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A GENERALIZATION OF THE WEINSTEIN-ARONSZAJN FORMULA

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This paper uses a technique of abstract spectral theory to reduce the study of certain eigenvalues, which are not necessarily isolated, to the case of isolated eigenvalues. By this method the Weinstein-Aronszajn formula for the change in multiplicity of an isolated eigenvalue of a self adjoint operator under a finite dimensional perturbation is extended.

The hypotheses of this generalization are studied in the abstract and also by demonstrative example.

The central result of this work is Corollary 3 to Theorem 1 which gives a generalization of the Weinstein-Aronszajn formula for the change of multiplicity of an eigenvalue under a finite dimensional perturbation. The reader should observe that the hypotheses of that corollary are trivially satisfied in the case of an isolated eigenvalue.

Although the hypotheses of this main result are easy to understand from the point of view of abstract spectral theory, there are obvious questions about their computability and about their applicability. These two questions are investigated in § 1 and § 3. Section 1 describes some Hilbert space geometries under which the hypotheses are satisfied. Section 3 gives two elementary examples.

The actual technique used to prove Theorem 1 is to remove a small deleted interval about the eigenvalue from the spectrum of the operator. This is accomplished by replacing the original Hilbert space with the orthogonal complement of the subspace causing the spectrum in that deleted interval. Such a constructive process requires the handling of complicated technical details. Then we apply the theory for isolated eigenvalues and we deduce from that conclusion some conclusions with the original hypotheses. This new technique, which seems to be very general in nature, is probably the most interesting feature of this paper.

1. Preliminaries.

NOTATION. Throughout this paper T_0 will be a self adjoint (not necessarily bounded) unperturbed operator and $V = \sum_{j=1}^{r} \langle \cdot, \phi_j \rangle c_j \phi_j$ is a self adjoint perturbation; both operators are defined on a complex Hilbert space H. So $T = T_0 + V$ is defined on the dense domain of T_0 and we write $R(z) = (T - zI)^{-1}$, $R_0(z) = (T_0 - zI)^{-1}$. The spectral measures and the resolutions of identify of the two operators T_0, T

are denoted $E_0(\cdot)$, $E(\cdot)$ and $E_0(t)$, $E(t)$, respectively. $\mathscr{R}(V)$ means the range of V and the Weinstein-Aronszajn matrix is denoted by $W(z) = [I + VR_0(z)]/\mathscr{R}(V)$ while $\omega(z) = \det W(z)$ is the $W-A$ determinant.

MAIN HYPOTHESES. This paper is concerned with a generalization of isolated eigenvalues which in many instances includes the so called embedded eigenvalues (eigenvalues which belong to an interval which is wholly contained in the spectrum). If λ_0 is not in the essential spectrum of T_0, i.e., λ_0 is in the resolvent set or λ_0 is an isolated eigenvalue of finite multiplicity, then by the stability of the essential spectrum under compact perturbations (the Weyl Theorem, see p. 367 of [5]) we get that λ_0 is not in the essential spectrum of $T_0 + V$. Thus there exists a $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ with $\delta > 0$ such that $E_0(D_\delta) = 0$ and $E(D_\delta) = 0$. Necessarily

$$(\,*\,)\ E_0(D_\delta)E(\{\lambda_0\}) = 0 \quad \text{and} \quad (**)\ E(D_\delta)E_0(\{\lambda_0\}) = 0\ .$$

However, the converse of the last statement is not true; in fact both (*) and (**) may be satisfied while λ_0 is actually an embedded eigenvalue of T_0. If both (*) and (**) are satisfied we say that λ_0 is *quasi-isolated*. Since (*) and (**) depend on V we should say "quasi-isolated with respect to V." However, we shall abuse notation and use the shorter phrase.

All isolated eigenvalues are quasi-isolated. The following propositions will demonstrate some of the Hilbert space geometries which produce quasi-isolated eigenvalues. These constructions exploit the easy fact that the spectral measure of a direct sum operator is the direct sum of the spectral measures of the operators in the sum. Thus if $T_0 = T_1 \oplus T_2$ is self adjoint and defined on $H = H_1 \oplus H_2$, then $E_0(D)H_i \subset H_i$ and in fact $E_0(D)/H_i$ is the spectral measure for T_i.

PROPOSITION 1. *Let $T_0 = T_1 \oplus T_2$ be self adjoint and let $V = V_1 \oplus V_2$ be a compact self adjoint operator on $H = H_1 \oplus H_2$. If λ_0 is not a point in the essential spectrum of T_1 and λ_0 is not an eigenvalue for T_2, then $E(D_\delta)E_0(\{\lambda_0\}) = 0$ for all sufficiently small δ.*

Proof. Let $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ for $\delta > 0$. Since λ_0 does not become a point of the essential spectrum of $T_1 + V_1$ there is some positive δ such that $E(D_\delta)H_1 = \{0\}$. Then $E(D_\delta)H = E(D_\delta)(H_1 \oplus H_2) \subset H_2$ since $E(D_\delta)H_2 \subset H_2$. By hypothesis $E_0(\{\lambda_0\})H \subset H_1$. Thus $E_0(\{\lambda_0\})H$ is orthogonal to $E(D_\delta)H$ and we have $E(D_\delta)E_0(\{\lambda_0\}) = 0$.

PROPOSITION 2. *Let T_0 be self adjoint and let V be a finite dimensional self adjoint operator on H. Let $\{\tau_j\}$ be a basis for $E(\{\lambda_0\})H$. If for each τ_j there exists some $\delta(j) > 0$ such that*

$$E_0(D_{\delta(j)})V\tau_j = 0\ , \quad then \quad E_0(D_\delta)E(\{\lambda_0\}) = 0$$

for all sufficiently small δ.

Proof. Since V is finite dimensional for only a finite subset of $\{\tau_j\}$ can V be nonzero. Let $\{\tau_1, \cdots, \tau_p\}$ be that subset; so $V\tau_j = 0$ for $j \geqq p+1$. Let $0 < \delta < \delta(j)$ then

$$0 = \langle E_0(D_{\delta(j)})V\tau_j\ V\tau_j\rangle \geqq \langle E_0(D_\delta)V\tau_j,\ V\tau_j\rangle = \| E_0(D_\delta)V\tau_j\|^2 \geqq 0\ .$$

Thus $E_0(D_\delta)V\tau_j = 0$ for all j. Hence

(†) $E_0(D_\delta)VE(\{\lambda_0\})H = \{0\}$.

If $\tau \in E(\{\lambda_0\})H$, then $0 = (T - \lambda_0)\tau = V\tau + (T_0 - \lambda_0)\tau$ or $V\tau = (\lambda_0 - T_0)\tau$. Using (†) above we see that

$$0 = E_0(D_\delta)V\tau = E_0(D_\delta)(\lambda_0 - T_0)\tau = (\lambda_0 - T_0)E_0(D_\delta)\tau\ .$$

This says that $E_0(D_\delta)\tau$ which is conspicuously a vector from $E_0(D_\delta)H$ is a λ_0-eigenvector of T_0. Since $E_0(\{\lambda_0\})E_0(D_\delta) = E_0(\{\lambda_0\} \cap D_\delta) = E_0(\phi) = 0$ the above is only possible if $E_0(D_\delta)\tau = 0$. By the arbitrariness of τ we have shown $E_0(D_\delta)E(\{\lambda_0\})H = \{0\}$ or $E_0(D_\delta)E(\{\lambda_0\}) = 0$.

COROLLARY 1. *Let T_0 and V be self adjoint operators on H. If $E_0(D_\delta)H \subset \ker V$, then $E_0(D_\delta)E(\{\lambda_0\}) = 0$.*

Proof. Taking orthogonal complements of $E_0(D_\delta)H$ and $\ker V$ while using that $E_0(D_\delta)$ and V are self adjoint we get that $\ker E_0(D_\delta) \supset \mathscr{R}(V)$. So the hypotheses of the preceding proposition are trivially satisfied.

2. A generalization of the Weinstein-Aronszajn formula.

In this section we shall prove a generalization of the formula given by Weinstein and Aronszajn and extended by Kuroda for the change in the multiplicity of an eigenvalue under perturbation. As in the work of Weinstein and Aronszajn the perturbation will be finite dimensional. Because we use Kuroda's form of the $W-A$ theorem the restriction that the operators be self adjoint is essential. For Kuroda in [4] uses the notion of algebraic multiplicity of an eigenvalue while the notion used here is that of geometric multiplicity. For self adjoint operators the two notions coincide.

Before giving the generalization of the Weinstein-Aronszajn-Kuroda theory, we reformulate the theory in a manner appropriate for this work. Proofs of the following facts may be found in [3, pp. 244–250].

The Weinstein-Aronszajn formula for isolated eigenvalues. Clearly $\mathscr{R}(V)$, the range of the perturbation, is invariant under the operator $I + VR_0(x)$; so it makes sense to consider $\omega(z) = \det\{I + VR_0(z)/\mathscr{R}(V)\}$ and the usual definition is available since $\mathscr{R}(V)$ is finite dimensional. We define a multiplicity function for a self adjoint operator S by

$$\nu(\zeta, S) = \begin{cases} 0 \text{ if } \zeta \text{ is in the resolvent set of } S \\ \text{dimension of the eigenspace for } S \\ \quad \text{if } \zeta \text{ is an isolated eigenvalue} \\ \infty \text{ otherwise.} \end{cases}$$

We define the multiplicity of $\omega(z)$ at ζ by

$$\nu(\zeta, \omega) = \begin{cases} k \text{ if } \zeta \text{ is a zero of } \omega(z) \text{ of order } k \\ -k \text{ if } \zeta \text{ is a pole of } \omega(z) \text{ of order } k \\ 0 \text{ otherwise .} \end{cases}$$

Then the $W-A$ formula is just $\nu(\zeta, T_0 + V) = \nu(\zeta, T_0) + \nu(\zeta, \omega)$ for $\zeta \in D$ where D is a region of the complex plane such that the only spectra of T_0 and $T_0 + V$ in D are isolated eigenvalues.

In what follows it will be convenient to have the $W-A$ formula in a slightly different form. A statement clearly equivalent to the one given above is the following: there exists an integer k such that $(\zeta - z)^k\omega(z)$ is bounded above and bounded away from zero in some neighborhood of ζ and $k = \nu(\zeta, T_0) - \nu(\zeta, T_0 + V)$. This statement follows from the well known behavior of a meromorphic function in every neighborhood of a pole and in every neighborhood of a zero. In fact the integer k can be specified by $0 < m \leqq \gamma^k \,|\, \omega(\zeta + i\gamma) \,|\, \leqq M < +\infty$ for $\gamma \leqq \gamma_0$ for some positive γ_0 where ζ is real. In the following we shall use this determination of k.

THEOREM 1. *Let P be the orthogonal projection onto $\mathscr{R}(V)$ which has $\{\phi_j\}_{j=1}^r$ as an orthonormal basis. Let T_0 and $V = \sum_{j=1}^r \langle\cdot, \phi_j\rangle c_j\phi_j$ be self adjoint operators on the complex Hilbert space H. If there exists a $\delta > 0$ such that*

(1) $E(D_\delta)E_0(\{\lambda_0\}) = 0$ *with* $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$

and

(2) $\| PR_0(\lambda_0 - i\gamma)/E(D_\delta)\mathscr{R}(V) \| \leqq M < +\infty$ *for all sufficiently small γ, say $\gamma < \gamma_0$, then the following are true:*

(a) *There exists an integer k such that*

$$(\lambda_0 - z)^k\omega(z) = (\lambda_0 - z)^k \det\,[(I + VR_0(z))/\mathscr{R}(V)]$$

is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ with γ sufficiently small and

(b) $\nu(\lambda_0, T_0 + V) \geqq \nu(\lambda_0, T_0) - k$ *where $\nu(\zeta, S)$ is the multiplicity of ζ as an eigenvalue for S.*

Step 1. Let $Q_\delta = I - E(D_\delta)$. Then $\nu(\lambda_0, T - Q_\delta VQ_\delta) \geqq \nu(\lambda_0, T_0)$.

Proof. It is sufficient to show that every solution of $T_0\tau = \lambda_0\tau$ is a solution of $(T - Q_\delta VQ_\delta)\tau = \lambda_0\tau$. If τ is a solution of $T_0\tau = \lambda_0\tau$, then $\tau \in E_0(\{\lambda_0\})H$ and by hypothesis (1) $E(D_\delta)\tau = 0$ or $Q_\delta\tau = \tau$ for δ sufficiently small. Thus $(T - Q_\delta VQ_\delta)\tau = (TQ_\delta - Q_\delta V)\tau = Q_\delta T_0\tau = \lambda_0\tau$ as required.

Step 2. Let P be the orthogonal projection onto $\mathscr{R}(V)$. For all sufficiently small $\delta > 0$ $\{Q_\delta\phi_j\}$ is a basis for $\mathscr{R}(-Q_\delta VQ_\delta)$ and in this basis the matrix of $I + (-Q_\delta VQ_\delta)R(z)$ restricted to $\mathscr{R}(-Q_\delta VQ_\delta)$ is identical with the matrix of $[P(Q_\delta - Q_\delta VQ_\delta R(z))]/\mathscr{R}(V)$ in the basis $\{\phi_j\}$.

Proof. First let us note that a straightforward consequence of the measure-theoretic properties of $E(\cdot)$ is that $E(D_\delta) \to 0$ strongly as $\delta \to 0$ and so $Q_\delta \to I$ strongly as $\delta \to 0$.

So for each ϕ_j, $1 \leqq j \leqq r$, there exists a $\delta(j)$ such that $\| (I - Q_\delta)\phi_j \| < 1/2r$ for $\delta \leqq \delta(j)$. If $\delta < \delta(j)$ for all j and $V = \sum_{k=1}^r \beta_k\phi_k$ and $1 = \| V \|^2 = \sum_{k=1}^r | \beta_k |^2$ then $| \beta_k | \leqq 1$ for each k, $1 \leqq k \leqq r$, and

$$\begin{aligned} \| Q_\delta v \| &\geqq \| Iv \| - \| (I - Q_\delta)v \| \geqq 1 - \left\| \sum_{k=1}^r \beta_k(I - Q_\delta)\phi_k \right\| \\ &\geqq 1 - \sum_{k=1}^r | \beta_k | \, \| (I - Q_\delta)\phi_k \| > 1 - (1/2r) \sum_{k=1}^r | \beta_k | \\ &\geqq 1 - (1/2r)r = 1/2 . \end{aligned}$$

So $Q_\delta v \neq 0$. Thus $\ker Q_\delta/\mathscr{R}(V) = \{0\}$ or $Q_\delta/\mathscr{R}(V)$ is one to one for all sufficiently small δ. Since $\{\phi_j\}$ is a basis for $\mathscr{R}(V)$ it must be that $\{Q_\delta\phi_j\}$ is a linearly independent set for all δ sufficiently small. We note that

$$-Q_\delta VQ_\delta = -Q_\delta\Big(\sum_{j=1}^r \langle Q_\delta\cdot, \phi_j\rangle c_j\phi_j\Big) = \sum_{j=1}^r \langle\cdot, Q_\delta\phi_j\rangle(-Q_\delta c_j\phi_j) ,$$

Clearly Span $\{Q_\delta\phi_j\} = \mathscr{R}(-Q_\delta VQ_\delta)$ and since $\{Q_\delta\phi_j\}$ is a linearly independent set for all sufficiently small δ, we get that $\{Q_\delta\phi_j\}$ is a basis for $(-Q_\delta VQ_\delta)$ for all sufficiently small δ.

A straightforward computation gives that both

$$\langle[I - Q_\delta VQ_\delta R(z)]Q_\delta\phi_i, Q_\delta\phi_j\rangle \quad \text{and} \quad \langle P[Q_\delta - Q_\delta VQ_\delta R(z)]\phi_i, \phi_j\rangle$$

are equal to $\langle Q_\delta\phi_i, \phi_j\rangle - \langle VR(z)Q_\delta\phi_i, Q_\delta\phi_j\rangle$ for $i, j = 1, \cdots, r$.

Step 3. For $\nu = \nu(\lambda_0, T) - \nu(\lambda_0, T - Q_\delta VQ_\delta)$, $(\lambda_0 - z)^\nu \det\{P(Q_\delta - Q_\delta VQ_\delta R(z))/\mathscr{R}(V)\}$ is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ with γ sufficiently small.

Proof. First we note that we may add to the hypotheses of the theorem the conclusion of the preceding step. This condition is guaranteed by simply choosing δ sufficiently small.

Consider $T/Q_\delta H$ as an unperturbed operator and $-Q_\delta VQ_\delta/Q_\delta H$ as a perturbation. Since $Q_\delta = I - E((\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta))$ it is clear that λ_0 is isolated from the spectrum of $T/Q_\delta H$. Thus we can apply the classical $W-A$ formula and we observe that the $W-A$ matrix is $[I + (-Q_\delta VQ_\delta)R(z)]/\mathscr{R}(-Q_\delta VQ_\delta)$. The $W-A$ theorem asserts the existence and uniqueness of an integer ν such that

$$(\lambda_0 - z)^\nu \det \{[I + (-Q_\delta VQ_\delta)R(z)]/\mathscr{R}(-Q_\delta VQ_\delta)\}$$

is bounded above and bounded away from zero for all $z = \lambda_0 + ir$ with γ sufficiently small and $\nu = \nu(\lambda_0, T) - \nu(\lambda_0, T - Q_\delta VQ_\delta)$. By the previous step we see that this ν is the unique integer such that

$$(\lambda_0 - z)^\nu \det \{[P(Q_\delta - Q_\delta VQ_\delta R(z))]/\mathscr{R}(V)\}$$

is bounded above and is bounded away from zero for all $z = \lambda_0 + i\gamma$ with γ sufficiently small. Thus Step 3 is proved.

Step 4.

$$[PQ_\delta(T - Q_\delta VQ_\delta - z)(T - z)^{-1}]/\mathscr{R}(V)[(T - z)(T_0 - z)^{-1}]/\mathscr{R}(V)$$

converges to $I_{\mathscr{R}(V)}$ in the norm topology as $\delta \to 0$ and the convergence is uniform in γ for all sufficiently small γ, say $\gamma < \gamma_0$. Thus the determinant of the composite operator is bounded above and is bounded away from zero for all sufficiently small δ uniformly in $z = \lambda_0 + i\gamma$ for $\gamma < \gamma_0$.

Proof. Note that $\mathscr{R}(V)$ is invariant under $(T - z)(T_0 - z)^{-1} = I + VR_0(z)$ and observe the following simplification

$$\begin{aligned}
&[PQ_\delta(T - Q_\delta VQ_\delta - z)(T - z)^{-1}]\mathscr{R}(V)[(T - z)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = [PQ_\delta(T - Q_\delta VQ_\delta - z)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = PQ_\delta[(T_0 - z)(T_0 - z)^{-1} + (V - Q_\delta VQ_\delta)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = P[Q_\delta + (Q_\delta V - Q_\delta VQ_\delta)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = P[Q_\delta + Q_\delta V(I - Q_\delta)(T_0 - z)^{-1}]/\mathscr{R}(V)\,.
\end{aligned}$$

Because $\mathscr{R}(V)$ is finite dimensional it would suffice for the conclusion of Step 4 to show convergence in the strong topology of $PQ_\delta/\mathscr{R}(V)$ to $I_{\mathscr{R}(V)}$ and $PQ_\delta V(I - Q_\delta)R_0(\lambda_0 + i\gamma)/\mathscr{R}(V)$ to 0 as approaches zero. The first limit is established by taking $x \in \mathscr{R}(V)$ and noting

$$\|(I - PQ_\delta)x\| = \|(P - PQ_\delta)x\| = \|PE(D_\delta)x\| \leqq \|E(D_\delta)x\|$$

and recalling that $E(D_\delta)$ converges strongly to 0. To establish the second limit observe

$$\begin{aligned} VE(D_\delta)R_0(\lambda_0 + i\gamma)x &= \sum_{k=1}^{r} < E(D_\delta)R_0(\lambda_0 + i\gamma)x, \phi_k) c_k \phi_k \\ &= \sum_{k=1}^{r} \langle x, PR_0(\lambda_0 - i\gamma)E(D_\delta)\phi_k \rangle c_k \phi_k \end{aligned}$$

and $\| PR_0(\lambda_0 - i\gamma)E(D_\delta)\phi_k \| \leqq \| PR_0(\lambda_0 - i\gamma)/E(D_\delta)\mathscr{R}(V) \| \, \| E(D_\delta)\phi_k \| \leqq M \| E(D_\delta)\phi_k \|$. Since r is finite this proves the second limit is 0 and thus it proves the conclusion of Step 4.

Step 5. Let ν be the same integer as in Step 3. Then

$$(\lambda_0 - z)^{-\nu} \det \{(I + VR_0(z)/\mathscr{R}(V)\}$$

is bounded above and bounded away from zero for $z = \lambda_0 + i\gamma$ and $\gamma < \gamma_0$. This proves the conclusion of the theorem.

Proof. Note the following equation

$$\begin{aligned} &[\gamma^{\nu} \det \{P(Q_\delta - Q_\delta VQ_\delta R(\lambda_0 + i\gamma))/\mathscr{R}(V)\}] \\ &\qquad \times [\gamma^{-\nu} \det \{(T - \lambda_0 - i\gamma)R_0(\lambda_0 + i\gamma)/\mathscr{R}(V)\}] \\ &\quad = \det \{PQ_\delta(T - Q_\delta VQ_\delta - \lambda_0 - i\gamma)R(\lambda_0 + i\gamma)/\mathscr{R}(V) \\ &\qquad \times (T - \lambda_0 - i\gamma)R_0(\lambda_0 + i\gamma)/\mathscr{R}(V)\} . \end{aligned}$$

By Step 3 the first bracketed factor is bounded above and bounded away from zero for $\gamma < \gamma_0$; by Step 4 the right side or second line of the equation is bounded above and bounded away from zero for $\gamma < \gamma_0$. Thus ν must be the unique integer such that the second bracketed factor is bounded above and bounded away from zero for $\gamma < \gamma_0$. Since $(T - \lambda_0 - i\gamma)R_0(\lambda_0 + i\gamma) = I + VR_0(\lambda_0 + i\gamma)$ this proves the first assertion of step 5.

By Step 3, $\nu = \nu(\lambda_0, T) - (\lambda_0, T - Q_\delta VQ_\delta)$. Recalling from Step 1 that $\nu(\lambda_0, T - Q_\delta VQ_\delta) \geqq \nu(\lambda_0, T_0)$ we get $\nu(\lambda_0, T) - \nu \geqq \nu(\lambda_0, T_0)$. By letting $k = -\nu$ we see that part (b) of the theorem is proved. Since part (a) was proved in the above paragraph this concludes the proof of Theorem 1.

COROLLARY 1. *Let* (1) *and* (2) *of Theorem* 1 *be satisfied. If* $\omega(\lambda_0 + i\gamma)$ *is bounded above for all sufficiently small* γ *then* λ_0 *is an eigenvalue for* $T_0 + V$ *with multiplicity at least as great as its multiplicity for* T_0.

Proof. This is immediate from Theorem 1.

By using the symmetry between the perturbed and the unperturbed operators—i.e., add $(-V)$ to $(T_0 + V)$ and get T_0—we can get a result symmetric to Theorem 1.

COROLLARY 2 (*to Theorem* 1). *Let* λ_0 *be an eigenvalue of* T_0. *If there exists* $\delta > 0$ *such that*

(1) $E_0(D_\delta)E(\{\lambda_0\}) = 0$ *with* $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ *and*

(2) $\| PR(\lambda_0 - i\gamma)/E_0(D_\delta)\mathscr{R}(V) \| \leqq M < +\infty$ *for all sufficiently small* γ, *say* $\gamma < \gamma_0$, *then the following are true*:

(a) *there exists a unique integer* k *such that* $(\lambda_0 - z)^k\omega(z) = (\lambda_0 - z)^k \det \{I + VR_0(z)/\mathscr{R}(V)\}$ *is bounded above and bounded away from zero for all* $z = \lambda_0 + i\gamma$ *and* γ *sufficiently small, and*

(b) $\nu(\lambda_0, T_0) \geqq \nu(\lambda_0, T_0 + V) + k$.

Proof. By a direct application of Theorem 1 considering $T_0 + V$ as the unperturbed operator and $(-V)$ as the perturbation one gets the existence of an integer $-k$ such that $(\lambda_0 - z)^{-k} \det \{I - VR(z)/\mathscr{R}(V)\}$ is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ and γ sufficiently small. Also

$$\nu(\lambda_0, (T_0 + V) - V) \geqq \nu(\lambda_0, (T_0 + V)) - (-k)$$

or

$$\nu(\lambda_0, T_0) \geqq \nu(\lambda_0, T_0 + V) + k \,.$$

$$1 = \{(\lambda_0 - z)^{-k} \det [I - VR(z)]/\mathscr{R}(V)\} \times \{(\lambda_0 - z)^k \det [I + VR_0(z)]/\mathscr{R}(V)\}$$

is obviously bounded above and bounded away from zero everywhere. Since the first factor in braces has this property also, it must be that the second factor in braces has this property. This proves the corollary.

COROLLARY 3. (*The generalized Weinstein-Aronszajn formula.*) *If* λ_0 *is a quasi-isolated eigenvalue for* T_0 *and if there exist positive numbers* γ_0 *and* M *such that* $\| PR_0(\lambda_0 - i\gamma)/E(D_\delta)\mathscr{R}(V) \| \leqq M$ *and* $\| PR(\lambda_0 - i\gamma)/E_0(D_\delta)\mathscr{R}(V) \| \leqq M$ *for* $\gamma < \gamma_0$ *then the following are true*:

(a) *there exists a unique integer* k *such that*

$$(\lambda_0 - z)^k\omega(z) = (\lambda_0 - z)^k \det \{I + VR_0(z)/\mathscr{R}(V)\}$$

is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ *and* $\gamma < \gamma_0$, *and*

(b) $\nu(\lambda_0, T) = \nu(\lambda_0, T_0) - k$.

Proof. Simply apply Theorem 1 and Corollary 2 both.

3. Examples. In this section two examples of the preceding theory will be given; however the verification that they are examples will only be outlined. Following each example, the significant facts about that example will be given.

No effort has been made to render the examples as general as possible; in fact many arbitrary choices have been made. The examples do demonstrate how the theory can be applied and Example 1 shows that the generalization is a proper generalization of the $W-A$ theorem.

EXAMPLE 1. Let H_1 and H_2 be the spaces of square integrable functions on the interval $(1, 2)$ with the measures $du(t)$ and dt, respectively. Here dt is Lebesgue measure while $du(t)$ agrees with dt on $(1, 5/4) \cup (7/4, 2)$ and $u([5/4, 3/2) \cup (3/2, 7/4]) = 0$ and $u(\{3/2\}) = 1$. Let $T_0(f_1(t), f_2(t)) = (tf_1(t), tf_2(t))$ where $(f_1(t), f_2(t))$ is an element of $H = H_1 \oplus H_2$. We obviously have a spectral representation space for T_0 and it is clear that $3/2$ is an eigenvalue for T_0 with a corresponding eigenvector $(\chi_{\{3/2\}}(t), 0) = \psi_0$. Set $V = \langle\cdot, \phi_1\rangle c_1\phi_1 + \langle\cdot, \phi_2\rangle c_2\phi_2$ with $\phi_1 = (1, 0)$, $\phi_2 = (0, 1)$.

Fact 1. $3/2$ is quasi-isolated.

Fact 2.

$$\langle R_0(z)\phi_1, \phi_1\rangle = \int_{(1,2)} (t - z)^{-1} du(t) = (3/2 - z)^{-1} + \int_{(1,5/4) \cup (7/4,2)} (t - z)^{-1} dt$$

and $\langle R_0(z)\phi_2, \phi_2\rangle = \int_{(1,2)} (t - z)^{-1} dt$ and $\int_{(1,2)} (t - \lambda - i\gamma)^{-1} dt$ approaches as a limit $ln[(2 - \lambda)/(\lambda - 1)] + \pi i$ provided $\lambda \in (1, 2)$ and γ approaches 0 from the right.

Fact 3. If $\omega(z)$ is the $W-A$ determinant $i\gamma\, \omega(_3/2 + i\gamma)$ is bounded above and is bounded away from zero for $|\gamma|$ sufficiently small.

Fact 4. For all $|\gamma|$ sufficiently small $\|PR(\lambda_0 - i\gamma)/E_0(D_{1/4})\mathscr{R}(V)\| \leqq M$ and thus by Corollary 2 to Theorem 1 we get $\nu(3/2, T_0 + V) = 0$.

Note. Although $3/2$ is quasi-isolated it is an embedded eigenvalue. Still the change in the multiplicity is given by the formula involving the $W-A$ determinant. Finally there is no triviality involved in the example in the sense that H has no proper subspace reducing T_0 and containing $\mathscr{R}(V)$.

EXAMPLE 2. Let H, H_1, H_2, and T_0 be as in Example 1. In the definition of V change ϕ_1 to $(\chi_D(t), 0)$ where $D = (1, 5/4) \cup (7/4, 2)$.

Fact 1. $3/2$ is quasi-isolated.

Fact 2.

$$\langle R_0(z)\phi_1, \phi_1\rangle = \int_D (t - z)^{-1} du(t) = \int_{(1,5/\ )\cup(7/4,2)} (t - z)^{-1} dt$$

and $\langle R_0(z)\phi_2, \phi_2\rangle = \int_{(1,2)} (t - z)^{-1}dt$ which approaches as a limit

$$ln(2 - \lambda)/(\lambda - 1)] + i\pi \quad \text{for} \quad \lambda \in (1, 2) .$$

Fact 3. If $\omega(z)$ is the $W-A$ determinant then $\omega(3/2 + i\gamma)$ is bounded above and is bounded away from 0 for $|\gamma|$ sufficiently small.

Fact 4. The 3/2-eigenvector of T_0, $(\chi_{\{3/2\}}(t), 0)$ is in ker V. Thus it is a 3/2-eigenvector for $T = T_0 + V$ and $\nu(3/2, T) \geqq 1$.

Fact 5. For all $|\gamma|$ sufficiently small

$$\| PR(\lambda_0 - i\gamma)/E_0(D_{1/4})\mathscr{R}(V) \| \leqq M .$$

Using Corollary 2 in addition to the preceding fact we get $\nu(3/2, T) = 1$.

Note. This example is guilty of some triviality since a 3/2-eigenvector of T_0 is in the kernel of V. Nevertheless the quasi-isolated eigenvalue 3/2 is embedded and is preserved by the finite dimensional perturbation. Also the new multiplicity is given by the $W-A$ formula.

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