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CONVERGENCE OF A SEQUENCE OF TRANSFORMATIONS OF DISTRIBUTION FUNCTIONS

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Let F be the distribution function (d.f.) of a nonnegative random variable (r.v.) X all of whose moments $\mu_n = \int_0^\infty x^n dF(x)$ exist and are finite. Define, recursively, the sequence $\{G_n\}$ of absolutely continuous d.f.'s as follows: put

$$G_1(x) = \mu_1^{-1}\int_0^x [1 - F(y)]dy \quad \text{for} \quad x > 0 \quad \text{and}$$

$$G_1(x) = 0 \quad \text{for} \quad x \le 0; \quad \text{for} \quad n > 1, \quad \text{let}$$

$$G_n(x) = \mu_{1,n-1}^{-1}\int_0^x [1 - G_{n-1}(y)]dy \quad \text{for} \quad x > 0 \quad \text{and}$$

$$G_n(x) = 0 \quad \text{for} \quad x \le 0, \quad \text{where}$$

$$\mu_{1,n-1} = \int_0^\infty [1 - G_{n-1}(y)]dy \ .$$

It is shown that if F is distributed on a finite interval, then the sequence $\{G_n(x/n)\}$ converges to the simple exponential d.f. On the other hand, if $F(x) < 1$ for all $x > 0$ and $G_n(c_n x) \to G(x)$, where G is a proper d.f. and $\{c_n\}$ is a sequence of constants such that $\{c_n/c_{n-1}\}$ is bounded, then (among other things) it is shown that (a) the convergence is uniform, (b) G is continuous and concave on $[0, \infty)$, (c) c_n is asymptotically equal to $\mu_{n+1}/b(n+1)\mu_n$ where $b = \int_0^\infty [1 - G(u)]du$ and (d) $\lim c_n/c_{n-1}$ exists. Finally, criteria for the existence of a sequence $\{c_n\}$ such that $\{G_n(c_n x)\}$ converges to a proper d.f. are given. In particular, it is shown that this sequence converges if F is absolutely continuous with probability density function (p.d.f.) f and F has increasing hazard rate.

The d.f. G_1 is obtained as an (integral) transform of the d.f. F. These transforms (although not explicitly labeled as such) have been encountered frequently in renewal theory. In particular, it is such a transform (d.f.) which, in a delayed renewal process, makes the renewal rate a constant. In order to verify that G_1 (and hence, inductively, G_n for $n > 1$) is indeed a d.f. one makes use of the well-known alternative computational formula for the mean μ_1 of a nonnegative r.v. X with d.f. F:

$$\mu_1 = \int_0^\infty x dF(x) = \int_0^\infty [1 - F(x)]dx \ .$$

More generally, it may be shown that (cf. Feller [3], p. 148)

$$\mu_n = \int_0^\infty x^n dF(x) = n\int_0^\infty x^{n-1}[1 - F(x)]dx \,. \tag{1.1}$$

A d.f. F is said to be a finite d.f. if there exists real numbers a and b such that $F(x) = 0$ for $x \leqq a$ and $F(x) = 1$ for $x \geqq b$. The d.f. G, given by $G(x) = 1 - e^{-bx}$ for $x > 0$ and zero for $x \leqq 0$, with $b > 0$, is called the exponential d.f. The c.f. φ of a d.f. F, defined for all real t by $\varphi(t) = \int_{-\infty}^\infty e^{itx}dF(x)$, is said to be an analytic c.f. if $\varphi(t)$ coincides in some (real) neighborhood of $t = 0$ with an analytic function $A(z)$ of a complex variable $z = t + iv$. In particular, if $A(z)$ is entire then φ is called an entire c.f.

In § 2 we calculate explicitly the moments and c.f. of G_n, $n \geqq 1$. We prove in § 3 that for a finite d.f. F the sequence $\{G_n(x/n)\}$ converges to the exponential d.f. The following three sections are devoted to the problem of existence of a sequence $\{c_n\}$ of constants such that the sequence $\{G_n(c_n x)\}$ converges to a proper d.f. Finally, in the last section some examples are given.

2. Moments, characteristic functions.

Throughout F will be a d.f. of a nonnegative r.v., all of whose moments μ_n are finite. Let φ_F and φ_{G_n} denote the c.f.'s of F and G_n, respectively; the k^{th} moment of G_n will be denoted by $\mu_{k,n}$. We first consider the case $n = 1$.

