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MINIMAL T_0-SPACES AND MINIMAL T_D-SPACES

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The family of all topologies on a set is a complete, atomic lattice. There has been a considerable amount of interest in topologies which are minimal or maximal in this lattice with respect to certain topological properties. Given a topological property P, we say a topology is minimal P (maximal P) if every weaker (stronger) topology does not possess property P. A topological space $(X, \mathscr{T})$ is called a T_D-space if and only if $[x]'$, (the derived set of $[x]$) is a closed set for every x in X [1]. It is known that a space is T_D if and only if for every x in X there exists an open set G and a closed set C such that $[x] = G \cap C$ [9]. The purpose of this paper is to characterize minimal T_0 and minimal T_D-spaces as follows: A T_0-space is minimal T_0 if and only if the family of open sets is nested and the complements of the point closures form a base for the topology. A T_D-space is minimal T_D if and only if the open sets are nested. These characterizations prove to be useful in gaining other results about minimal T_0 and minimal T_D-spaces.

The following are examples of characterizations of some minimal and maximal topological spaces. A space is mininum T_1 if and only if the closed sets are precisely the finite sets. A T_2-space is minimal T_2 if and only if every open filter which has a unique cluster point converges to that point [4]. A T_2-space is minimal T_2 if and only if it is semi-regular and absolutely H-closed [6]. A T_{2a}-space (Urysohn space) is minimal T_{2a} if and only if for every two open filters $\mathscr{F}_1$ and $\mathscr{F}_3$ such that there exists a closed filter $\mathscr{F}_2$ with $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$ and such that $\mathscr{F}_1$ has a unique cluster point, it follows that $\mathscr{F}_3$ converges to that point [5]. A T_3-space is minimal T_3 if and only if every regular filter (a filter which has both an open filter base and a closed filter base) which has a unique cluster point converges to that point [3]. A space is minimal T_{3a} (Tychonoff) and minimal T_4 if and only if it is T_2 and bicompact [2]. A space is maximal bicompact if and only if the bicompact subsets of the space are precisely the closed subsets of the space [8].

In [8] it is mentioned that there exist minimal T_0-spaces which are not bicompact, also the fact that in a minimal T_0-space every open set is dense was known to N. Symthe and C. A. Wilkins. At the time I wrote this paper, I was not aware of any other mention of these spaces. However, since that time, a recent paper by Ki-Hyun Pahk has been brought to my attention [7]. By a different sequence of lemmas, he obtained the result given in Theorem 1, but his characterization of minimal T_D-spaces contains an unnecessary somewhat cumbersome condition. The results of Lemmas 1 and 2 as well as Theorems 3 through 7 are not discussed in Pahk's paper.

LEMMA 1. *If $(X, \mathscr{T})$ is a T_0 or T_D topological space and B is an open set in $\mathscr{T}$, then the family $\mathscr{T}(B) = [G: G \in \mathscr{T}, G \subset B$ or $B \subset G]$ is, respectively, a T_0 or T_D topology on X.*

Proof. One may easily see that $\mathscr{T}(B)$ is a topology on X by making the following observations. $\varnothing \subset B$ and $B \subset X$ imply that $\varnothing$, $X \in \mathscr{T}(B)$. If G_1 and G_2 are elements of $\mathscr{T}(B)$, then $G_1 \cap G_2 \in \mathscr{T}(B)$ since either both G_1 and G_2 contain B, in which case $B \subset G_1 \cap G_2$, or one of the sets G_1, G_2 is a subset of B, in which case $G_1 \cap G_2 \subset B$. If $[G_\alpha: \alpha \in A]$ is an arbitrary family of open sets in $\mathscr{T}(B)$, then $\cup [G_\alpha: \alpha \in A]$ is an element of $\mathscr{T}(B)$ since either every $G_\alpha \subset B$, in which case $\cup [G_\alpha: \alpha \in A] \subset B$, or B is a subset of some G_α, in which case $B \subset \cup [G_\alpha: \alpha \in A]$.

To see that $\mathscr{T}(B)$ is T_0 if $\mathscr{T}$ is a T_0 topology on X, we consider the following three cases.

Case 1. If $x, y \in B$, and there exists an open set $G \in \mathscr{T}$ such that $x \in G$ and $y \notin G$, then $x \in G \cap B$, $y \notin G \cap B$, and $G \cap B \in \mathscr{T}(B)$.

Case 2. If $x, y \notin B$, and there exists an open set $G \in \mathscr{T}$ such that $x \in G$ and $y \notin G$, then $x \in G \cup B$, $y \notin G \cup B$, and $G \cup B \in \mathscr{T}(B)$.

