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SINGULARITY OF GAUSSIAN MEASURES IN FUNCTION SPACES WITH FACTORABLE COVARIANCE FUNCTIONS

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Singularity of Gaussian measures μ_1 and μ_2 on the function space R^D of real valued functions $x(t)$ on an arbitrary interval D with factorable covariance functions $r_i(s, t)$, i.e., $r_i(s, t) = u_i(s)v_i(t)$ for $s \leq t$ and $r_i(s, t) = v_i(s)u_i(t)$ for $s \geq t$, $i = 1, 2$, is treated. Local conditions on the factor functions $u_i(t)$ and $v_i(t)$ which insure the singularity of μ_1 and μ_2 are given.

Consider the measurable space $(R^D, \mathfrak{F})$ where R^D is the space of all real valued functions $x(t)$ on a fixed but unspecified interval D of the real line and $\mathfrak{F}$ is the smallest σ-field of subsets of R^D with respect to which all real valued functions $Y(t, x) = x(t)$ defined on R^D with parameter $t \in D$ are measurable. A probability measure μ on $(R^D, \mathfrak{F})$ is called a Gaussian measure on the function space R^D if the stochastic process $Y(t, x) = x(t)$ on the probability space $(R^D, \mathfrak{F}, \mu)$ with the domain of definition D is a Gaussian process. From the viewpoint of stochastic processes if $X(t, \omega)$ is a stochastic process on an arbitrary probability space $(\Omega, \mathfrak{B}, P)$ with the domain of definition D then a probability measure μ_X is induced on the measurable space $(R^D, \mathfrak{F})$ by embedding the sample functions $X(\cdot, \omega), \omega \in \Omega$, in R^D. The stochastic process $Y(t, x) = x(t)$ defined on the probability space $(R^D, \mathfrak{F}, \mu_X)$ with the domain of definition D is equivalent to the original process $X(t, \omega)$ so that if $X(t, \omega)$ is a Gaussian process so is $Y(t, x)$. Thus a Gaussian measure on $(R^D, \mathfrak{F})$ can be defined equivalently as the probability measure μ_X induced on $(R^D, \mathfrak{F})$ by a Gaussian process $X(t, \omega)$.

J. Feldman [3] and J. Hájek [4], [5] showed independently that any two Gaussian measures are either equivalent or singular. In [7] we applied Hájek's criterion for equivalence or singularity to investigate the singularity of Gaussian measures induced by Brownian motion processes with nonstationary increments. In the present paper we consider the singularity of Gaussian measures μ_1 and μ_2 on $(R^D, \mathfrak{F})$ for which the covariance functions $r_i(s, t)$ of the stochastic process $Y(t, x) = x(t)$ are factorable. Our main result is the following theorem

THEOREM. *Let μ_1 and μ_2 be Gaussian measures on $(R^D, \mathfrak{F})$ with zero mean functions and factorable covariance functions $r_i(s, t)$, $i = 1, 2$, given by*

$$(1.1)\qquad r_i(s, t) = \begin{cases} u_i(s)v_i(t) & s \leqq t,\ s, t \in D,\ i = 1, 2 \\ v_i(s)u_i(t) & s \geqq t,\ s, t \in D,\ i = 1, 2 \end{cases}$$

where $u_i(t)$ *and* $v_i(t)$ *are nonnegative functions on* D *satisfying*

$$(1.2)\quad u_i(t'')v_i(t') - u_i(t')v_i(t'') \geqq 0 \qquad t', t'' \in D, t' < t'', i = 1, 2 .$$

If there exists $t_0 \in D$ *such that* $v_i(t) > 0$ *and* $u_i(t)[v_i(t)]^{-1}$ *are strictly increasing on* $(t_0, t_0 + \delta)$ *for some* $\delta > 0$, *the right derivatives* $D^+u_i(t_0)$ *and* $D^+v_i(t_0)$ *of* $u_i(t)$ *and* $v_i(t)$ *at* t_0 *exist and*

$$(1.3)\qquad u_i(t_0) = 0, D^+u_i(t_0) = \lambda_i > 0 , \qquad i = 1, 2$$

$$(1.4)\qquad v_i(t_0) = r_i > 0 , \qquad i = 1, 2$$

then the condition

$$\lambda_1 r_1 \neq \lambda_2 r_2$$

implies the singularity of μ_1 *and* μ_2.

We remark that the above theorem can also be stated in terms of the left derivatives of $u_i(t)$ and $v_i(t)$. When $v_i(t)$ are positive on D the condition (1.2) is equivalent to the condition that $u_i(t)[v_i(t)]^{-1}$ be nondecreasing on D. For a symmetric function $r(s, t)$, $s, t \in D$, defined as in (1.1) by means of two nonnegative functions $u(t)$ and $v(t)$ on D to be the covariance function of a Gaussian process it is necessary and sufficient that for any $t_1, \cdots, t_n \in D, t_1 < \cdots < t_n$, the $n \times n$ matrix $[r(t_k, t_l), k, l = 1, 2, \cdots, n]$ be nonnegative definite. The condition (1.2) is equivalent to this condition (see p. 525, [1]). In particular for every $n \times n$ matrix $[r(t_k, t_l), k, l = 1, 2, \cdots, n]$ to be positive definite it is necessary and sufficient that $u(t)$ and $v(t)$ be positive on D and the strict inequality in (1.2) hold. In connection with our theorems we mention an earlier result by G. Baxter, corollary [1], which showed that if $u_i(t)$ and $v_i(t)$ have bounded second derivatives on $D = [0, 1]$ then for the two subsets E_i, $i = 1, 2$, of R^D defined by

$$E_i = \left\{x \in R^D; \lim_{n\to\infty} \sum_{k=1}^{2^n} \left[x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right)\right]^2 = \int_0^1 \{u_i'(t)v_i(t) - u_i(t)v_i'(t)\}dt\right\}$$

the equalities $\mu_i(E_i) = 1$, $i = 1, 2$, hold so that the condition

$$\int_0^1 \{u_1'(t)v_1(t) - u_1(t)v_1'(t)\}dt \neq \int_0^1 \{u_2'(t)v_2(t) - u_2(t)v_2'(t)\}dt$$

implies $E_1 \cap E_2 = \emptyset$ as well as $\mu_i(E_j) = \delta_{ij}$.

The proof of the theorem is given in § 3. For some examples of factorable covariance functions to which our theorem can be applied see J. A. Beekman, pp. 805–806, [2].

2. A lemma concerning the inversion of a class of symmetric matrices.

LEMMA. *Given real or complex numbers*

$$a_1, a_2, \cdots, a_n \quad \text{and} \quad b_1, b_2, \cdots, b_n \,.$$

Let $M = [m_{k,l}, k, l = 1, 2, \cdots, n]$ *be an* $n \times n$ *symmetric matrix with entries*

$$m_{k,l} = a_k b_l \qquad \text{for } k \leqq l \,, \qquad k, l = 1, 2, \cdots, n \,.$$

Let

$$\begin{aligned} C_j &= a_j b_{j-1} - a_{j-1} b_j \qquad j = 2, 3, \cdots, n \\ D_j &= a_j b_{j-2} - a_{j-2} b_j \qquad j = 3, 4, \cdots, n \end{aligned}$$

then

$$\det M = a_1 b_n \prod_{j=2,\cdots,n} C_j \,.$$

For the determinants $M_{k,l}$ *of the minor matrices corresponding to the entries* $m_{k,l}$ *we have*

$$\begin{aligned} M_{1,1} &= a_2 b_n \prod_{j=3,\cdots,n} C_j \,, & M_{1,2} &= a_1 b_n \prod_{j=3,\cdots,n} C_j \,, \\ M_{1,l} &= 0 \text{ for } l = 3, \cdots, n \,, & M_{k,k} &= a_1 b_n D_{k+1} \prod_{\substack{j=2,\cdots,n \\ j \neq k, k+1}} C_j \,, \\ M_{k,k+1} &= a_1 b_n \prod_{\substack{j=2,\cdots,n \\ j \neq k+1}} C_j \,, & M_{k,l} &= 0 \text{ for } l = k+2, \cdots, n \,, \end{aligned}$$

for $k = 2, \cdots, n-1$, *and finally*

$$M_{n,n} = a_1 b_{n-1} \prod_{j=2,\cdots,n-1} C_j \,.$$

In particular M *is invertible if and only if* $a_1, b_n, C_j \neq 0$ *for* $j = 2, \cdots, n$. *In this case*

$$M^{-1} = \left[\frac{(-1)^{k+l}}{\det M} M_{k,l}, k, l = 1, 2, \cdots, n \right] . \tag{2.1}$$

The proof of this lemma is lengthy and will not be given here. We merely mention that the expression (2.1) for M^{-1} can be verified by direct multiplication with M.

3. Proof of the theorem.

The J-divergence of two probability measures P and Q on a measurable space $(\Omega, \mathfrak{B})$ is defined to be

$$(3.1)\quad J(P, Q) = \begin{cases} E_P\left[\log \frac{dP}{dQ}(\omega)\right] + E_Q\left[\log \frac{dQ}{dP}(\omega)\right] & \text{when } P \text{ and } Q \text{ are equivalent} \\ \infty & \text{otherwise} \end{cases}$$

where E_P and E_Q denote integration with respect to the probability measures P and Q.

Let $\mathfrak{F}$ be the smallest σ-field of subsets of the function space R^D with respect to which the real valued function $Y(t, x) = x(t)$ on R^D is measurable for every $t \in D$. For $t_1, \cdots t_n \in D$, $t_1 < \cdots < t_n$, let

$$p_{t_1 \cdots t_n}(x) = [x(t_1), \cdots, x(t_n)] \qquad x \in R^D$$
$$p^{-1}_{t_1 \cdots t_n}(B) = \{x \in R^D; [x(t_1), \cdots, x(t_n)] \in B\} \qquad B \in \mathfrak{B}^n$$

where $\mathfrak{B}^n$ is the σ-field of Borel sets in the n-dimensional Euclidean space R^n, and let

$$(3.2)\qquad \mathfrak{F}_{t_1 \cdots t_n} = \{p^{-1}_{t_1 \cdots t_n}(B), B \in \mathfrak{B}^n\} .$$

Then $\mathfrak{F}_{t_1 \cdots t_n}$ is a σ-field of subsets of R^D and $\mathfrak{F}$ is the σ-field generated by the union of all the σ-fields $\mathfrak{F}_{t_1 \cdots t_n}$. Given two probability measures μ_i, $i = 1, 2$, on $(R^D, \mathfrak{F})$, let $\mu_{i, t_1 \cdots t_n} = \mu_i \mid \mathfrak{F}_{t_1 \cdots t_n}$, i.e., the restrictions of μ_i to the σ-field $\mathfrak{F}_{t_1 \cdots t_n}$. Let $J = J(\mu_1, \mu_2)$ and

$$J_{t_1 \cdots t_n} = J(\mu_{1, t_1 \cdots t_n}, \mu_{2, t_1 \cdots t_n}) .$$

According to J. Hájek [4], [5], $J = \sup J_{t_1 \cdots t_n}$ where the supremum is taken over the collection of $\{t_1, \cdots t_n\}$, i.e., over the collection of all the σ-fields $\mathfrak{F}_{t_1 \cdots t_n}$, and $J < \infty$ implies the equivalence of μ_1 and μ_2. Furthermore if μ_1 and μ_2 are Gaussian then $J = \infty$ implies the singularity of μ_1 and μ_2.

Let $t_1, \cdots, t_n \in D$, $t_0 < t_1 < \cdots < t_n < t_0 + \delta$. For the fixed $\{t_1, \cdots, t_n\}$ there is one-to-one correspondence between the members of $\mathfrak{F}_{t_1 \cdots t_n}$ and the members of $\mathfrak{B}^n$ according to the definition (3.2). Since the measures μ_i, $i = 1, 2$, are Gaussian, i.e., the stochastic process $Y(t, x) = x(t)$ is a Gaussian process on each of the two probability spaces $(R^D, \mathfrak{F}, \mu_i)$, we have

$$(3.3)\qquad \mu_i(p^{-1}_{t_1 \cdots t_n}(B)) = \Phi_{i, t_1 \cdots t_n}(B) , \qquad B \in \mathfrak{B}^n, i = 1, 2$$

where $\Phi_{i, t_1 \cdots t_n}$ are n-dimensional (regular or degenerate) normal distributions on $(R^n, \mathfrak{B}^n)$.

Now since $v_i(t) > 0$ and $u_i(t)[v_i(t)]^{-1}$ are strictly increasing on $(t_0, t_0 + \delta)$ we have

$$u_i(t'')v_i(t') - u_i(t')v_i(t'') > 0 \quad \text{for} \quad t', t'' \in (t_0, t_0 + \delta), t' < t'', i = 1, 2 .$$

Then the covariance matrices $[r_i(t_k, t_l), k, l = 1, 2, \cdots, n]$, $i = 1, 2$, of the n-dimensional normal distributions $\Phi_{i, t_1 \cdots t_n}$ are positive definite and consequently $\Phi_{it_1\cdots t_n}$ are regular with density functions given by

$$\begin{aligned}&\Phi'_{i,t_1\cdots t_n}(\xi)\\ (3.4)\qquad &= \frac{1}{\{(2\pi)^n \det W_{i,t_1\cdots t_n}\}^{1/2}} \exp\left\{-\frac{1}{2}(W^{-1}_{i,t_1\cdots t_n}\xi, \xi)\right\}, \ \xi \in R^n,\ i = 1, 2\end{aligned}$$

where $W_{i,t_1\cdots t_n} = [w_{i,k,l}, k, l = 1, 2, \cdots, n]$ are $n \times n$ symmetric and positive definite matrices with entries

$$(3.5)\quad w_{i,k,l} = u_i(t_k)v_i(t_l) \quad \text{for} \quad k \leqq l,\ k, l = 1, 2, \cdots, n,\ i = 1, 2\ .$$

Now

$$(3.6)\qquad \Phi_{i,t_1\cdots t_n}(B) = \int_{R^n} \Phi'_{i,t_1\cdots t_n}(\xi) m_L(d\xi)\ , \qquad B \in \mathfrak{B}^n,\ i = 1, 2$$

where m_L is the Lebesgue measure on $(R^n, \mathfrak{B}^n)$. The regularity of $\Phi_{1,t_1\cdots t_n}$ and $\Phi_{2,t_1\cdots t_n}$ implies their equivalence. This in turn implies the equivalence of $\mu_{1,t_1\cdots t_n}$ and $\mu_{2,t_1\cdots t_n}$ on account of the one-to-one correspondence between the members of $\mathfrak{F}_{t_1\cdots t_n}$ and the members of $\mathfrak{B}^n$ and the relation (3.3) between $\mu_{i,t_1\cdots t_n}$ and $\Phi_{i,t_1\cdots t_n}$. From (3.6) and (3.4) we obtain the Radon-Nikodym derivatives

$$\begin{aligned}(3.7)\qquad &\frac{d\mu_{j,t_1\cdots t_n}}{d\mu_{i,t_1\cdots t_n}}(x) = \frac{d\Phi_{j,t_1\cdots t_n}}{d\Phi_{i,t_1\cdots t_n}}(\xi) = \frac{\Phi'_{j,t_1\cdots t_n}(\xi)}{\Phi'_{i,t_1\cdots t_n}(\xi)}\\ &= \left[\frac{\det W_{i,t_1\cdots t_n}}{\det W_{j,t_1\cdots t_n}}\right]^{1/2} \exp\left\{\frac{1}{2}([W^{-1}_{i,t_1\cdots t_n} - W^{-1}_{j,t_1\cdots t_n}]\xi, \xi)\right\},\ i, = 1, 2\ .\end{aligned}$$

According to (3.1), (3.3) and (3.7)

$$\begin{aligned}(3.8)\qquad J_{t_1\cdots t_n} &= E_{\mu_{2,t_1\cdots t_n}}\left[\log \frac{d\mu_{2,t_1\cdots t_n}}{d\mu_{1,t_1\cdots t_n}}(x)\right] + E_{\mu_{1,t_1\cdots t_n}} \log\left[\frac{d\mu_{1,t_1\cdots t_n}}{d\mu_{2,t_1\cdots t_n}}(x)\right]\\ &= E_{\Phi_{2,t_1\cdots t_n}}\left[\log \frac{\Phi'_{2,t_1\cdots t_n}(\xi)}{\Phi'_{1,t_1\cdots t_n}(\xi)}\right] + E_{\Phi_{2,t_1\cdots t_n}}\left[\log \frac{\Phi'_{1,t_1\cdots t_n}(\xi)}{\Phi'_{2,t_1\cdots t_n}(\xi)}\right].\end{aligned}$$

In evaluating the integrals in (3.8) we quote the well known equality that for any $n \times n$ matrices A and B where A is symmetric and B is positive definite

$$(3.9)\qquad \frac{1}{\{2\pi)^n \det B\}^{1/2}} \int_{R^n} (A\xi, \xi) \exp\left\{-\frac{1}{2}(B^{-1}\xi, \xi)\right\} m_L(d\xi) = Tr(C)$$

where $C = AB$ and $Tr(C) = \sum_{k=1}^n c_{k,k}$ for $C = [c_{k,l}, k, l = 1, 2, \cdots, n]$. Applying (3.9) to (3.8) remembering (3.7), (3.6) and (3.4)

$$(3.10)\quad J_{t_1\cdots t_n} = \frac{1}{2} Tr[W^{-1}_{1,t_1\cdots t_n} W_{2,t_1\cdots t_n} + W^{-1}_{2,t_1\cdots t_n} W_{1,t_1\cdots t_n} - 2I]\ .$$

We proceed to evaluate the diagonal entries of the two product matrices in (3.10). Let us consider $W^{-1}_{1,t_1\cdots t_n}W_{2,t_1\cdots t_n}$ for example. The entries of $W_{i,t_1\cdots t_n}$ are given by (3.5). Let $M_{i,k,l}$ be the determinant of the minor matrix corresponding to $w_{i,k,l}$. According to our lemma, § 2, the 1st diagonal entry of $W^{-1}_{1,t_1\cdots t_n}W_{2,t_1\cdots t_n}$ is given by

$$\begin{aligned}(3.11)\qquad &\frac{M_{1,1,1}w_{2,1,1} - M_{1,1,2}w_{2,1,2}}{\det W_{1,t_1\cdots t_n}} \\ &= [u_1(t_2)v_1(t_n)u_2(t_1)v_2(t_1) - u_1(t_1)v_1(t_n)u_2(t_1)v_2(t_2)] \\ &\cdot[u_1(t_1)v_1(t_n)\{u_1(t_2)v_1(t_1) - u_1(t_1)v_1(t_2)\}]^{-1} .\end{aligned}$$

The k-th diagonal entry, $k \neq 1, n$, is given by

$$\begin{aligned}&\frac{-M_{1,k-1,k}w_{2,k-1,k} + M_{1,k,k}w_{2,k,k} - M_{1,k,k+1}w_{2,k,k+1}}{\det W_{1,t_1\cdots t_n}} \\ &= u_1(t_1)v_1(t_n)[-\{u_1(t_{k+1})v_1(t_k) - u_1(t_k)v_1(t_{k+1})\}u_2(t_{k-1})v_2(t_k) \\ &+ \{u_1(t_{k+1})v_1(t_{k-1}) - u_1(_{k-1})v_1(t_{k+1})\}u_2(t_k)v_2(t_k) \\ (3.12)\qquad &- \{u_1(t_k)v_1(t_{k-1}) - u_1(t_{k-1})v_1(t_k)\} \\ &\cdot u_2(t_k)v_2(t_{k+1})][u_1(t_1)v_1(t_n)\{u_1(t_k)v_1(t_{k-1}) \\ &- u_1(t_{k-1})v_1(t_k)\}\{u_1(t_{k+1})v_1(t_k) - u_1(t_k)v_1(t_{k+1})\}]^{-1} .\end{aligned}$$

Finally, the n-th diagonal entry is given by

$$\begin{aligned}&\frac{-M_{1,n-1,n}w_{2,n-1,n} + M_{1,n,n}w_{2,n,n}}{\det W_{1,t_1\cdots t_n}} = [-u_1(t_1)v_1(t_n)u_2(t_{n-1})v_2(t_n) \\ (3.13)\qquad &+ u_1(t_1)v_1(t_{n-1})u_2(t_n)v_2(t_n)][u_1(t_1)v_1(t_n)\{u_1(t_n)v_1(t_{n-1}) \\ &- u_1(t_{n-1})v_1(t_n)\}]^{-1} .\end{aligned}$$

Now according to (1.3)

$$u_i(t) = \lambda_i(t - t_0) + \varepsilon_i(t - t_0) \qquad \text{where} \qquad \lim_{t\downarrow t_0} \varepsilon_i = 0,\ i = 1, 2 .$$

For fixed n let p be a sufficiently large positive integer so that $t_k = t_0 + k/p \in (t_0, t_0 + \delta)$ for $k = 1, 2, \cdots, n$. Then

$$(3.14)\qquad u_i(t_k) = \frac{k}{p}(\lambda_i + \varepsilon_i) = k\frac{\lambda_i}{p}\{1 + o(1)\} , \qquad k = 1, 2, \cdots, n,\ p \to \infty,\ i = 1, 2 .$$

From (1.4), writing ν_i for $D^+v_i(t_0)$,

$$(3.15)\qquad v_i(t_k) = r_i + k\frac{\nu_i}{p}\{1 + o(1)\} = r_i\Big\{1 + O\Big(\frac{n}{p}\Big)\Big\} , \qquad k = 1, 2, \cdots n,\ p \to \infty,\ i = 1, 2 .$$

If we apply (3.14) and (3.15) to (3.11), the 1st diagonal entry of $W^{-1}_{1,t_1\cdots t_n}W_{2,t_1\cdots t_n}$ is reduced to

$$(3.16)\qquad \frac{\lambda_1\lambda_2 r_1 r_2 p^{-2}\{2-1\}\{1+o(1)\}\left\{1+O\left(\frac{n}{p}\right)\right\}}{\lambda_1^2 r_1^2 p^{-2}\{2-1\}\{1+o(1)\}\left\{1+O\left(\frac{n}{p}\right)\right\}} = \frac{\lambda_2 r_2}{\lambda_1 r_1}\{1+o(1)\}\ .$$

Similarly the k-th diagonal entry, $k \neq 1, n$, is reduced to

$$(3.17)\qquad \begin{aligned} &\frac{\lambda_1^2\lambda_2 r_1^2 r_2 p^{-3}[-\{(k+1)-k\}(k-1)+\{(k+1)-(k-1)\}}{\lambda_1^3 r_1^3 p^{-3}\{k-(k-1)\}\{(k+1)-k\}} \\ &\times \frac{k-\{k-(k-1)\}k]\{1+o(1)\}\left\{1+O\left(\frac{n}{p}\right)\right\}}{\{1+o(1)\}\left\{1+O\left(\frac{n}{p}\right)\right\}} = \frac{\lambda_2 r_2}{\lambda_1 r_1}\{1+o(1)\} \end{aligned}$$

and finally the n-th diagonal entry is reduced to

$$(3.18)\qquad \begin{aligned} &\frac{\lambda_1\lambda_2 r_1 r_2 p^{-2}\{-(n-1)+n\}\{1+o(1)\}\left\{1+O\left(\frac{n}{p}\right)\right\}}{\lambda_1^2 r_1^2 p^{-2}\{n-(n-1)\}\{1+o(1)\}\left\{1+O\left(\frac{n}{p}\right)\right\}} \\ &\qquad = \frac{\lambda_2 r_2}{\lambda_1 r_1}\{1+o(1)\}\ . \end{aligned}$$

From (3.16), (3.17) and (3.18)

$$T_r[W^{-1}_{1,t_1\cdots t_n} W_{2,t_1\cdots t_n}] = n\frac{\lambda_2 r_2}{\lambda_1 r_1}\{1+o(1)\}\ , \qquad p\to\infty\ .$$

Similarly

$$T_r[W^{-1}_{2,t_1\cdots t_n} W_{1,t_1\cdots t_n}[= n\frac{\lambda_1 r_2}{\lambda_2 r_2}\{1+o(1)\}\ , \qquad p\to\infty\ .$$

Substituting these estimates in (3.10) we obtain

$$J_{t_1\cdots t_n} = \frac{n}{2}\left\{\sqrt{\frac{\lambda_2 r_2}{\lambda_1 r_1}} - \sqrt{\frac{\lambda_1 r_2}{\lambda_2 r_2}}\right\}^2 + no(1)\ , \qquad p\to\infty\ .$$

Since n is fixed, $no(1)\to 0$ as $p\to\infty$. Thus for sufficiently large p chosen for the given n, $no(1)$ is as small as we wish. Therefore

$$\sup J_{t_1\cdots t_n} = \infty\ .$$

This proves the singularity of μ_1 and μ_2.

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