THEOREM 2.1. *The* c.f. φ_{G_1} *of* G_1 *is given by*

$$\varphi_{G_1}(t) = (it\mu_1)^{-1}[\varphi_F(t) - 1] \,, \quad \text{for} \quad t \neq 0 \,, \quad \text{and} \quad \varphi_{G_1}(0) = 1 \,. \tag{2.1}$$

Proof. Clearly, $\varphi_{G_1}(0) = 1$. For $t \neq 0$,

$$\begin{aligned}
\varphi_{G_1}(t) &= \int_0^\infty e^{itx}dG_1(x) = \mu_1^{-1}\int_0^\infty e^{itx}[1 - F(x)]dx \\
&= \mu_1^{-1} \lim_{a\to\infty} \int_0^a e^{itx}[1 - F(x)]dx \\
&= \lim_{a\to\infty} (it\mu_1)^{-1}\left\{e^{ita}[1 - F(a)] + \int_0^a e^{itx}dF(x) - 1\right\} \\
&= (it\mu_1)^{-1}[\varphi_F(t) - 1] \,.
\end{aligned}$$

THEOREM 2.2. *The moments* $\mu_{k,1}$ *of* G_1 *are given by*

$$\mu_{k,1} = \mu_{k+1}/(k + 1)\mu_1 \,, \qquad k = 1, 2, \cdots \,.$$

Proof. By definition,

$$\begin{aligned}
\mu_{k,1} &= \int_0^\infty x^k dG_1(x) = \mu_1^{-1}\int_0^\infty x^k[1 - F(x)]dx \\
&= \mu_{k+1}/(k + 1)\mu_1
\end{aligned}$$

using (1.1).

Using the recurrence relation

$$\mu_{k,n} = \int_0^\infty x^k dG_n(x) = \mu_{1,n-1}^{-1}\int_0^\infty x^k[1 - G_{n-1}(x)]dx$$
$$= \mu_{k+1,n-1}/(k + 1)\mu_{1,n-1}$$

and mathematical induction, we obtain the following result.

THEOREM 2.3. *For all positive integers n and k,*

$$\mu_{k,n} = k!n!\mu_{n+k}/(n + k)!\mu_n = \binom{n+k}{k}^{-1}\mu_n^{-1}\mu_{n+k} \; . \tag{2.2}$$

The result given in Theorem 2.3 is of some independent interest, inasmuch as it asserts that if $\{\mu_n\}$ is the moment sequence of a non-negative r.v. then the sequence $\{\mu_{k,n}\}$, given in (2.2), is also a moment sequence for every n.

Similarly, using the recurrence relation (for $t \neq 0$)

$$\varphi_{G_n}(t) = (it\mu_{1,n-1})^{-1}[\varphi_{G_{n-1}}(t) - 1] \; ,$$

obtained by replacing G_1 by G_n and F by G_{n-1} in Theorem 2.1, and induction, we can prove the following theorem, which again is mostly of independent interest.

THEOREM 2.4. *The* c.f. *of G_n is given by*

$$\varphi_{G_n}(t) = \frac{n!}{\mu_n(it)^n}\left[\varphi_F(t) - \sum_{j=0}^{n-1}\mu_j\frac{(it)^j}{j!}\right], \; t \neq 0 \tag{2.3}$$

with $\varphi_{G_n}(0) = 1$.

Finally, we observe that if $\{c_n\}$ is a sequence of positive real numbers then $\{H_n\}$, $H_n = G_n(c_n x)$, is a sequence of d.f.'s, with the k^{th} moment of H_n being given by

$$\mu_k(H_n) = \mu_{k,n}/c_n^k \; . \tag{2.4}$$

3. Convergence of $\{G_n\}$ for finite d.f.'s. We shall prove in this section that for finite d.f.'s $\lim_{n\to\infty} G_n(x/n)$ is an exponential d.f. We need some lemmas.

LEMMA 3.1. *If all the moments μ_n of F are finite then $\{\mu_{n+1}/\mu_n\}$ is a monotonically nondecreasing sequence.*

Proof. For n a positive integer and t real,

$$0 \leqq \int_0^\infty (x^{(n+1)/2} + tx^{(n-1)/2})^2 dF(x) = \mu_{n+1} + 2t\mu_n + t^2\mu_{n-1} .$$

Hence $\mu_n^2 - \mu_{n+1}\mu_{n-1} \leqq 0$ implying that $\mu_n\mu_{n-1} \leqq \mu_{n+1}/\mu_n$, $n \geqq 1$.

