5)=B\oplus 1+(u^5, u^5)$	$C\oplus h=C\oplus h+u^5$	
			$3h\oplus 1$		
			$3h\oplus 1+(p_2^*)^{-1}u_5$		
			$4h$		
			$4h+(u^5, 0)$		
			$4h+(0, u^5)$		
			$4h+(u^5, u^5)=\sim(4h+(0, u^5))$		
			$C=C+(0,u^5); C\|P_4=2h\oplus 2+(0,\bar{u}^4)$		
			$D=D+(0, u^5)=\sim C$		
			$C+(u^5, 0)=C+(u^5, u^5)$		
			$D+(u^5, 0)=$ $\sim(C+(u^5, 0))=D+(u^5, u^5)$		

4.6. Similarly, we can classify all complex vector bundles over P_k, for $k \leqq 5$. We give a table of homotopy groups:

	$BU(1)$	$BU(2)$	$BU(n)$ for $3 \leqq n \leqq \infty$
π_1	0	0	0
π_2	Z	Z	Z
π_3	0	0	0
π_4	Z	Z	Z
π_5	0	Z_2	0

The only nonzero k-invariant in this range is k^6 of $BU(2)$, which is $\Pi_*(c_1c_2) + Sq^2\Pi_*c_2$, where $c_i \in H^{2i}(BU(2); Z)$ are the Chern classes. We thus have:

REMARK (4.6.1). For any space X, all complex line bundles over X correspond to $H^2(X; Z)$.

REMARK (4.6.2). For any space X of dimension $\leqq 5$, all complex n-bundles, for $n \geqq 3$, over X correspond to $KU(X)$, satisfying the exact sequence $0 \to H^4(X; Z) \to KU(X) \to H^2(X; Z) \to 0$.

REMARK (4.6.3). If $f: X \to (BU(2))_5$ is a map, then

$$d_2(x) = \Pi_*(c_1x) + Sq^2\Pi_*x \in H^5(X; Z_2)$$

for all $x \in H^3(X; Z)$; $d_3 = 0$; $d_4(x) = \Pi_*(f^*c_2 \cup x) + \operatorname{Im} d_2$ for all

$$x \in H^1(X; Z) .$$

Proof. Let $S: S^2 = \Sigma K(Z, 1) \to BU$ be the generator of $\pi_2(BU)$; then $S^*c_1 = \sigma$, the fundamental class of S^2, and $S^*c_2 = 0$. The result follows just as in (4.4.4).

TABLE (4.6.4). We summarize complex n-bundles over P_k, $2n - 1 \leqq k \leqq 5$. The reallification is given in square brackets.

Over P_2				Over P_3			
1	[2]	2	[4]	1	[4]	2	[4]
H	$[2h]$	$H \oplus 1 = 2 + u^2$	$[2h \oplus 2]$	H	$[2h]$	$H \oplus 1$	$[2h \oplus 2]$

Over P_4					
1	[2]	2	[4]	3	[6]
H	$[2h]$	$H \oplus 1$	$[2h \oplus 2]$	$H \oplus 2$	$[2h \oplus 4]$
		$2H = 2 + \bar{u}^4$	$[4h]$	$2H \oplus 1 = 3 + \bar{u}^4$	$[4h \oplus 2]$
		$H \oplus 1 + \bar{u}^4$	$[2h \oplus 2 + (\bar{u}^4, 0)]$	$3H = H \oplus 2 + \bar{u}^4$	$[6h]$
		Stable class $3H - 1$			

Over P_5			
1	$[2]$	2	$[4]$
H	$[2h]$	$2 + u^5$	$[4 + (0, u^5)]$
		$H \oplus 1$	$[2h \oplus 2]$
		$H \oplus 1 \oplus u^5$	$[2h \oplus 2 + (0, u^5)]$
		$2H$	$[4h]$
		$2H + u^5$	$[4h + (0, u^5)]$
		C	$[C]$
		$C + u^5$	$[C]$

4.7. We give a few representative examples of evaluating those difficult differentials. Is $f: P_5 \to (BO_4)_5$ is a map representing a 4-plane bundle ξ, then $d_4^f(u)$ is defined if and only if

$$d_2^f(u) = (j_1)_*\beta(u^3 + uf^*w_2) = 0 \in H^4(P_5; (Z + Z)^T[f^*w_1]) .$$

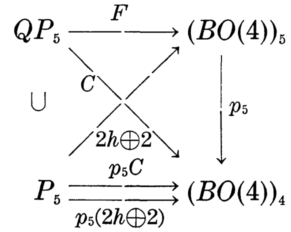
If $d_2(u) = 0$, then $d_4^f(u) = 0$ if and only if there is a map $F: QP_5 \to (BO_4)_5$ which represents a homotopy of f with itself, such that $F^*w_2 = qf^*w_2 + Qu$, where QX is as given in [4; 5].

EXAMPLE (4.7.1). If $\xi = 4$ or $4h$, then $f^*w_2 = 0$, so $d_2(u) = (\bar{u}^4, 0)$ and $d_4(u)$ is not defined. Thus $4, 4 + (u^5, 0), 4 + (0, u^5)$, and $4 + (u^5, u^5)$ are all distinct oriented vector bundles.

