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AN EMBEDDING THEOREM FOR LATTICE-ORDERED FIELDS

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In this paper we develop a method for constructing lattice-ordered fields ("$\mathscr{L}$-fields") which are not totally ordered ("o-fields") and hence are not f-rings. We show that many of these fields admit a Hahn type embedding into a field of formal power series with real coefficients. In order to establish such an embedding we make use of the valuation theory for abelian $\mathscr{L}$-groups and prove the "well known" fact that each o-field can be embedded in an o-field of formal power series.

Let G be an $\mathscr{L}$-field that contains n disjoint elements, but not $n+1$ such elements. An element $0 < s \in G$ is *special* if there is a unique $\mathscr{L}$-ideal of $(G, +)$ that is maximal without containing s. We show that the set S of special elements of G form a multiplicative group if and only if $S \neq \emptyset$ and $s^{-1} > 0$ for each $s \in S$. If this is the case, then there is a natural mapping of S onto the set Γ of all values of the elements of G. Thus Γ is a po-group and if, in addition, Γ is torsion free, then there exists an $\mathscr{L}$-isomorphism of G into the $\mathscr{L}$-field $V(\Gamma, R)$ of all functions v of Γ into the real field R whose support $\{\gamma \in \Gamma \mid v(\gamma) \neq 0\}$ satisfies the ascending chain condition. If G is an o-field, then the above hypotheses are satisfied and hence the embedding theorem for o-fields is a special case of our embedding theorem. The authors wish to thank the referee for many constructive suggestions.

NOTATION. If S is a subset of a group G, then $[S]$ will denote the subgroup of G that is generated by S. If G is a po-group, then G^+ will denote the set $\{g \in G \mid g \geqq 0\}$ of positive elements. A *disjoint* subset of an $\mathscr{L}$-group G is a set S of strictly positive elements such that $a \wedge b = 0$ for all pairs $a, b \in S$.

2. A method for constructing lattice-ordered rings.

A po-set Γ is called a *root system* if for each $\gamma \in \Gamma$, the set $\{\alpha \in \Gamma \mid \alpha \geqq \gamma\}$ is totally ordered. A nonvoid subset Δ of a root system Γ is called a *W-set* if it is the join of a finite number of inversely well ordered subsets of Γ, and an *I*-set if it is infinite and trivially ordered or well ordered with order type ω. In [2] it is shown that Δ is a W-set if and only if Δ does not contain an I-set; while in [10] five other conditions are derived which are equivalent to Δ not containing an I-set.

If Γ is a root system and if $v: \Gamma \rightarrow R$ is a function into the real field R, then the support of v is defined as $\operatorname{supp} v = \{\gamma \in \Gamma \mid v(\gamma) \neq 0\}$. The set $V = V(\Gamma, R)$ of all v whose support satisfies the ascending chain condition (A.C.C.) is a *po*-group if one defines v to be positive if $v(\gamma) > 0$ for each maximal element γ in supp v. Such a $v(\gamma)$ will be referred to as a *maximal component* of v. In [5] it is shown that V is an $\mathscr{L}$-group for an arbitray *po*-set Γ if and only if Γ is a root system. For a root system Γ

$$W = W(\Gamma, R) = \{v \in V(\Gamma, R) \mid \operatorname{supp} v \text{ is a } W\text{-set}\}$$

is an $\mathscr{L}$-subgroup of V.

Now suppose that the root system Γ is also a *strictly po-semigroup*:

$$\alpha < \beta \rightarrow \alpha + \gamma < \beta + \gamma \quad \text{and} \quad \gamma + \alpha < \gamma + \beta$$

for all $\alpha, \beta, \gamma \in \Gamma$. For $u, v \in W$ define $uv \in W$ by

$$(uv)(\gamma) = \sum_{\alpha+\beta=\gamma} u(\alpha)v(\beta) .$$

Then W is a ring (see [2], p. 76, or [10], p. 333). If $0 < u, v \in W$, then $0 < uv$ and so W is an $\mathscr{L}$-ring and also a real vector lattice. If Γ is an *o*-group, then $V = W$ is a totally ordered division ring (see [8], p. 137]). Throughout, a "field" is always commutative while a "division ring" is not necessarily commutative.

In § 6, there are two examples of strictly *po*-semigroups which are root systems and hence can be used to construct $\mathscr{L}$-rings. Although it does not appear likely that all such semigroups can be reasonably characterized, the next lemma completely characterizes all *po*-groups which are also root systems.

LEMMA 2.1. *Suppose that a group Γ has a totally ordered subgroup H with positive cone H^+. If $H^+ \triangleleft \Gamma$, then Γ with this positive cone H^+ is a po-group and a root system. Conversely, each po-group that is a root system is of this form.*

Proof. Clearly, Γ is just the join of disjoint totally ordered cosets and so in this partial order Γ becomes a root system and a *po*-group. Conversely, suppose that Γ is a *po*-group and a root system. Let $[\Gamma^+]$ be the subgroup of Γ generated by its positive cone Γ^+. Then $H = [\Gamma^+] \triangleleft \Gamma$ is a directed *po*-group. If H were not an *o*-group, then there would exist $\alpha, \beta, \gamma \in H$ such that $\alpha \geqq \beta, \alpha \geqq \gamma$ and such that β and γ are not comparable (notation $\beta \parallel \gamma$). But then $-\beta \parallel -\gamma$, and $-\beta, -\gamma \in \{\delta \in \Gamma \mid \delta \geqq -\alpha\}$ which contradicts the fact that Γ is a root system.

Now let Γ be a *po*-group and a root system and suppose that $H = [\Gamma^+] \lhd \Gamma$ is the unique totally ordered normal subgroup such that Γ is the disjoint union of totally ordered cosets of H. It is well known that if Γ is abelian and torsion free then the given partial order can be extended in a not necessarily unique way to yield a totally ordered group. The latter may fail for nonabelian groups. However, if Γ is torsion free with $H \lhd \Gamma$ and Γ/H finite, then the given total order on H can be extended uniquely to a total group order on Γ (see [14], p. 326). The hypothesis that Γ/H is finite can in fact be weakened to require merely that any finite set of elements of Γ/H generate a finite subgroup (see [14], p. 325).

PROPOSITION 2.2. *Suppose that Γ is a torsion free po-group, and $H = [\Gamma^+]$ is a totally ordered subgroup with Γ/H finite. Then $W(\Gamma, R) = V(\Gamma, R)$ is a lattice ordered division ring. Moreover, the lattice order of $V(\Gamma, R)$ can be extended to a total ring order on $V(\Gamma, R)$.*

Proof. Let Γ_1 be the totally ordered group having the same underlying set of elements as Γ given by the unique extension of the partial order of Γ to a total one. As has already been remarked ([8], p. 137), $V(\Gamma_1, R)$ is a totally ordered division ring. Since the support of $v \in V(\Gamma, R)$ is the join of a finite number of inversely well ordered sets in Γ, when $\operatorname{supp} v$ is viewed as a subset of Γ_1, it will satisfy the A.C.C. Thus $v \in V(\Gamma_1, R)$ and $V(\Gamma, R) \subseteqq V(\Gamma_1, R)$. Clearly, $V(\Gamma_1, R) \subseteqq V(\Gamma, R)$. Since $V(\Gamma, R) = V(\Gamma_1, R)$ as sets, the lattice order of $V(\Gamma, R)$ can be extended to a total order.

COROLLARY. *In the previous proposition $V(\Gamma, R)$ satisfies the following three conditions:*

(i) *$V(\Gamma, R)$ contains n pairwise disjoint elements but not $n + 1$ such elements.*

(ii) *If $0 < v \in V(\Gamma, R)$ has just one maximal component (such a v is called special), then so does its inverse. All the special elements form a multiplicative group.*

(iii) *The multiplicative group of special elements is torsion free.*

In § 4, we show that, conversely, an $\mathscr{L}$-field with these three properties can be embedded in $V(\Gamma, R)$.

3. Special elements in an $\mathscr{L}$-ring. In order to obtain an embedding theorem for an $\mathscr{L}$-field G, we assume that the special elements in G form a multiplicative group. In this section we investigate what this hypothesis means. In particular, we show that such special

elements behave like elements in f-rings in that they distribute over joins and intersections.

Let G be an abelian $\mathscr{L}$-group. A convex subgroup of G which is also sublattice is called an *$\mathscr{L}$-ideal*. An $\mathscr{L}$-ideal L of G is called *regular* if it is maximal with respect to not containing some element $g \in G$. If this is the case, then G/L is an o-group (see [4] or [5]) and hence there exists a unique $\mathscr{L}$-ideal that covers L. Let $\Gamma = \Gamma(G)$ be the set of all pairs of $\mathscr{L}$-ideals (G^γ, G_γ) such that G_γ is regular and G^γ covers G_γ. We shall frequently identify Γ with the set of pairs (G^γ, G_γ). In particular, define $\alpha < \beta$ in Γ if $G^\alpha \subseteqq G_\beta$. Then $(\Gamma, \leqq)$ is a root system. If $g \in G^\gamma \backslash G_\gamma$, then we say that γ is a *value* of g. If $0 < g$ has exactly one value, then g is called *special* and in this case its unique value will be denoted by $v(g)$. If $g \in G$ has exactly one value then g is comparable with zero and so either g or $-g$ is special. If $a, b, \in G$ are special, then $a \wedge b = 0$ if and only if $v(a) \| v(b)$. If L is an $\mathscr{L}$-ideal of G such that G/L is an o-group and $0 < g \in G \backslash L$ implies that $g > L$, then G is called a *lex-extension* of L. It follows that each coset $L \neq L + x$ consists entirely of positive elements or entirely of negative elements. If a and b are positive elements of an $\mathscr{L}$-ring G, then $a \ll b$ will mean that $na < b$ for all integers $n > 0$. If $a \ll b, c > 0$, and $bc \neq 0$, then $nac < bc$ for all n and so $ac \ll bc$.

3.1. In [4] it is shown that for $0 < g \in G$, the following are equivalent:

(1) g is special;

(2) $G(g) = \{z \in G \mid |z| \leqq ng$ for some integer $n > 0\}$ has exactly one maximal $\mathscr{L}$-ideal;

(3) $G(g)$ is a lex-extension of a proper $\mathscr{L}$-ideal L.

Consequently, if a is special and L is the unique maximal $\mathscr{L}$-ideal of $G(a)$, then $G(a)/L$ is an archimedian o-group and $G(a)$ is a lex-extension of L.

LEMMA 3.2. *If G is an abelian $\mathscr{L}$-group and $0 < g \in G$, then $Tg = \{z \in G \mid 0 \leqq z \ll g\}$ is a convex semigroup that contains* 0 *but not g and so $[Tg] = \{y - z \mid y, z \in Tg\}$ is an $\mathscr{L}$-ideal of G and $[Tg]^+ = Tg$.*

Proof. By Theorem 11 on page 81 in [8] it suffices to show that Tg is a semigroup. But this is well known for o-groups, and since G is a subdirect sum of o-groups, it follows that Tg is a semigroup.

COROLLARY. *$[Tg]$ is the largest (proper) $\mathscr{L}$-ideal of $G(g)$ if and only if g is special.*

Proof. If $[Tg]$ is the largest $\mathscr{L}$-ideal of $G(g)$, then g is special by 3.1 (2). Conversely, suppose that g is special and let L be the largest

$\mathscr{L}$-ideal of $G(g)$. Since $g \notin [Tg]$ it follows that $[Tg] \subseteqq L$ and since $nL^+ \subseteqq L^+ < g$ for all positive integers $n, L \subseteqq [Tg] = L$.

LEMMA 3.3. *Suppose that a and b are special elements in an $\mathscr{L}$-ring G with an identity and that a^{-1} and b^{-1} exist.*

(i) *If $a^{-1} \in G^+$, then a^{-1} is special.*

(ii) *If $a^{-1}, b^{-1} \in G^+$, then $Tab = TaG(b)^+ = G(a)^+Tb$ and $[Tab]$ is the largest $\mathscr{L}$-ideal in $G(ab)$. Thus ab is special.*

Proof. (i) Let L be a proper $\mathscr{L}$-ideal of $G(a^{-1})$ and consider $0 < q \in L$. Since $q < na^{-1}$ for some $n > 0$, we have $qa^2 < na$ and so $qa^2 \in G(a)$. If $qa^2 \notin Ta$, then since $G(a)/[Ta]$ is an archimedian o-group, $[Ta] + nqa^2 > [Ta] + a$ for some $n > 0$. Then since $G(a)$ is a lex-extension of $[Ta]$, $nqa^2 > a$ and so $nq > a^{-1}$. But then $L \supseteqq G(a^{-1})$, a contradiction. Thus $qa^2 \in Ta$ and so $qa^2 \ll a$, and hence $q \ll a^{-1}$. Therefore $L^+ \subseteqq Ta^{-1}$ and hence by the above corollary a^{-1} is special.

(ii) If $x \in Ta$ and $y \in G(b)^+$, then $knx < a$ and $y < kb$ for some $k > 0$ and all $n > 0$. Thus $knxy \leqq ay \leqq kab$ and hence $nxy \leqq ab$ for all n, and since $ab \neq 0, nxy < ab$ for all n. Thus $TaG(b)^+ \subseteqq Tab$. If $z \in Tab$, then $z \ll ab$ and $zb^{-1} \ll a$. Then $z = (zb^{-1})b \in TaG(b)^+$. Therefore $Tab = TaG(b)^+$ and similarly $Tab = G(a)^+Tb$.

Now suppose that L is a proper $\mathscr{L}$-ideal of $G(ab)$ and $0 < q \in L$. Since $q < nab$ for some n, $qb^{-1} < na$ shows that $qb^{-1} \in G(a)$. If $qb^{-1} \notin Ta$, then as above $mqb^{-1} > a$ *for some* $m > 0$ and so $L \supseteqq G(ab)$, a contradiction. Thus $qb^{-1} \in Ta$ and hence $q = (qb^{-1})b \in TaG(b)^+ = Tab$. Hence $L^+ \subseteqq Tab$ and ab is special.

Conditions (4) and (5) in the next theorem show that special elements behave like elements from an f-ring. A commutative $\mathscr{L}$-field is totally ordered if and only if the positive cone is closed under division (see [8], p. 139). This is one reason for putting requirements on the special elements, rather than on all the positive elements.

THEOREM I. *For a lattice ordered division ring G with an identity the following are equivalent.*

(1) *The special elements form a multiplicative group or the null set.*

(2) *If a is special, then $a^{-1} > 0$.*

(3) *If a is special, then a^{-1} is special.*

(4) $a(c \vee 0) = ac \vee 0$ *for special elements a and all $c \in G$.*

(5) $a(y \vee z) = ay \vee az$ *for special elments a and all $y, z \in G$. Six additional conditions each equivalent to* (4) *and* (5) *are obtained by writing $(c \vee 0)a = ca \vee 0$, $(y \vee z)a = ya \vee za$, and by replacing "$\vee$" with "$\wedge$".*

Proof. The implications (1) $\rightarrow$ (3) $\rightarrow$ (2) and (5) $\rightarrow$ (4) are trivial and (2) $\rightarrow$ (1) follows from Lemma 3.3.

(2) $\rightarrow$ (4). Since a and a^{-1} are both positive, the left multiplications by a and a^{-1} are inverse order preserving mappings and hence are lattice automorphisms.

(4) $\rightarrow$ (2). If a is special, then $a(1 \vee 0) = a \vee 0 = a$ and so $1 \vee 0 = 1$. Thus $a(a^{-1} \vee 0) = 1 \vee 0 = 1$ and hence $a^{-1} = a^{-1} \vee 0 > 0$.

(4) $\rightarrow$ (5).

$$\begin{aligned} a(y \vee z) &= a[((y - z) \vee 0) + z] \\ &= ((ay - az) \vee 0) + az = ay \vee az \, . \end{aligned}$$

The equation

$$\begin{aligned} a(y \wedge z) &= a(-(-y \vee -z)) = -(a(-y \vee -z)) \\ &= -(-ay \vee -az) = ay \wedge az \end{aligned}$$

shows that "$\vee$" may be replaced by "$\wedge$" throughout. Finally, each of the above arguments applies equally well to $(c \vee 0)a$, $(y \vee z)a$, $(c \wedge 0)a$, and $(y \wedge z)a$.

Suppose that each element in the lattice ordered division ring G has at most a finite number of values and that the special elements in G form a multiplicative group S. Then each $\gamma \in \Gamma$ is the value of a special element (see [4], p. 118) and the map v of $s \in S$ onto its value $v(s)$ is an o-homomorphism of S onto Γ. In particular, Γ is a partially ordered group and of course a root system.

PROPOSITION 3.4. *If G is a finite valued $\mathscr{L}$-field, i.e., each element has only a finite number of values, and if the special elements of G form a group and the associated value group Γ of G is torsion free, then the order of G can be extended to a total order.*

Proof. Extend the partial order of Γ to a total order. An element $0 \neq g \in G$ has a unique representation $g = g_1 + \cdots + g_n$ where each g_i or $-g_i$ is special and $|g_i| \wedge |g_j| = 0$ if $i \neq j$ (see [4]). One of the $v(g_i)$ will be the largest in the total ordering of Γ, say $\gamma = v(g_j)$. Define g to be positive if $G_\gamma + g > G_\gamma$. Clearly this is a total order of the set G that extends the given lattice order and a straightforward computation shows that G is an o-field.

An element $0 < b$ of an $\mathscr{L}$-group G is *basic* if $\{g \in G \mid 0 \leqq g \leqq b\}$ is totally ordered. A *basis* for G is a maximal pairwise disjoint subset of G which, in addition, consists of basic elements. G has a finite basis if there exists a basis consiting of n elements or equivalently if G contains n disjoint elements but not $n + 1$ such elements. For a structure theorem for a group with a finite basis see ([8], p. 86).

If G is a lattice ordered division ring with a finite basis, if the special elements form a group and if $\Gamma(G)$ is torsion free, then there exists an extension of the lattice order of G to a total order of G. The proof of this fact is the same as the proof of the last proposition.

4. An embedding theorem for *o*-fields. In this section it is shown that an arbitrary totally ordered field F can be embedded in the *o*-field $V(\Gamma(F), R)$. Only the statement of this embedding theorem and not the method of proof will be used in subsequent sections. The proof assumes some familiarity with the valuation theory of fields.

Let F be an *o*-field and F^* be the multiplcative group of all strictly positive elements of F. Then F^* is the set of all special elements, and the mapping v of $f \in F^*$ upon its value $v(f)$ in $\Gamma = \Gamma(F)$ is an *o*-homomorphism. Thus Γ may be regarded as an additive *o*-group with identity θ, and v is the natural order valuation of F (see [1] or [11]). Note that $1 \in F^{\theta} \backslash F_{\theta}$, F^{θ} is the *valuation ring* of F, and F^{θ}/F_{θ} is the residue class field. Also, F^{θ}/F_{θ} is an archimedian o-field and hence essentially a subfield of the real numbers. As before, $V = V(\Gamma, R)$ is the *o*-field of formal power series with exponents from Γ and with real coefficients. For $\gamma \in \Gamma$, let x^{γ} be the element in V such that

$$x^{\gamma}(\alpha) = \begin{cases} 1 \text{ if } \alpha = \gamma \\ 0 \text{ otherwise} . \end{cases}$$

Note that $x^{\theta} = 1$. Although $V(\Gamma, R)$ in general contains several *o*-isomorphic copies of the reals, it contains Rx^{θ} as a distinguished copy, and $V(\Gamma, R)$ is an *o*-algebra over the reals under component-wise multiplication by Rx^{θ}.

Let E be a not necessarily ordered division ring with a valuation $w: E\backslash\{0\} \rightarrow \Gamma(E)$ in the sense of [16] except with the order of $\Gamma(E)$ reversed. Thus in case E is ordered w would be an order preserving map. If $E \subset D$ where D is another valuated division ring whose valuation extends w, then D is called an *immediate extension* of (E, w) provided the value group of E, that is $w(E)$, is also the value group of D, and if the residue class fields of E and D are isomorphic. By Zorn's lemma, every (E, w) has a maximal immediate extension.

THEOREM II. (i) *If F is an o-field with value group Γ, then there exists a value and order preserving isomorphism π of F into the o-field $V(\Gamma, R)$.* (ii) *Moreover, if $\Delta \subset \Gamma$ is a rationally independent basis for the divisible hull of Γ, and for each $\delta \in \Delta$, $0 < x_{\delta} \in F$ is arbitrary with value δ, then π can be chosen so that $x_{\delta}\pi = x^{\delta}$.* (iii) *Now assume in addition that $\bar{R} \subset F$ is any o-isomorphic copy* *of the reals and* $r \to \bar{r}$ *is the unique o-isomorphism of* R *onto* $\bar{R}$. *Then in addition to satisfying* (ii), π *can be so chosen that* $\bar{r}\pi = rx^{\theta}$.

Proof. We only outline a proof in the sense that [13] and [16] are quoted for all the difficult steps (also see [1], p. 328). By [13], any totally ordered field F can be embedded in a totally ordered field E so that the order induced on F from $F \subset E$ is the orginal order of F, both E and F have the same value group $\Gamma = \Gamma(F) = \Gamma(E)$, and E contains an isomorphic copy of the reals, i.e., $R \cong \bar{R} \subset E$. Since $v(1) = \theta$, necessarily, $\bar{R}\backslash\{0\} \subsetneqq E^{\theta}\backslash E_{\theta}$ and also $E^{\theta}/E_{\theta} \cong R$. The reader should recall that the real field R has no nontrivial automorphisms, since R admits exactly one total order. Let $r \to \bar{r}$ denote the o-isomorphism of R onto $\bar{R}$. The field E with the natural order valuation $v\colon E/\{0\} \to \Gamma$ has a maximal immediate extension $E \subsetneqq M$. Denote the valuation on M also by v. We define $d \in M$ with value γ to be positive if $M_{\theta} + df^{-1}$ is positive in M^{θ}/M_{θ} for some $0 < f \in F$ with value γ. This is the unique extension of the order of E to M. Let $M^* = \{d \mid 0 < d \in M\}$.

It will be shown next that the subgroup

$$M^* \cap (M^{\theta}\backslash M_{\theta}) = \{0 < d \in M \mid v(d) = \theta\}$$

is divisible. If $0 < d \in M$ with $v(d) = \theta$, define $\bar{c} \in \bar{R}$ by $\bar{c} = \inf\{\bar{r} \mid d < r1\}$. Then $v(d - \bar{c}1) < \theta$ and $d = \bar{c}(1 + \lambda)$, $\lambda = (1/\bar{c})d - 1$ with $v(\lambda) < \theta$. If $m > 1$ is any integer, then in order to show that $d^{1/m} \in M^* \cap (M^{\theta}\backslash M_{\theta})$ take $\bar{c} = 1$ and define $p_n \in M^* \cap (M^{\theta}\backslash M_{\theta})$ by taking terms up to λ^n from the formal power series expansion of $(1 + \lambda)^{1/m}$. Then $\{p_n \mid n = 1, 2, \cdots\}$ defines a so called pseudo convergent sequence (see [16], p. 39]). If this sequence has a pseudo limit ([16], p. 47), then that limit is $d^{1/m}$. However, by ([16], p. 51, Th. 8), M contains a pseudo limit for each of its pseudo convergent sequences. Thus $d^{1/m} \in M^* \cap (M^{\theta}\backslash M_{\theta})$ and hence M^* splits, $M^* = T \times M^* \cap (M^{\theta}\backslash M_{\theta})$, where T is some complement of $M^* \cap (M^{\theta}\backslash M_{\theta})$. For $t \in T$ define $t\,\pi = x^{v(t)}$ and for $\bar{r} \in \bar{R}$ define $\bar{r}\pi = rx^{\theta}$. Then this determines a value and order preserving isomorphism π of the subfield K of M that is generated by $\bar{R} \cup T$ into V. Moreover, M and V are maximal immediate extensions of K and $K\pi$ respectively. By ([16], p. 222, Th. 4), π can be extended to a value preserving isomorphism of M onto V so that the following diagram commutes:

$$\begin{array}{ccc} K & \longrightarrow & M \\ \Big\downarrow & & \cong\Big\downarrow \pi \\ K\pi & \longrightarrow & V(\Gamma, R)\,. \end{array}$$

It is asserted that $\pi\colon M \to V$ preserves order. Since each element of M^{θ} is congruent modulo M_{θ} to an element of the form $\bar{r}$ with $r \in R$, and since $\bar{r}\pi = rx^\theta$, it follows that π induces an order preserving isomorphism $\pi^\theta\colon M^\theta/M_\theta \to V^\theta/V_\theta = R$. But $d \in M$ is positive by definition, provided for any $0 < k \in K$ with $v(k) = v(d)$, we have $M_\theta < M_\theta + dk^{-1}$. However, since $k\pi > 0$, and since $(M_\theta + dk^{-1})\pi^\theta = (V_\theta + d\pi)(V_\theta + k^{-1}\pi)$, it necessarily follows that $d\pi > 0$.

The set $\{x_\delta \mid \delta \in \Delta\}$ described in the theorem generates a subgroup of M^* whose intersection with $M^* \cap (M^\theta \backslash M_\theta)$ is zero and so we may pick $T \supset \{x_\delta \mid \delta \in \Delta\}$. Finally, in performing the embedding any subfield $\bar{R} \subset M$ isomorphic to R could have been used.

REMARK. Hahn's theorem for an abelian o-group G states that G can be o-embedded in $V(\Gamma, R)$. (See [8], p. 60). There are now several short elementary proofs of this result in the literature. It would be a considerable achievement to also have such a direct proof of the above theorem.

5. An embedding theorem for a class of $\mathscr{L}$-fields. The embedding theorem for an o-field is actually a special case of the more general embedding theorem for $\mathscr{L}$-fields which is developed in this section.

Suppose that H is a subgroup of finite index in a torsion free abelian group Γ. Then Γ/H is a direct sum of cyclic groups.

$$\Gamma/H = [H + s(1)] \oplus \cdots \oplus [H + s(k)]$$

where the order of $[H + s(i)] = d(i)$, $d(1) \geqq \cdots \geqq d(k)$ and $d(i+1) \mid d(i)$.

LEMMA 5.1. *The subgroup of Γ generated by the $s(i)$ is a direct sum* $[s(1)] \oplus \cdots \oplus [s(k)]$. *In particular,* $d(1)s(1), \cdots, d(k)s(k)$ *are rationally independent elements of H.*

Proof. Suppose that $\sum m(i)s(i) = 0$, where the integers $m(i)$ are not all zero. Since Γ is torsion free, the g.c.d. of the $m(i)$ can be factored out and so we may assume that the $m(i)$ have g.c.d. 1. But since the linear combination must become trivial modulo H, $d(i) \mid m(i)$ and hence $d(k) \mid m(i)$ for all i, a contradiction.

THEOREM III. *Suppose that G is an $\mathscr{L}$-field with a finite basis and that the special elements of G form a group. Then the set of values Γ of G is a po-group and a root system. If Γ is torsion free then there exists a value preserving $\mathscr{L}$-isomorphism of G into the $\mathscr{L}$-field $V(\Gamma, R)$.*

Proof. It follows from § 3 that Γ is a po-group and a root system. Then by Lemma 2.1 there exists a totally ordered subgroup H of Γ

such that the index $|\Gamma: H| = n$ of H in Γ is finite and H^+ is the positive cone for Γ. Thus $\Gamma = \bigcup\{H + \gamma_k \mid k = 1, \cdots, n\}$ is a disjoint union of totally ordered cosets, where $\gamma_1 \in \Gamma$ is chosen as $\gamma_1 = \theta$. Just as in the proof of Proposition 3.4 each $g \in G$ is uniquely of the form $g = g_1 + \cdots g_n$, where if $g_i \neq 0$, then either g_i or $-g_i$ is special, where $|g_i| \wedge |g_j| = 0$ if $i \neq j$, and where g_i "lives" on $H + \gamma_i$, that is $g_i \in G_\gamma$ for all $\gamma \in \Gamma \backslash (H + \gamma_i)$. Let F be the set of all elements that "live" on H, that is

$$F = \{g \in G \mid g \in G_\gamma \text{ for all } \gamma \in \Gamma \backslash H\} .$$

Then F is a totally ordered subfield of G. For clearly, F is a totally ordered convex subring of G, and if g is special, then by hypothesis g^{-1} is also special. Thus g^{-1} lives on $H + \gamma_i$ for some i. If $i \neq 1$, then $g\, g^{-1} = 1$ lives on $H + \gamma_i$ which is impossible. Therefore $g^{-1} \in F$ and thus F is a field. Now assume that Γ is torsion free; then by Proposition 2.2 $V(\Gamma, R)$ is an $\mathscr{L}$-field. As before, for each $\gamma \in \Gamma$ define $x^\gamma \in V$ by

$$x^\gamma(\alpha) = \begin{cases} 1 \text{ if } \alpha = \gamma \\ 0 \text{ otherwise} . \end{cases}$$

In particular $x^\theta = 1$. As previously

$$\Gamma/H = [H + s(1)] \oplus \cdots \oplus [H + s(k)]$$

with orders $d(1) \geqq \cdots \geqq d(k)$ so that $d(i + 1) \mid d(i)$. The reader should note that $n = d(1) \cdots d(k)$ and that $d(k)^k \mid n$. For each $i = 1, \cdots, k$ pick $0 < z_i \in G$ that lives on $H + \gamma_i$ and has value γ_i. In particular, each z_i is special. By Lemma 5.1, the $d(1)s(1), \cdots, d(k)s(k)$ are rationally independent elements of H and hence by Theorem II there exists a value and order preserving isomorphism π of the o-field F into $V(\Gamma, R)$ such that

(i) the support of $f\pi$ is contained in H for each $f \in F$, and

(ii) $z_i^{d(i)}\pi = x^{d(i)s(i)}$.

We shall extend π to an isomorphism of G into V. Consider

$$g(1)(H + s(1)) + \cdots + g(k)(H + s(k)) \in \Gamma/H$$

where $g(i)$ are integers in $0 \leqq g(i) < d(i)$ and let $g \in G$ live on this coset. Then

$$g = \bar{g} z_1^{g(1)} z_2^{g(2)} \cdots z_k^{g(k)}$$

where $\bar{g} \in F$. Since g lives on one of the n distinct cosets of Γ/H, g is special and conversely every special element is of the above form. Define

$$g\pi = \bar{g}\pi x^{g(1)s(1)}x^{g(2)s(2)} \cdots x^{g(k)}s^{(k)} .$$

Thus we have extended π to a one to one mapping of all special elements S of G. (Note that the map $S \to F, g \to \bar{g}$ is *not* a homomorphism of multiplicative groups unless $\Gamma = H$, while $g \to g(i)$ is a homomorphism of S into the integers modulo $d(i)$.) If $h \in S$ also lives on this same coset as g, then so does $h + g$ and $g(i) = h(i) = (h + g)(i)$ for all i, thus

$$h + g = \bar{h}z_1^{h(1)} + \cdots z_k^{h(k)} + \bar{g}z_1^{g(1)} \cdots z_k^{g(k)} = (\bar{h} + \bar{g})z_1^{g(1)} \cdots z_k^{g(k)} .$$

Therefore $(h + g)^- = \bar{h} + \bar{g}$ and so $(h + g)\pi = h\pi + g\pi$. Next it will be shown that $\pi\colon S \to F$ is a homomorphism of multiplicative groups. Take $g, h \in S$ and write

$$g(i) + h(i) = n(i)d(i) + r(i), 0 \leqq r(i) < d(i), i = 1, \cdots, k .$$

Then since

$$\begin{aligned} hg &= \overline{hg}z_1^{g(1)+h(1)} \cdots z_k^{g(k)+h(k)} \\ &= \overline{hg}z_1^{n(1)d(1)} \cdots z_k^{n(k)d(k)}z_1^{r(1)} \cdots z_k^{r(k)} , \end{aligned}$$

it follows that

$$(hg)^- = \overline{hg}z_1^{n(1)d(1)} \cdots z_k^{n(k)d(k)}; (hg)(i) = r(i), i = 1, \cdots k .$$

Thus

$$(hg)\pi = (\overline{hg})^- z_1^{r(1)} \cdots z_k^{r(k)} = (h\pi)(g\pi) .$$

Now each $a \in G$ has the above mentioned unique representation $a = a_1 + \cdots + a_n$ where a_i lives on $H + \gamma_i$; define $a\pi = a_1\pi + \cdots + a_n\pi$. Clearly, π is a map of G into V that preserves addition and values. If $b \in G$ with $b = b_1 + \cdots + b_n$ and $ab = c_1 + \cdots + c_n$ where b_i, c_i live on $H + \gamma_i$, it remains to show that $c_1\pi + \cdots + c_n\pi = \sum (a_i\pi)(b_j\pi)$. Each c_t is of the form $c_t = \sum' a_ib_j$ where $\sum'$ denotes the sum over those distinct pairs (i, j) for which $\gamma_i\gamma_j \in H + \gamma_t$. It suffices to show that $c_t\pi = \sum' (a_i\pi)(b_j\pi)$. However, first, since a_i, $b_j \in -S \cup S$ we have $(a_ib_j)\pi = (a_i\pi)(b_j\pi)$; and, secondly, since π preserves addition, $\sum' (a_i\pi)(b_j\pi) = (\sum' a_ib_j)\pi$. Thus it follows that $(ab)\pi = (a\pi)(b\pi)$. Therefore π is a homomorphism of the field G into the field $V(\Gamma, R)$ that is clearly not zero and so it must be an isomorphism. If $a = a_1 + \cdots + a_n$ where the a_i live on $H + \gamma_i$, then $a \vee 0$ is just the sum of the positive a_i. Therefore $(a \vee 0)\pi = a\pi \vee 0$ and π is a value preserving $\mathscr{L}$-isomorphism of G into V. This completes the proof of the theorem.