COROLLARY. $\mu_{n+k}/\mu_n \geqq (\mu_{n+1}/\mu_n)^k$, $n, k \geqq 1$.

The following result is easily proved (see Boas [2]).

LEMMA 3.2. *If F is a finite* d.f. *on $[a, b]$ where $0 \leqq a < b < \infty$ and $b = \inf\{x \mid F(x) = 1\}$ then $\lim_{n\to\infty}\mu_n^{1/n} = b$.*

LEMMA 3.3. *Under the hypothesis of Lemma* 3.2

$$\lim_{n\to\infty} \mu_{n+k}/\mu_n = b^k,\ k \geqq 0 .$$

Proof. Since $\{\mu_{n+1}/\mu_n\}$ is a nondecreasing sequence $\lim_{n\to\infty}\mu_{n+1}/\mu_n = L \leqq \infty$ exists. By a well-known theorem this implies that $\lim_{n\to\infty}\mu_n^{1/n} = L$ and by Lemma 3.2 it follows that $L = b < \infty$. Hence

$$\lim_{n\to\infty} \mu_{n+k}/\mu_n = (\lim_{n\to\infty} \mu_{n+1}/\mu_n)^k = L^k = b^k .$$

The lemma is proved.

THEOREM 3.1. *Let F be a finite* d.f. *on $[a, b]$, where*

$$b = \inf\{x \mid F(x) = 1\} .$$

Then $\lim_{n\to\infty} G_n(x/n) = G(x) \equiv 1 - e^{-x/b}$ *for $x > 0$ and zero for $x \leqq 0$.*

Proof. Letting $c_n = 1/n$ we get from (2.2) and (2.4) that $\mu_k(H_n) = n^k\mu_{k,n} \to k!b^k$ as $n \to \infty$. The limit is the k^{th} moment of the exponential d.f. $G(x)$ given in the statement of the theorem. The present theorem new follows from the moment convergence theorem (cf. Loeve [7], p. 185).

4. D.F.'s on an infinite range. For finite d.f.'s Theorem 3.1 gives an explicit sequence $\{c_n\}$, namely, $c_n = 1/n$, of normalizers such that $G_n(c_n x)$ converges to an exponential d.f. We investigate, in the remainder of this paper the problem of existence of the sequence $\{c_n\}$ for d.f.'s F on $[0, \infty)$ (i.e., $F(x) < 1$ for all x) all of whose moments μ_n are finite. Henceforth F will stand for such a d.f.

THEOREM 4.1. $\lim_{n\to\infty} \mu_{n+1}/\mu_n = +\infty$

Proof. Since $\{\mu_{n+1}/\mu_n\}$ is nondecreasing its limit $L \leqq +\infty$ exists. Assume to the contrary that $L < +\infty$. This implies $\lim_{n\to\infty}\mu_n^{1/n} = L < \infty$. We now show that this is a contradiction by proving that $\lim_{n\to\infty}\mu_n^{1/n} = +\infty$. Let $A > 0$ be an arbitrary (fixed) number. Then

$$\mu_n = \int_0^\infty x^n dF(x) \geqq \int_A^\infty x^n dF(x) \geqq A^n[1 - F(A)] .$$

Hence, $\lim_{n\to\infty} \inf \mu_n^{1/n} \geqq A$. Since A was arbitrary we conclude that $\mu_n^{1/n} \to +\infty$ as $n \to \infty$. Hence $L = +\infty$.

We shall need the following result which is easily proven by induction on k.

LEMMA 4.2. *If* $\lim \mu_{n+2}\mu_n/\mu_{n+1}^2 = l < \infty$ *then for each* $k \geqq 0$ *and* $0 \leqq r \leqq k$, $\lim_{n\to\infty} \mu_{n+k}\mu_n/\mu_{n+r}\mu_{n+k-r} = l^{r(k-r)}$.

LEMMA 4.3. *If* $F(x)$ *has an analytic* c.f., $\liminf_{n\to\infty} \mu_{n+2}\mu_n/\mu_{n+1}^2 = 1$.

Proof. By Lemma 3.1, $\liminf_{n\to\infty} \mu_{n+2}\mu_n/\mu_{n+1}^2 = \alpha \geqq 1$. If $\alpha > 1$, choose β such that $\alpha > \beta > 1$. Then there exists N such that

$$\mu_{n+2}\mu_n/\mu_{n+1}^2 > \beta$$

for $n \geqq N$. Hence $\mu_{N+2}\mu_N/\mu_{N+1}^2 > \beta$; i.e., $\mu_{N+2} > \beta a_N b_N$ where $a_N = \mu_{N+1}/\mu_N$ and $b_N = \mu_{N+1}$. Similarly, $\mu_{N+3} > \beta^3 a_N^2 b_N$, and, by induction, it follows that $\mu_{N+k+1} > \beta^{k(k+1)/2} a_N^k b_N$, $k \geqq 1$. Letting $N + k + 1 = n$ this becomes $\mu_n > \beta^{(n-N)(n-N-1)/2} a_N^{n-N-1} b_N$ for $n \geqq N + 2$ and so

$$\limsup_{n\to\infty} \mu_n^{1/n}/n \geqq \limsup_{n\to\infty} \beta^{(n-N-1)/2} a_{N/n} = +\infty$$

since $\beta > 1$. This is a contradiction since $\limsup_{n\to\infty} \mu_n^{1/n}/n < \infty$ if $F(x)$ has an analytic c.f. (see [8] p. 136). The lemma is proved.

The main theorem of this section is the following.

THEOREM 4.1. *Let* $\{c_n\}$ *be a sequence of positive real numbers such that* $H_n(x) \equiv G_n(c_n x) \to G(x)$, *with* $G(x)$ *a proper* d.f. *and assume that* $\limsup_{n\to\infty} c_n/c_{n-1} = l < \infty$. *Then*

(i) $\{b_n\}$ *is a bounded sequence where* $b_n = \int_0^\infty [1 - H_n(u)]du$.

(ii) $b = \int_0^\infty [1 - G(u)]du < \infty$.

(iii) $b_n \to b$ as $n \to \infty$.

(iv) $\lim_{n\to\infty} c_n/c_{n-1} = l$.

(v) *If* $\lim_{n\to\infty} a_n = \lambda$ *where* $a_n > 0, \lambda > 0$ *then* $\lim_{n\to\infty} H_n(a_n x) = G(\lambda x)$.

(vi) $G(x)$ *is continuous and concave on* $[0, \infty)$ *and* $\lim_{n\to\infty} H_n'(x) = G'(x)$ *for* $x > 0$.

(vii) $l \geqq 1$ *and equality holds if* F *has an analytic* c.f.

(viii) $\lim_{n\to\infty} \mu_k(H_n) = \mu_k(G)$ *for* $k \geqq 0$, *where* $\mu_k(G)$ *is the* k^{th} *moment of* G.

Proof. Let $b_n = \int_0^\infty [1 - H_n(u)]du$ and $b = \int_0^\infty [1 - G(u)]du$.

(i) Since $\limsup_{n\to\infty} c_n/c_{n-1} = l < \infty$, there exists N' such that $c_n/c_{n-1} < l + 1$ for $n > N'$. Also there exists x_0 and N'' such that $G(x_0) > \frac{1}{2}$ and $H_n(x_0) > \frac{1}{4}$ for $n > N''$. This last assertion is valid since $H_n(x_0) \to G(x_0) > \frac{1}{2}$. Now

$$\begin{aligned} H_n(x) = G_n(c_n x) &= \int_0^{c_n x} [1 - G_{n-1}(u)]du \Big/ \int_0^\infty [1 - G_{n-1}(u)]du \\ &= b_{n-1}^{-1} \int_0^{c_n x/c_{n-1}} [1 - H_{n-1}(u)]du \end{aligned} \tag{4.1}$$

by a simple change of variable. For $n > \max(N', N'')$ (4.1) yields

$$H_n(x_0) \leqq b_{n-1}^{-1} \int_0^{(l+1)x_0} [1 - H_{n-1}(u)]du .$$

Hence

$$\begin{aligned} \limsup_{n\to\infty} b_{n-1} &\leqq \limsup_{n\to\infty} \int_0^{(l+1)x_0} [1 - H_{n-1}(u)]du \Big/ H_n(x_0) \\ &= \int_0^{(l+1)x_0} [1 - G(u)]du \Big/ G(x_0) < \infty . \end{aligned} \tag{4.2}$$

(ii) By Fatou's lemma and (4.2)

$$b = \int_0^\infty [1 - G(u)]du \leqq \liminf_{n\to\infty} \int_0^\infty [1 - H_n(u)]du < \infty .$$

(iii) Let $\sup_n b_n = D < \infty$ and $\sup_n c_n/c_{n-1} = M < \infty$. From (4.1),

$$1 - H_n(x) = b_{n-1}^{-1} \int_{c_n x/c_{n-1}}^\infty [1 - H_{n-1}(u)]du ,$$

so that for $n \geqq 2$ and $x \geqq 0$

$$\begin{aligned} \int_{Mx}^\infty [1 - H_{n-1}(u)]du &\leqq \int_{c_n x/c_{n-1}}^\infty [1 - H_{n-1}(u)]du \\ &= b_{n-1}[1 - H_n(x)] \leqq D[1 - H_n(x)] . \end{aligned} \tag{4.3}$$

Now let $\varepsilon > 0$ be arbitrary. Pick x_0 such that $1 - G(x_0) < \varepsilon/2D$ and pick $x_1 > Mx_0$ such that $\int_{x_1}^\infty [1 - G(u)]du < \varepsilon$. Then there exists N such that for $n > N$, $1 - H_n(x_0) < \varepsilon/D$ and also

$$\left| \int_0^{x_1} [1 - H_{n-1}(u)]du - \int_0^{x_1} [1 - G(u)]du \right| < \varepsilon .$$

Since (4.3) holds for all x,

$$\int_{x_1}^\infty [1 - H_{n-1}(u)]du \leqq \int_{Mx_0}^\infty [1 - H_{n-1}(u)]du \leqq D[1 - H_n(x_0)] < \varepsilon .$$

Then for $n > N$,

$$\begin{aligned}\left|\int_0^\infty [1 - H_{n-1}(u)]du - \int_0^\infty [1 - G(u)]du\right| \\ \leqq \left|\int_0^{x_1} [1 - H_{n-1}(u)]du - \int_0^{x_1} [1 - G(u)]du\right| \\ + \int_{x_1}^\infty [1 - H_{n-1}(u)]du + \int_{x_1}^\infty [1 - G(u)]du < 3\varepsilon .\end{aligned}$$

Hence $b_n \to b$ as $n \to \infty$. In other words the sequence of first moments of $G_n(c_n x)$ converges to the first moment of $G(x)$. Recalling (2.2) and (2.4) we conclude that $c_n \sim \mu_{n+1}/b(n+1)\mu_n$. Note that $b > 0$ since $G(x)$ is a proper d.f.

(iv) Since $\{c_n/c_{n-1}\}$ is bounded the sequence will fail to have a limit only if $\infty > \limsup_{n\to\infty} c_n/c_{n-1} = p > q = \liminf_{n\to\infty} c_n/c_{n-1}$. It suffices to assume $p > 0$ and $p > q$ and obtain a contradiction.

Let $0 < \varepsilon < (p - q)/2$. Then $p - \varepsilon > q + \varepsilon$. Let $\{c_{n'}/c_{n'-1}\}$ be a subsequence converging to p and let $c_{n'}/c_{n'-1} > p - \varepsilon$ for $n' > N$. Then, from (4.1) for $x > 0$ and $n' > N$ we have

$$H_{n'}(x) \geqq b_{n'-1}^{-1}\int_0^{(p-\varepsilon)x} [1 - H_{n'-1}(u)]du$$

so that $G(x) \geqq b^{-1}\int_0^{(p-\varepsilon)x} [1 - G(u)]du$. In a like manner,

$$G(x) \leqq b^{-1}\int_0^{(q+\varepsilon)x} [1 - G(u)]du .$$

Since $p - \varepsilon > q + \varepsilon$ these inequalities together imply that

$$\int_{(q+\varepsilon)x}^{(p-\varepsilon)x} [1 - G(u)]du = 0$$

or that $G(u) = 1$ a.e. on $[(q+\varepsilon)x, (p-\varepsilon)x]$. If $G(x)$ is continuous on $(0, \infty)$ it would follow that $G(u) = 1$ on $[(q+\varepsilon)x, (p-\varepsilon)x]$ for $x > 0$ and so $G(x)$ is degenerate at the origin which is a contradiction of our hypothesis. Hence $p = q$ and $\lim_{n\to\infty} c_n/c_{n-1} = l$ exists. Hence it suffices to show that $G(x)$ is continuous on $(0, \infty)$. It is clear that $G_n(x)$ has derivatives for $x > 0$ up to order $n - 1$. Therefore, for $n \geqq 3$. we obtain from (4.1) that

$$\begin{aligned} H_n'(x) &= (c_n/c_{n-1})[1 - G_{n-1}(c_n x)]/b_{n-1} \geqq 0 \\ H_n''(x) &= -c_n^2[1 - G_{n-2}(c_n x)]/c_{n-1}c_{n-2}b_{n-1}b_{n-2} \leqq 0 \end{aligned} \tag{4.4}$$

for $x > 0$. Hence $H_n(x)$ is a concave d.f. for $n \geqq 3$ and $x > 0$. So

$$\begin{aligned} G(\theta x + (1-\theta)y) &= \lim_{n\to\infty} H_n(\theta x + (1-\theta)y) \\ &\geqq \lim_{n\to\infty} [\theta H_n(x) + (1-\theta)H_n(y)] = \theta G(x) + (1-\theta)G(y) \end{aligned}$$

and $G(x)$ is concave for $x > 0$ and hence continuous on $(0, \infty)$.

(v) Let $\varepsilon > 0$ be given and let $x > 0$. Let $a_n > 0$ and $a_n \to \lambda > 0$. Since G is continuous at λx, there is a $\delta > 0$ such that

$$|G(\lambda x + h) - G(\lambda x)| < \varepsilon \qquad \text{for} \qquad |h| < \delta .$$

Pick ε' such that $0 < \varepsilon' < \min(\varepsilon, \delta/x)$. Then for n large $H_n(a_n x) - H_n(\lambda x) < H_n((\lambda + \varepsilon')x) - H_n(\lambda x) \to G(\lambda x + \varepsilon' x) - G(\lambda x) < \varepsilon$. Hence for large enough n, $H_n(a_n x) - H_n(\lambda x) < 2\varepsilon$. Similarly, for large enough n, $H_n(a_n x) - H_n(\lambda x) > -2\varepsilon$. Thus

$$\begin{aligned} |H_n(a_n x) - G(\lambda x)| &\leqq |H_n(a_n x) - H_n(\lambda x)| \\ &+ |H_n(\lambda x) - G(\lambda x)| < 2\varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$

for sufficiently large n. That is, $\lim_{n\to\infty} H_n(a_n x) = G(\lambda x)$.

(vi) By virtue of (4.4) and (iii), (iv), and (v),

$$H_n'(x) \to lb^{-1}[1 - G(lx)]$$

for $x > 0$. On the other hand we see from (4.1) that

$$G(x) = b^{-1}\int_0^{lx} [1 - G(u)]du,\ x > 0 \tag{4.5}$$

and so for $x > 0$

$$G'(x) = lb^{-1}[1 - G(lx)] = \lim_{n\to\infty} H_n'(x) . \tag{4.6}$$

Since $G(x)$ is continuous it follows from (4.5) that $G'(x)$ exists for all $x > 0$.

It remains to show that $G(x)$ is continuous at the origin. Clearly, by definition, $G(0) = 0$. Also, from (4.5), $G(x) \leqq lb^{-1}x < \varepsilon$ for $x > 0$ sufficiently small. Together with the argument in (iv), this completes the proof that $G(x)$ is continuous and concave on $[0, \infty)$.

(vii) From the remark at the end of (iii) we know that $c_n \sim \mu_{n+1}/b(n+1)\mu_n$. Since $c_n/c_{n-1} \to l$ it follows that $\mu_{n+2}\mu_n/\mu_{n+1}^2 \to l$. Hence $l \geqq 1$ (Lemma 3.1) and $l = 1$ if F has an analytic c.f. (Lemma 4.3).