EXAMPLE (4.7.2). If $\xi = 2h \oplus 2$, then $f^*w_2 = u^2$, so $d_2(u) = 0$.

Let η_1 be that line bundle over QP_5 such that $w_1(\eta_1) = qu$; now 2-plane bundles over a space X with $w_1 = x$ are classified by $H^2(X; Z^T[x])$; let η_2 be that 2-plane bundle over QP_5 with $w_1(\eta_2) = qu$ classified by $Q\bar{u}$. Then $w_2(\eta_2) = Qu$. Let $c: QP_5 \to BO(4)$ be the classifying map of $\eta_1 \oplus \eta_2 \oplus 1$; $c^*w_2 = qu^2 + Qu$ and $(\eta_1 \oplus \eta_2 \oplus 1) \mid P_5 = 2h \oplus 2$. Thus F, the projection of c onto $(BO(4))_5$, and $d_4^f(u) = 0$.

EXAMPLE (4.7.3). If $\xi = C$, then $f^*w_2 = u^2$, so $d_2^f(u) = 0$, and $d_4^f(u)$ is defined. Now $p_5C = p_5(2h \oplus 2) + (0, \bar{u}^4)$,



and so $d_4(u) = 0$ if and only if we can lift the map

$$p_5F + q(0, \bar{u}^4): Qp_5 \longrightarrow ((BO(4))_4$$

to $(BO(4))_5$, where F is the map given in (4.7.2). Now the k-invariant k^6 is based on the relation $S_q^2\Pi_*k^5 + w_2 \cup \Pi_*k^5 = 0$, and $(p_5F)^*k^6 = 0$, so $(p_5F + a)^*k^6 = S_q^2\Pi_*a + (p_5F)^*w_2 \cup \Pi_*a$ which, when $a = q(0, \bar{u}^4)$, equals $S_q^2q(0, u^4) + (qu^2 + Qu) \cup q(0, u^4) = Q(0, u^5)$. So, by [4; 5.2], $d_4(u) = (0, u^5)$. Thus $C + (0, u^5) = C$, but $C + (u^5, 0)$ is different. We also have that there are two complex structures on C, because since C is the reallification of the complex bundle C, $C = C + (0, u^5)$ is the reallification of $C + u^5$.

4.8. We would like to know how vector bundles behave under tensor products. If L is any line bundle over any space, $L \otimes L = 1$. Furthermore:

REMARK (4.8.1). If η_1 and η_2 are locally oriented real n-plane bundles over a space X, which agree on X^{k-1}, and if ξ is a locally oriented real m-plane bundle over X, then $i_*\delta^k(\eta_1, \eta_2) = \delta^k(\eta_1 \oplus \xi, \eta_1 \oplus \xi)$ and $j_*\delta^k(\eta_1, \eta_2) = d^k(\eta_1 \otimes \xi, \eta_2 \otimes \xi)$, where $i: BO(n) \to BO(n + m)$ and $j: BO(n) \subset BO(nm)$ are the maps induced by the inclusion of $O(n)$ in $O(n + m)$ and $O(nm)$. Similarly for complex vector bundles.

REMARK (4.8.2). If ξ is an oriented real vector bundle which has a complex structure, and if η is any other locally oriented real vector bundle, then $\xi \otimes \eta$ also has a complex structure.

Proof. Let $C(\eta)$ be the complexification of η, and let ξ' be a complex bundle whose reallification is ξ. Then we can see routinely that the reallification of $\xi' \otimes C(\eta)$ is $\xi \otimes \eta$.

With the above information, we can almost completely determine the action of "$\oplus$" and "$\otimes$" on all locally oriented real vector bundles over P_k, $k \leqq 5$. For example,

$$A \otimes h = B, C \otimes h = C, 4 \otimes h = 4h, (4 + (0, u^5)) \otimes h = 4h + (0, u^5),$$
$$T_p \otimes h = T_p, E_p \otimes h = F_p, (4h + (u^5, u^5)) \oplus 1 = 4h \oplus 1 + u^5 .$$

The only unsolved questions are whether $A \oplus h = B \oplus 1$; it is also possible that $A \oplus h = B \oplus 1 + (0, u^5)$; and whether $B \oplus 2$ equals $2h \oplus 3$ or $2h \oplus 3 + u^5$.

BIBLIOGRAPHY

1. G. E. Bredon, *Sheaf theory*, McGraw Hill, 1967.
2. A. Dold & H. Whitney, *Classification of oriented sphere bundles over a 4-complex*, Ann. of Math. **69** (1959).

3. M. A. Kervaire, *Some nonstable homotopy groups of Lie groups*, Illinois J. Math. **4** (1960), 161-169.
4. L. L. Larmore, *Map-cohomology operations and enumeration of vector bundles*, J. Math. Mech. **17** (1967), 199-208.
5. H. Samelson, *A note on the bokstein operator*, Proc. Amer. Math. Soc. **15** (1964), 450-453.

Received January 20, 1968, and in revised form January 27, 1969. This research was supported by N.S.F. grant GP-6560.

OCCIDENTAL COLLEGE AND
UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE