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PONTRYAGIN SQUARES IN THE THOM SPACE OF A BUNDLE

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The object of this note is to determine the action of the Pontryagin squares in the cohomology of the Thom space of a vector bundle. This computation is then applied to the case of the normal bundle of a manifold imbedded in Euclidean space to give simplified proofs of some theorems of Mahowald.

The first of Mahowald's theorems [3] was inspired by some 1940 results of Whitney [9], who showed that in certain cases the Euler class (with twisted integer coefficients) of the normal bundle of a nonorientable surface imbedded in Euclidean 4-space could be nonzero. This contrasts with the well-known theorem that the Euler class of the normal bundle of an orientable manifold in Euclidean space is always zero.

2. **Notation and statement of results.** For any space X, we will use integral cohomology $H^q(X, \mathbf{Z})$; cohomology with integers mod n as coefficients, $H^q(X, \mathbf{Z}_n)$; cohomology with twisted integer coefficients, $H^q(X, \mathscr{Z})$ cohomology with twisted integers mod n coefficients, $H^q(X, \mathscr{Z}_n)$; and rational cohomology, $H^q(X, Q)$. In the third and fourth cases the local system of groups which is used for coefficients will be determined by the Stiefel-Whitney class $w_1 \in H^1(X, \mathbf{Z}_2)$. Note that for the case $n=2$, we have

$$H^q(X, \mathscr{Z}_2) = H^q(X, Z_2) .$$

since a cyclic group of order 2 admits no nontrivial automorphisms.

Let (E, p, B, S^{n-1}) be an $(n-1)$-sphere bundle over the base space B with structure group $0(n)$. We will use the following notation for characteristic classes of such a bundle:

Stiefel-Whitney classes:

$$w_i \in H^i(B, Z_2) , \qquad 1 \leqq i \leqq n$$
$$W_i \in H^i(B, \mathscr{Z}) , \qquad 1 \leqq i \leqq n, \ i \text{ odd} .$$

Pontrjagin classes:

$$p_i \in H^{4i}(B, Z) , \qquad 1 \leqq i \leqq n/2 .$$

Euler class:

$$X_n \in H^n(B, \mathscr{Z}) . \qquad (\text{If } n \text{ is odd, then } X_n = W_n.)$$

Let (A, π, B, D^n) be the associated n-dimensional disc bundle; we will call the pair (A, E) or the single space A/E the *Thom space* of the bundle. The *Thom class*, $U \in H^n(A, E, \mathscr{Z})$, has twisted integer coefficients; by taking cup products with U, we obtain the Thom isomorphism (see Thom [6]).

$$H^q(A, \mathscr{Z}) \approx H^{q+n}(A, E, Z) ,$$
$$H^q(A, Z) \approx H^{q+n}(A, E, \mathscr{Z}) ,$$
$$H^q(A, \mathscr{Z}_n) \approx H^{q+n}(A, E, Z_n) , \text{ etc.}$$

Recall also that the projection $\pi: A \to B$ is a deformation retraction, and hence induces isomorphisms of cohomology groups with any coeffients (even local coefficients!). For the sake of convenience, we will often identify the cohomology groups of A and B by means of this isomorphism; similarly we will identify the cohomology groups of the pair (A, E) and the space (A/E) (except in dimension 0) with ordinary coefficients (the local coefficient systems $\mathscr{Z}$ and $\mathscr{Z}_n$ do not exist in the space A/E).

The obvious epimorphism $\rho_n: Z \to Z_n$ and monomorphism $\theta: Z_2 \to Z_4$ induce homomorphisms of cohomology groups wich will be denoted as follows:

$$\rho_n: H^q(X, Z) \longrightarrow H^q(X, Z_n) ,$$
$$\tilde{\rho}_n: H^q(X, \mathscr{Z}) \longrightarrow H^q(X, \mathscr{Z}_n) ,$$
$$\theta: H^q(X, Z_2) \longrightarrow H^q(X, Z_4) ,$$
$$\tilde{\theta}: H^q(X, Z_2) \longrightarrow H^q(X, \mathscr{Z}_4) .$$

For convenience, we will let $U_2 = \tilde{\rho}_2(U)$, the Thom class reduced mod 2.

Our main concern will be the Pontryagin squaring operation,

$$\mathscr{P}: H^q(X, Z_2) \longrightarrow H^{2q}(X, Z_4) .$$

If q is odd, the Pontryagin square can be expressed in terms of simpler cohomology operations. (see formula (4.2) below); this is not true for q even. For a list of papers describing this operation, see the first paragraph of [7]. Our main result is the following, which describes the Pontryagin square of the mod 2 Thom class, U_2.

THEOREM I. *Let (E, p, B, S^{n-1}) be a (not necessarily orientable) $(n-1)$-sphere bundle with structure group $O(n)$, n even. Then*

$$\mathscr{P}(U_2) = [\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1 \cdot w_{n-1})] \cdot U .$$

As a corollary, we obtain the following result which was proved by Whitney [9] in 1940 for the case $n = 2$; the general case is due to Mahowald, [3, Th. I]:

COROLLARY 1. *Let M^n be a compact, connected, nonorientable n-manifold (n even) which is imbedded differentiably in R^{2n}. Then the twisted Euler class of the normal bundle, X_n, satisfies the following condition*:

$$\tilde{\rho}_4(X_n) + \tilde{\theta}(\bar{w}_1\bar{w}_{n-1}) = 0 .$$

(Here $\bar{w}_i$ denotes the ith dual Stiefel-Whitney class of M^n.)

In particular, if $\bar{w}_1\bar{w}_{n-1} \neq 0$ (which can only happen if n is a power of 2, cf. [4]) then $X_n \neq 0$. Apparently this is the only general result known about the twisted Euler class of the normal bundle to a nonorientable manifold.

The corollary may be derived from the theorem as follows: Let (E, p, B, S^{n-1}) denote the normal sphere bundle of the imbedding, and (A, π, B, D^n) the associated disc bundle. It is well known that the top homology group of the Thom space,

$$H_{2n}(A/E, Z) = H_{2n}(A, E, Z) ,$$

is infinite cyclic, and the Hurewicz homomorphism

$$\pi_{2n}(A/E) \longrightarrow H_{2n}(A/E)$$

is an epimorphism. From this it follows that $\langle \mathscr{P}(U_2), x\rangle = 0$ for any $x \in H_{2n}(A/E, Z)$, and hence $\mathscr{P}(U_2) = 0$. Applying the formula for $\mathscr{P}((U_2)$ in Theorem I, we obtain the corollary.

Next, we give formulas for the Pontryagin square of an *arbitrary* mod 2 cohomology class of even degree in the Thom space of a vector bundle.

THEOREM II. *Let (E, p, B, S^{n-1}) be an $(n-1)$-sphere bundle with structure group $0(n)$, and let $x \in H^m(B, Z_2)$, $m + n$ even. Then if m and n are both even,*

$$\mathscr{P}(U_2x) = \{\mathscr{P}(x)[\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1w_{n-1})] + \tilde{\theta}[w_{n-1}xSq^1x + w_1w_nSq^{m-1}x]\}\cdot U$$

while if m and n are odd,

$$\mathscr{P}(U_2x) = \{\mathscr{P}(x)[\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1w_{n-1} + w_1^2w_{n-2})] + \tilde{\theta}[w_{n-1}xSq^1x + w_1w_nSq^{m-1}x]\}\cdot U .$$

As a corollary, we derive a necessary condition due to Mahowald [3] for the imbeddability of an *orientable* manifold in Euclidean space of dimension $4k$ with codimension n.

COROLLARY 2. *Let M be a compact, connected, orientable manifold of dimension q which is differentiably imbedded in Euclidean space of dimension $q+n=4k$. Then for any $x \in H^m(M, Z_2)$, where $m = 1/2(q-n)$, we must have*

$$\bar{w}_{n-1}xSq^1x = 0 .$$

Proof of corollary. One applies Theorem II with $B = M$ and (E, p, B, S^{n-1}) the normal bundle of the imbedding. Since M is assumed orientable, $\bar{w}_1 = 0$, $\bar{w}_n = 0$, $X_n = 0$, and $\bar{W}_n = 0$. Exactly as in the proof of the previous corollary we know that $\mathscr{P}(U_2 \cdot x) = 0$ in this case. Thus we conclude that

$$\theta(\bar{w}_{n-1}xSq^1x) = 0$$

for any $x \in H^m(M, Z_2)$. Since M is orientable, the homomorphism

$$\theta\colon H^q(M, Z_2) \longrightarrow H^q(M, Z_4)$$

is a monomorphism, and therefore we must have $\bar{w}_{n-1}xSq^1x = 0$, as desired.

Perhaps the neatest application of this corollary is to prove that q-dimensional real projective space does not imbed in R^{2q-2} for $q = 2^r + 1$. A discussion of the possibilities of using this theorem to prove non-imbedding results is given in § 5.

COROLLARY 3. *Let M be a compact, connected, nonorientable manifold of dimension q which is differentiably imbedded in Euclidean space of dimension $q+n=4k$, q and n even. Then for any element $x \in H^m(M, Z_2)$, where $m = (1/2)(q-n)$, we must have*

$$\mathscr{P}(x)\cdot[\tilde{\rho}_4(X_n) + \tilde{\theta}(\bar{w}_1\bar{w}_{n-1})] + \tilde{\theta}(\bar{w}_{n-1}xSq^1x) = 0 .$$

This is a generalization of Corollary 1, and the proof is similar. Presumably this theorem would enable one to prove in certain cases that $\tilde{\rho}_4(X_n) \neq 0$, and hence $X_n \neq 0$, but the author knows of on examples to illustrate this possibility. Perhaps the most likely case in which this theorem could be applied is the case where $n = q - 4$, $m = 2$.

3. Proof of Theorem I. As is usual in such cases, one only need prove Theorem I in the case of the universal example, where $B = B0(n)$, n even. Then E has the same homotopy type as $BO(n-1)$. Consider the following commutative diagram for this universal example:

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta} & H^*(A, E, Z_k) & \xrightarrow{j^*} & H^*(A, Z_k) & \xrightarrow{i^*} & H^*(E, Z_k) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle 1} & \nearrow & \uparrow & & \uparrow & & \\
 & & H^*(A, \mathscr{X}_k) & & \Big| {\scriptstyle \pi^*} & & \Big| {\scriptstyle 2} & & \\
 & & \uparrow{\scriptstyle \pi^*} & & \Big| & & \downarrow & & \\
\cdots & \xrightarrow[\psi]{} & H^*(B, \mathscr{X}_k) & \xrightarrow[\mu]{} & H^*(B, Z_k) & \xrightarrow[p^*]{} & H^*(E, Z_k) & \xrightarrow[\psi]{} & \cdots .
\end{array}
$$

The top line of this diagram is the mod k cohomology sequence of the pair (A, E) while the bottom line is the Gysin sequence of fibration. All vertical arrows are isomorphisms; arrow No. 1 denotes the Thom isomorphism, and arrow No. 2 is the identity. It is well known that in these exact sequences for the case $k=2$ (i.e., mod 2 cohomology), the following statements are true:

p^* and i^* are epimorphisms,

μ and j^* are monomorphisms, and

ψ and δ are zero.

We assert that these statements are also true in case $k = 4$. In order to prove this, it suffices to prove that j^* is a monomorphism, and for this purpose consider the following commutative diagram:

$$
\begin{array}{ccccc}
0 & \longrightarrow & H^{q-1}(A, E, Z_2) & \xrightarrow{j_2} & H^{q-1}(A, Z_2) \\
 & & \downarrow{\scriptstyle Sq^1} & & \downarrow{\scriptstyle Sq^1} \\
0 & \longrightarrow & H^q(A, E, Z_2) & \xrightarrow{j_2} & H^q(A, Z_2) \\
 & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} \\
\cdots & \longrightarrow & H^q(A, E, Z_4) & \xrightarrow{j_4} & H^q(A, Z_4) \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
0 & \longrightarrow & H^q(A, E, Z_2) & \xrightarrow{j_2} & H^q(A, Z_2) .
\end{array}
$$

The vertical lines are exact sequences corresponding to the following short exact sequence of coefficients:

$$0 \longrightarrow Z_2 \xrightarrow{\theta} Z_4 \xrightarrow{\eta} Z_2 \longrightarrow 0 .$$

Let $x \in H^q(A, E, Z_4)$ and assume that $j^*(x) \equiv j_4(x) = 0$. Therefore

$$j_2\eta(x) = \eta j_4(x) = 0$$

and since j_2 is a monomorphism, $\eta(x) = 0$. By exactness, there exists an element $y \in H^q(A, E, Z_2)$ such that

$$\theta(y) = x .$$

Since $\theta j_2(y) = 0$, there exists an element $z \in H^{q-1}(A, Z_2)$ such that

$$Sq^1(z) = j_2(y) .$$

We wish to show that z can be chosen so that $z \in$ image j_2. For this purpose, recall that we are identifying $H^*(A, Z_2)$ with $H^*(B, Z_2) = Z_2[w_1, w_2, \cdots, w_n]$; using this identification, the image of j^* is the ideal generated by w_n. We may split $H^*(A, Z_2)$ into the (vector space) direct sum of this ideal and a supplementary subspace as follows: one subspace is spanned by all monomials which have w_n as a factor, the other subspace is spanned by those monomials which do not have w_n as a factor. It is readily verified that the homomorphism

$$Sq^1 \colon H^*(A, Z_2) \longrightarrow H^*(A, Z_2)$$

maps each of these summands into itself (this depends on the fact that n is even). Since $j_2(y)$ belong to this ideal generated by w_n, we can choose z so it also belongs to this ideal. Therefore $z = j_2(u)$ for some element $u \in H^{q-1}(A, E, Z_2)$. It follows that

$$j_2(y - Sq^1 u) = 0 .$$

Since j_2 is a monomorphism, $y = Sq^1 u$, and

$$x = \theta(y) = \theta Sq^1 u = 0$$

as asserted.

Next, let $X_n \in H^n(BO(n), \mathscr{Z})$ denote the Euler class (n even). We assert that

$$X_n^2 = p_{n/2} \in H^{2n}(BO(n), Z) .$$

To prove this, we make use of the fact that all torsion in $H^*(BO(n), Z)$ is of order 2 (cf. Borel and Hirzebruch, [2]). Hence it suffices to prove that the following two equations:

$$\rho_2(X_n^2) = \rho_2(p_{n/2}) \text{ and}$$
$$\rho_0(X_n^2) = \rho_0(p_{n/2}) ,$$

where ρ_0 is the homomorphism of cohomology groups induced by the coefficient map $Z \to Q$.

As to the first equation, it is well known that $\rho_2(X_n) = w_n$ and $\rho_2(p_i) = w_{2i}^2$, hence

$$\rho_2(X_n^2) = w_n^2 = \rho_2(p_{n/2}) .$$

To prove the second equation, consider the following commutative diagram.

$$\begin{array}{ccc} H^{2n}(BO(n), Z) & \xrightarrow{f^*} & H^{2n}(BSO(n), Z) \\ \downarrow \rho_0 & & \downarrow \rho_0 \\ H^{2n}(BO(n), Q) & \xrightarrow{f^*} & H^{2n}(BSO(n), Q) \end{array} .$$

Here $f: BSO(n) \to BO(n)$ is the 2-fold covering induced by the inclusion of $SO(n)$ in $O(n)$. It is well known that $\rho_0 f^*(X_n^2) = \rho_0 f^*(p_{n/2})$ and that f^* is a monomorphism on rational cohomology (see Borel and Hirzebruch [2]). Hence $\rho_0(X_n^2) = \rho_0(p_{n/2})$ as required.

With these preliminaries out of the way, we will now prove Theorem I by consideration of the following commutative diagram:

$$\begin{array}{ccc} H^n(A, E, Z_2) & \xrightarrow{j_2} & H^n(A, Z_2) \\ \downarrow \mathscr{P} & & \downarrow \mathscr{P} \\ H^{2n}(A, E, Z_4) & \xrightarrow{j_4} & H^{2n}(A, Z_4) \end{array} .$$

It is well known that $j_2(U_2) = w_n$, and according to Thomas [8], Theorem C,

$$\mathscr{P}(w_n) = \rho_4(p_{n/2}) + \theta(w_1 Sq^{n-1} w_n) .$$

Since j_4 is a monomorphism, it suffices to prove that

$$j_4\{\tilde{\rho}_4(X_n) + [\tilde{\theta}(w_1 w_{n-1})] \cdot U\} = \rho_4(p_{n/2}) + \theta(w_1 Sq^{n-1} w_n)$$

in order to complete the proof. Now

$$\tilde{\rho}_4(X_n) \cdot U = \rho_4(X_n \cdot U)$$

and

$$\begin{aligned} j_4\{\tilde{\rho}_4(X_n) \cdot U\} &= j_4 \rho_4(X_n \cdot U) = \rho_4 j(X_n \cdot U) \\ &= \rho_4(X_n^2) = \rho_4(p_{n/2}) \end{aligned}$$

since $j(U) = X_n$. Similarly,

$$[\tilde{\theta}(w_1 w_{n-1})] \cdot U = \theta(w_1 w_{n-1} \cdot U_2) = \theta(w_1 Sq^{n-1} U_2) ,$$

hence

$$\begin{aligned} j_4\{\tilde{\theta} w_1 w_{n-1}) \cdot U\} &= j_4 \theta(w_1 Sq^{n-1} U_2) \\ &= \theta j_2(w_1 Sq^{n-1} U_2) \\ &= \theta(w_1 Sq^{n-1} w_n) \end{aligned}$$

since $j_2(U_2) = w_n$. This completes the proof.

4. **Proof of Theorem 2.** The proof is a routine application of the following two formulas. For the first formula, assume that X is

a topological space, $u \in H^m(X, Z_2)$, $v \in H^n(X, Z_2)$, and $m \equiv n \bmod 2$; then the Pontryagin square of the cup product uv is given by the following formula:

$$\mathscr{P}(uv) = (\mathscr{P}u)(\mathscr{P}v) + \theta[(Sq^{m-1}u)vSq^1v + uSq^1u(Sq^{n-1}v)] \ . \tag{4.1}$$

For the case where m and n are both odd, this formula is given by Thomas [8], formula (10.5); in case m and n are even, the formula is given by Nakaoka [5], Theorem III. Our second formula expresses the Pontryagin square of an odd dimensional cohomology class in terms of more usual cohomology operations. Assume $u \in H^{2q+1}(X, Z_2)$; then

$$\mathscr{P}(u) = \rho_4\beta Sq^{2q}u + \theta Sq^{2q}Sq^1u \ , \tag{4.2}$$

where β is the Bockstein coboundary operator associated with the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$. In particular, if we apply (4.2) to the computation of $\mathscr{P}(U_2)$ for an m-dimensional vector bundle, m odd, and make use of the formula $Sq^iU_2 = w_iU_2$, we obtain the formula

$$\mathscr{P}(U_2) = [\tilde{\rho}_4(W_m) + \tilde{\theta}(w_1w_{m-1} + w_1^2w_{m-2})]\cdot U \ . \tag{4.3}$$

The proof of Theorem II is now a direct application of formula (4.1); one also uses Theorem I in case m and n are even, and (4.3) in case m and n are odd.

5. Critique of corollary 2. We propose to discuss the following question: Under what conditions does Corollary 2 enable one to prove nonimbedding theorems not provable by more standard and/or elementary methods? We will assume, as in the statement of the corollary, that M is a compact, connected, orientable manifold of dimension q, that $\bar{w}_{n-1} \neq 0$, and

$$q + n \equiv 0 \bmod 4 \ .$$

We wish to prove that M can not be imbedded differentiably in Euclidean space of dimension $q + n$. We may as well assume that $\bar{w}_i = 0$ for all $i > n - 1$, otherwise the proof would be trivial.

We assert that *if n is even*, then for any $x \in H^m(M, Z_2)$, $m = (1/2)(q - n)$,

$$\bar{w}_{n-1}xSq^1x = 0$$

under the above hypotheses, and hence Corollary 2 can not be applied to prove nonimbedding results.

Proof of assertion. By Lemma 1 of Massey and Peterson [4],

$$\begin{aligned}\bar{w}_{n-1}xSq^1x &= Q^{n-1}(xSq^1x)\\ &= Q^{n-1}(xQ^1x)\\ &= \sum_{i+k=n-1}(Q^ix)(Q^jQ^1x)\ .\end{aligned}$$

But

$$Q^jQ^1 = \begin{cases} Q^{j+1} & \text{if } j \text{ is even },\\ 0 & \text{if } j \text{ is odd }.\end{cases}$$

Hence

$$\begin{aligned}\bar{w}_{n-1}xSq^1x &= \sum_{i+j=n-1}(Q^ix)(Q^{j+1}x)\\ &= \sum_{i+k=n}(Q^ix)(Q^kx)\ .\end{aligned}$$

where the summations are restricted to even values of j and odd values of k respectively.

If $n \equiv 0 \bmod 4$, then i must also be odd in this sum, and the nonzero terms occur in pairs which cancel. If $n \equiv 2 \bmod 4$, then all terms cancel in pairs except for the term where $i = k = n/2$, and one sees that in this case

$$\bar{w}_{n-1}xSq^1x = Q^n(x^2) = \bar{w}_n \cdot x^2\ .$$

But by our hypothesis, $\bar{w}_n = 0$; hence $\bar{w}_{n-1}xSq^1x = 0$ in this case also.

Thus this method is only of interest in case n and q are odd. Perhaps the first case of interest is the case where q is odd and $n = q - 2$. In this case $m = 1$, $x \in H^1(M, Z_2)$, $Sq^1x = x^2$, and

$$\bar{w}_{n-1}xSq^1x = Q^{n-1}(x^3) \in H^q(M, Z_2)\ .$$

The question is, for what values of n can $Q^{n-1}(x^3)$ be nonzero? Now it is easy to prove that for any 1-dimensional cohomology class x,

$$Q(x) = x + x^2 + x^4 + x^8 + \cdots + x^{2^k} + \cdots ,$$

(see Atiyah and Hirzebruch [1], pp. 168–169), hence

$$\begin{aligned}Q(x^3) = (Qx)^3 &= x^3 + (x^4 + x^5) + (x^8 + x^9)\\ &\quad + \cdots + (x^{2^k} + x^{2^k+1}) + \cdots\ .\end{aligned}$$

Therefore the only case for which $Q^{n-1}(x^3)$ can possibly be nonzero is the case $q = n + 2 = 2^k + 1$, and in this case

$$Q^{n-1}(x^3) = x^q\ .$$

Thus the example $M = q$-dimensional real projective space is typical for this situation.

The next case of interest would be the case q odd, $n = q - 6$, $m = 3$. The author knows no nontrivial examples to illustrate this case.

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