Case 3. If $x \in B$ and $y \notin B$, we are done, since $B \in \mathscr{T}(B)$.

Similarly, we see that $\mathscr{T}(B)$ is T_D if $\mathscr{T}$ is a T_D topology on X.

Case 1. If $x \in B$, then since $\mathscr{T}$ is T_D, there exists an open set $G \in \mathscr{T}$ and a closed set C such that $[x] = G \cap C$. Then $G \cap B \in \mathscr{T}(B)$ and $\sim C \cup B \in \mathscr{T}(B)$; therefore, $C \cup \sim B$ is closed with respect to $\mathscr{T}(B)$ and

$$(G \cap B) \cap (C \cup \sim B) = (G \cap B \cap C) \cup (G \cap B \cap \sim B) = [x] .$$

Case 2. If $x \notin B$ with G and C as before, then $G \cup B \in \mathscr{T}(B)$ and $\sim C \cup B \in \mathscr{T}(B)$; therefore, $C \cap \sim B$ is closed with respect to $\mathscr{T}(B)$ and

$$(G \cup B) \cap (C \cap \sim B) = (G \cap C \cap \sim B) \cup (B \cap C \cap \sim B) = [x] .$$

LEMMA 2. *If $(X, \mathscr{T})$ is a topological space, then the following three conditions are equivalent:*

(1) *The open sets in the topology are nested.*
(2) *The closed sets in the topology are nested.*
(3) *Finite unions of point closures are point closures.*

Proof. It is clear that the first and second conditions are equivalent. It is also clear that the second condition implies the third since given a finite number of nested point closures, their union is simply the largest. In order to see that the third condition implies the second, assume that C and D are closed sets in $(X, \mathscr{T})$. If $C \neq D$, then either $C \sim D \neq \varnothing$ or $D \sim C \neq \varnothing$. Since these two cases are symmetrical, we will assume $C \sim D \neq \varnothing$ and show that this implies $D \subset C$. Choose $x \in C \sim D$, $y \in D$. If $[\bar{x}] \cup [\bar{y}] = [\bar{z}]$, then either $z \in [\bar{x}]$ or $z \in [\bar{y}]$; but $y \in [\bar{z}]$ and $x \in [\bar{z}]$; therefore, $[\bar{z}] = [\bar{x}]$ or $[\bar{z}] = [\bar{y}]$. How ever, if $[\bar{z}] = [\bar{y}]$, then $[\bar{x}] \subset [\bar{y}] \subset D$ and this is a contradiction since $x \in C \sim D$. Therefore, $[\bar{z}] = [\bar{x}]$, and $[\bar{y}] \subset [\bar{x}] \subset C$, which implies that $y \in C$ and $D \subset C$. Therefore the proof is complete since for any two closed sets, one of them must be contained in the other.

THEOREM 1. *A T_0 topological space, $(X, \mathscr{T})$, is minimal T_0 if and only if the family $[\sim[\bar{x}]: x \in X]$ is a base for $\mathscr{T}$ and finite unions of point closures are point closures.*

Proof. Necessity: Assume $(X, \mathscr{T})$ is a minimal T_0-space. If there exist open sets A and B in $\mathscr{T}$ such that $A \not\subset B$ and $B \not\subset A$, then by Lemma 1, $\mathscr{T}(B)$ is a T_0 topology on X, $\mathscr{T}(B) \subset \mathscr{T}$, and $A \notin \mathscr{T}(B)$; but, this contradicts the minimality of $\mathscr{T}$. Therefore, for every two open sets in $\mathscr{T}$, one is contained in the other, and by Lemma 2, finite unions of point closures are point closures. To see that the family $[\sim[\bar{x}]: x \in X]$ is a base for $\mathscr{T}$, we observe that since $\mathscr{T}$ is a nested family, $[\sim[\bar{x}]: x \in X]$ is closed under finite intersections, so it is a base for some topology on X, say $\mathscr{T}^*$. $\mathscr{T}^*$ is clearly T_0 since all the point closures are distinct. Therefore, since $\mathscr{T}^* \subset \mathscr{T}$ and $\mathscr{T}$ is minimal T_0, $\mathscr{T}^* = \mathscr{T}$.