An $\mathscr{L}$-field F is an a-extension of an $\mathscr{L}$-field G, if for each

$0 < f \in F$, there exists an element $0 < g \in G$ such that $f < mg$ and $g < nf$ for some positive integers m and n, and G is *a-closed* if it does not admit such an extension.

The next corollary shows that the field V, into which G was embedded in the last theorem, has an intrinsic characterization.

COROLLARY. *Under the same hypotheses as in the previous theorem, V is the unique a-closed a-extension of $G\pi$.*

Proof. V is a-closed as an $\mathscr{L}$-group and, clearly, it is an a-extension of $G\pi$. In order to prove the uniqueness of V, let $G \subset D$ be any other a-extension of G. Since D satisfies all the hypotheses of Theorem III, for $g \in D\backslash G$, $g\pi$ can be defined exactly as in the proof of Theorem III to yield and $\mathscr{L}$-embedding of D into V that extends π. Furthermore, $D\pi \subseteqq V$ is an a-extension. Finally, if D is a-closed, then so is also $D\pi$ and hence $D\pi = V$. Thus π extends to an $\mathscr{L}$-isomorphism of D onto V leaving G elementwise fixed.

REMARK. Under the hypotheses of Theorem III we can extend the order of G to a total order (Proposition 3.4) and hence by Theorem II there is an o-isomorphism of the o-field G into the $\mathscr{L}$-field $V(\Gamma, R)$. It would be nice to be able to prove that this isomorphism is also an $\mathscr{L}$-isomorphism, but this we have not been able to do.

6. Examples and questions.

The first example shows that Γ need not be torsion free even if G is an $\mathscr{L}$-field with a finite basis in which the special elements form a multiplicative group. Similar examples exist in which G is actually a real algebra.

6.1. Take an algebraic extension $G = Q[w]$ of the rationals Q, where $w \in R$, $w^n = 2$, i.e., $w = 2^{1/n}$ for some $n \geqq 2$. For

$$y = c_0 + c_1 w + \cdots + c_{n-1}w^{n-1} \in Q[w]$$

with $c_i \in Q$ define $y \geqq 0$ if and only if all $c_i \geqq 0$. Note that this order differs from the natural order of $Q[w]$ as a subset of R. Then in the context of the notation of § 5, the multiplicative group of special elements S is generated by $S = [\{cw \mid 0 < c \in Q\}]$, $H = \{\theta\}$; Γ is the cyclic group of order n and hence not torsion free.

6.2. Take $n = 2$ above in 6.1 but redefine $y = c_0 + c_1 w > 0$ if and only if $c_1 \leqq 0$ and $c_0 \geqq 0$.

6.3. Let Γ be a cancellative multiplicative semigroup with identity that contains an element k in the center such that $k^m \neq k^n$ for all distinct positive integers m and n. For $a, b \in \Gamma$, define $a \geqq b$ if $a = k^n b$ for some integer $n \geqq 0$ where $k^0 = 1$. Then a straightforward computation shows that Γ is a strictly *po*-semigroup and a root system; in fact, Γ is the join of disjoint totally ordered sets each of which is countable.

6.4. In the multiplicative abelian semigroup Γ generated by a, b, k with $k^0 = 1$, define $a^i b^j k^n > a^p b^q k^m$ provided one of the following four cases holds.

Case 1. $i > p$.
Case 2. $i = p = 0, j = q$, but $n > m$.
Case 3. $i = p > 0$ and $j > q$.
Case 4. $i = p > 0, j = q$, but $n > m$.

Note that the subsemigroup $\{a^i b^j k^n \mid i \geqq 1\}$ is lexiographically ordered. Aside from being a strictly *po*-semigroup and a root system, Γ has two noteworthy features. It is not the union of disjoint chains such that the elements from distinct chains are incomparable, and it has no convex semigroup ideals.

In conclusion we list some questions we could not answer.

(a) Can the partial order of each $\mathscr{L}$-field be extended to a total order?

(b) If F is an $\mathscr{L}$-field in which each square is positive, then is F an o-field?

(c) Does each $\mathscr{L}$-field contain a unique maximal totally ordered subfield?

(d) When can a lattice order of a commutative integral domain be extended to a lattice order of its field of fractions?

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