Sufficiency: Assume $(X, \mathscr{T})$ is a T_0-space such that $\mathscr{T}$ is a nested family, and $[\sim[x]: x \in X]$ is a base for T. Assume $T^* \subset T$, where T^* is a T_0-space. Let $[\bar{x}]^*$ be the closure of $[x]$ with respect to the topology $\mathscr{T}^*$. If there exists an $x \in X$ such that $[\bar{x}] \neq [\bar{x}]^*$, choose $y \in X$ such that $y \in [\bar{x}]^*$ and $y \notin [\bar{x}]$. Then, since $\mathscr{T}$ and $\mathscr{T}^*$ are T_0-spaces, $\mathscr{T}^* \subset \mathscr{T}$, and $\mathscr{T}$ is nested, the following inclusions hold: $[\bar{x}] \subset [\bar{y}] \subset [\bar{y}]^* \subset [\bar{x}]^*$. However, since $[\bar{x}]^*$ is the smallest closed set in $(X, \mathscr{T}^*)$ which contains x, and $x \in [\bar{y}]^*$, we have $[\bar{x}]^* = [\bar{y}]^*$. This contradicts the fact that $\mathscr{T}^*$ is T_0. Therefore, $[\bar{x}]^* = [\bar{x}]$ for every $x \in X$ and $\mathscr{T}^* = \mathscr{T}$ since $[\sim[\bar{x}]: x \in X]$ is a base for $\mathscr{T}$. This completes the proof that $\mathscr{T}$ is minimal T_0.

THEOREM 2. *A T_D topological space, $(X, \mathscr{T})$, is minimal T_D if and only if finite unions of point closures are point closures.*

Proof. The argument for necessity is identical to the argument given in Theorem 1.

Sufficiency: Assume $(X, \mathscr{T})$ is a T_D topological space such that $\mathscr{T}$ is a nested family. Assume $\mathscr{T}^* \subset \mathscr{T}$, where $\mathscr{T}^*$ is a T_D-space. If $[x]'^*$ is the derived set of $[x]$ in $\mathscr{T}^*$, then since every T_D-space is T_0, and $[\bar{x}] = [x]' \cup [x]$, where $[x]' \cap [x] = \varnothing$, we can apply the same argument given in Theorem 1 to conclude that $[\bar{x}]^* = [\bar{x}]$ and $[x]'^* = [x]'$ for every $x \in X$. If $\mathscr{T}^* \neq \mathscr{T}$, then there exists a closed set C in $(X, \mathscr{T})$ such that C is not a point closure, or a derived set of a point, or the intersection of these. Therefore, the following inclusion is proper: $C \subset C^* = \cap [D: D$ is closed with respect to $\mathscr{T}^*, C \subset D]$. Since $\mathscr{T}^*$ is T_0, it follows that $C^* \sim C$ contains exactly one point, say x. In fact, since C^* is closed with respect to $\mathscr{T}^*$, and there can be no smaller closed set in $(X, \mathscr{T}^*)$ which contains x, we have $C^* = [\bar{x}]^* = C \cup [x] = [x]'^* \cup [x]$. However, since $C \cap [x] = \varnothing$ and $[x]'^* \cap [x] = \varnothing$, it follows that $C = [x]'^*$, which is a contradiction, since we assumed that C was not the derived set of a point. Therefore, $\mathscr{T}^* = \mathscr{T}$, and $\mathscr{T}$ is minimal T_D.

EXAMPLE 1. Let X be the real numbers, let

$$\mathscr{T} = [(-\infty, x): x \in X] \cup [(-\infty, x]: x \in X] \cup [\varnothing, X] .$$

EXAMPLE 2. Let $X = [a, b, c]$, let $\mathscr{T} = [\varnothing, [b], [c], [b, c], X]$. Then $[\bar{a}] = [a]$, $[\bar{b}] = [a, b]$, and $[\bar{c}] = [a, c]$.

In general, neither of the two conditions of Theorem 1 imply the other. Example 1, as well as being an example of a minimal T_D-space, is an example of a T_0-space in which the open sets are nested, and yet it is not minimal T_0. Example 2 is an example of a T_0-space in which the complements of the point closures form a base for the topology and yet it is not minimal T_0. However, if X is a finite set, it is easy to show that the T_0 and T_D axioms are equivalent, and the following combined version of Theorems 1 and 2 is easily proved.

COROLLARY 1. *If X is a finite set, and $(X, \mathscr{T})$ is a T_0 topological space, then the following four conditions are equivalent*:

(1) *$(X, \mathscr{T})$ is minimal T_0.*
(2) *$(X, \mathscr{T})$ is minimal T_D.*
(3) *Finite unions of point closures are point closures.*
(4) *Every nonempty closed set in $(X, \mathscr{T})$ is a point closure.*

Requiring that the open sets be nested is a severe restriction on a topological space, as can be seen from the following theorem, which applies to both minimal T_0 and minimal T_D-spaces.

THEOREM 3. *If $(X, \mathscr{T})$ is a topological space in which the open sets are nested, then $(X, \mathscr{T})$ is connected, normal, and every open set in the space is dense. Furthermore, if X has more than one element, $(X, \mathscr{T})$ is not regular and not a T_1-space.*

Proof. Each part clearly follows from the nestedness of $\mathscr{T}$.

EXAMPLE 3. Let X be the real numbers, let

$$\mathscr{T} = [(-\infty, x): x \in X] \cup [\varnothing, X] .$$

EXAMPLE 4. Let X be the "half-open" interval on the real line, $(0, 1]$. Let $\mathscr{T} = [(0, x): x \in X] \cup [\varnothing, X]$.

Investigations in some of the other separation axioms have led to results such as the fact that every minimum T_1-space, minimal T_{3a}-space, and minimal T_4-space is bicompact [2]. It has been shown that there exist maximal bicompact spaces which are not T_2, as well as minimal T_2-spaces which are not bicompact [8]. It has also been shown that there exist minimal T_{2a}-space and minimal T_3-spaces which are not bicompact [5] [3]. Examples 1 and 3 are respectively examples of a minimal T_D-space and a minimal T_0-space which are not bicompact. Example 4 is an example of a minimal T_0 topology on an infinite set which is bicompact, and a similar example can be given for minimal T_D-spaces. Note that in Example 4, $[1]$ is a closed set. This leads to the following theorem.

THEOREM 4. *If $(x, \mathscr{T})$ is a minimal T_0 or minimal T_D topological space, then the two following conditions are equivalent:*

(1) *$(X, \mathscr{T})$ is bicompact.*

(2) *There exists exactly one singleton which is a closed set.*

Proof. To show that the first condition implies the second, assume $(X, \mathscr{T})$ is bicompact. Let $[G_\alpha: \alpha \in A]$ be an open cover for X. This can be reduced to a finite subcover $[G_{\alpha_i}: i = 1, 2, \cdots, n]$. However, since the open sets are nested and since X is the union of a finite number of these nested open sets, it must be equal to one of them. Therefore, every open cover of X must contain X as one of the open sets in the cover. Let $C = X \sim (\cup [\sim[\bar{x}]: x \in X])$. Since $\sim [\bar{x}] \neq X$ for any $x \in X$, $[\sim[\bar{x}]: x \in X]$ cannot be a cover for X and therefore, $C \neq \varnothing$. C contains exactly one point since $\mathscr{T}$ is T_0, and C is closed since it is the complement of an open set. Since the closed sets are nested, it is clear that there cannot exist two closed sets consisting of one point each.

To show that the second condition implies the first, assume that $(X, \mathscr{T})$ contains a singleton closed set, which implies that the only closed set not containing x is $\varnothing$. Therefore, the only open set containing x is X, and given any open cover of X, one of the open sets mst be X itself, and $(X, \mathscr{T})$ is bicompact.

The behavior of filters is of significant importance in minimal T_2, T_{2a}, T_3, T_{3a}, and T_4-spaces, as is partially seen in the introduction. However, the following easily proved remarks show that the same type of statements about filters cannot be made in minimal T_0 and minimal T_D-spaces.

(1) In a minimal T_0 or minimal T_D-space, every point in the space is a cluster point of every open filter.

(2) In a minimal T_0 or minimal T_D-space, if a filter converges to a point x in the space, and $[\bar{y}] \subset [\bar{x}]$, then the filter converges to y also.

One similarity between minimal T_0, minimal T_D, and minimum T_1-spaces is that in each case, the nonempty open sets form a filter base.

Any subspace of a minimum T_1-space is minimum T_1. Any closed subspace of a minimal T_4-space is minimal T_4 [2]. Any nonclosed subspace of a minimal T_4-space is not minimal T_4. A subspace of a minimal T_3 (minimal T_2) space which is both open and closed is minimal T_3 (minimal T_2) [3], [2]. There exist closed subspaces of minimal T_3 (minimal T_2) spaces which are not minimal T_3 (minimal T_2) [3] [2]. The following example and two theorems show that any subspace of a minimal T_D-space is minimal T_D, and that while minimal T_0-ness is not hereditary, an open or closed subspace of a minimal T_0-space is minimal T_0.

EXAMPLE 5. Let $(X, \mathscr{T})$ be as in Example 3, let

$$B = [(-\infty, 0] \cup (1, \infty)] .$$

Then $(X, \mathscr{T})$ restricted to B is not a minimal T_0-space. This can be seen by observing that in this relativized topology, $(-\infty, 0]$ is an open set.

THEOREM 5. *If B is an open or closed subset of a minimal T_0-space $(X, \mathscr{T})$, then $\mathscr{T}$ relativized to B is minimal T_0.*

Proof. Let $(B, \mathscr{U})$ be B with the relativized topology. Suppose $\mathscr{U}^* \subset \mathscr{U}$, where $\mathscr{U}^*$ is a T_0 topology on B, and $\mathscr{U}^* \neq \mathscr{U}$. If B is open, then $\mathscr{U}^* \cup [G: G \in \mathscr{T}, B \subset G]$ is a proper subtopology of $\mathscr{T}$ on X. If B is closed, then $[G: G \in \mathscr{T}$ and $G \subset \sim B] \cup [G \subset \sim B: G \in \mathscr{U}^*]$ is a proper subtopology of $\mathscr{T}$ on X. In both cases, these subtopologies are T_0 and this contradicts the fact that $\mathscr{T}$ is minimal T_0. The proof that these are topologies on X is similar to the proof of Lemma 1.

THEOREM 6. *Any subspace of a minimal T_D topological space is minimal T_D.*

Proof. Any subspace of a T_D-space is T_D [8]. It is clear that nestedness of the open sets is hereditary; therefore, by Theorem 2, any subspace of a minimal T_D-space is minimal T_D.

Given a topological space with property P, it would be of interest to know if the space could be written as the least upper bound of all minimal P-spaces weaker than it, or the greatest lower bound of all maximal P-spaces stronger than it. Unfortunately, this seems almost never to be the case. There exist bicompact spaces which are not weaker than any maximal bicompact space [7]. There exist T_2, T_{2a}, T_3, T_{3a}, and T_4-spaces, which are not stronger than any minimal T_2, minimal T_{2a}, minimal T_3, minimal T_{3a}, or minimal T_4-spaces, respectively [5].

EXAMPLE 6. Let X be the real numbers, let

$$\mathscr{T}_1 = [(-\infty, x): x \in X] \cup [\emptyset, X] ,$$

and let

$$\mathscr{T}_2 = [(x, \infty): x \in X] \cup [\emptyset, X] .$$

If $\mathscr{T}$ is the usual topology on the reals, then $\mathscr{T}$ is not only stronger than a minimal T_0 topology on the reals, it is the least upper bound of the two minimal T_0 topologies $\mathscr{T}_1$ and $\mathscr{T}_2$. However, not every T_0 or T_D topology may be written as the least upper bound of minimal T_0 or T_D topologies. As an example of this, consider the minimum T_1 topology on the reals. If it were stronger than some minimal T_0 or T_D topology on the reals, it would have to contain an uncountable family of nested closed sets; but, this is not the case since every closed set is finite. The following theorem gives a more desirable result when X is a finite set. As mentioned before, the T_0 and T_D axioms are equivalent in finite sets.

THEOREM 7. *Let X be a finite set, let $\mathscr{T}$ be a T_0 topology on X. Then $\mathscr{T}$ may be written as the least upper bound of minimal T_0 topologies on X.*

Proof. It is sufficient to show that for every open set B in $\mathscr{T}$, there exists a minimal T_0 topology on X which is weaker than $\mathscr{T}$ and which contains B. To show this, choose an open set $B \in \mathscr{T}$ and let $\mathscr{T}^*$ be a maximal chain of open sets in $\mathscr{T}$, one of which is B. $\mathscr{T}^*$ forms a topology on X, and since X is finite, $\mathscr{T}^*$ is T_0. (Note that if X is not finite, $\mathscr{T}^*$ may not be T_0 as is the case when $(X, \mathscr{T})$ is the minimum T_1 topology on the real numbers.) $\mathscr{T}^*$ is minimal T_0

by Corollary 1.

As a final remark, the product of minimal T_0 or minimal T_D topologies on sets of cardinality greater than one is never minimal T_0 or minimal T_D. Also, minimal T_0-spaces are not absolutely T_0-closed and minimal T_0-spaces are not absolutely T_D-closed, where a space is absolutely T_n-closed if it is closed in every T_n-space in which it can be embedded [5].

This paper is a result of a seminar given by W. J. Thron during the spring semester of 1968 at the University of Colorado. I am indebted to him for his help in the preparation of the paper. The terminology and notation is that of Thron [9].

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