(viii) Since $\mu_{n+2}\mu_n/\mu_{n+1}^2 \to l$ by (vii) we have from Lemma 4.2 that

$$(\mu_n/\mu_{n+1})^k \mu_{n+k}/\mu_n = \prod_{i=1}^k \mu_{n+i}\mu_n/\mu_{n+i-1}\mu_{n+1} \to \prod_{i=1}^k l^{i-1} = l^{k(k-1)/2} .$$

Further, from (iii) $\mu_{n+1}/(n+1)c_n\mu_n \to b$. Hence

$$\begin{aligned} \mu_k(H_n) &= \binom{n+k}{n}^{-1} \mu_{n+k}/\mu_n c_n^k \\ &\sim k!(\mu_{n+1}/(n+1)c_n\mu_n)^k(\mu_n/\mu_{n+1})^k\mu_{n+k}\mu_n,\ n \to \infty \\ &\to k! b^k l^{k(k-1)/2} = \nu_k , \qquad \text{say.} \end{aligned}$$

The proof of (viii) would be completed on showing that ν_k is the k^{th} moment of the limiting d.f. $G(x)$ and this we show now. From (4.5) $dG(x) = lb^{-1}[1 - G(lx)]dx$ and hence by (1.1)

$$\mu_k(G) = \int_0^\infty x^k dG(x) = k\int_0^\infty x^{k-1}[1 - G(x)]dx$$

$$= kb\int_0^\infty x^{k-1}dG(x/l) = kbl^{k-1}\mu_{k-1}(G) .$$

It follows by induction that, for $k > 0$, $\mu_k(G) = k!b^k l^{k(k-1)/2} = \nu_k$.

5. **Remarks on Theorem 4.1.** The above theorem yields among other things a complete picture about the sequence $\{c_n\}$ (when it exists satisfying the conditions of the theorem). It shows (a) that asymptotically $c_n \sim \mu_{n+1}/b(n + 1)\mu_n$ thus bringing out explicitly the connection between the normalizers and the growth rates of moments; (b) that the limiting d.f. $G(x)$ is continuous everywhere, concave and differentiable on $(0, \infty)$ and has finite moments of all orders which are the limits of the corresponding moments of $G_n(c_n x)$; (c) that the convergence $G_n(c_n x) \to G(x)$ is uniform (cf. Parzen [9], p. 438) and (d) that the limiting d.f. satisfies the integral equation (4.5). The solutions of this equation will therefore yield the class of limit d.f.'s. This is an interesting equation and will be discussed elsewhere.

Note that if $l > 1$ we can conclude from the form of ν_k that the limit d.f. is *not* exponential.

We close this section with the following simple result which is an easy consequence of Theorem 4.1.

THEOREM 5.1. *Under the hypothesis of Theorem* 4.1, $G(x)$ *is an expotential* d.f. *provided that any one of the following equivalent conditions holds* (a) $\limsup_{n\to\infty} c_n < \infty$ (b) $\mu_{n+1}/\mu_n = 0(n)$.

6. **An existence criterion for $\{c_n\}$.** It was shown in Theorem 4.1 that if the c_n are normalizers and $\limsup_{n\to\infty} c_n/c_{n-1} < \infty$ then $\lim_{n\to\infty} \mu_{n+2}\mu_n/\mu_{n+1}^2$ exists and is finite. In Theorem 6.1 below we show that this latter requirement is a sufficient condition for the existence of $\{c_n\}$ and in Theorem 6.2 we show that this condition is satisfied for a very wide class of distributions.

THEOREM 6.1. *Let F be a* d.f. *of a nonnegative* r.V. *having finite moments* $\mu_1, \mu_2, \cdots$, *such that* $\delta_n = \mu_{n+2}\mu_n/\mu_{n+1}^2 \to 1$ *as* $n \to \infty$. *If* $c_n = \mu_{n+1}/(n + 1)\mu_n$, *then* $G_n(c_n X) \to G(X)$, *where* $G(X) = 1 - e^{-X}$ *for* $X \geqq 0$ *and* $G(X) = 0$ *for* $X \leqq 0$.

Proof. From the assumption $\delta_n \to 1$ as $n \to \infty$ it is clear that

$c_n/c_{n-1} \to 1$. Furthermore, from the proof of Theorem 4.1 (iii), it follows that $\mu_k(H_n) \to k!$ as $n \to \infty$ for $k = 1, 2, \cdots$. The sequence $\{k!\}$ is a moment sequence uniquely determining the asserted d.f. G and therefore the assertion of the theorem is a consequence of the moment convergence theorem.

In order to exhibit distributions for which $\mu_n\mu_{n+2}/\mu_{n+1}^2$ converges (to one) we need the following definitions and facts.

A function $g(x, y)$ of two real variables ranging over linearly ordered one-dimensional sets $\underline{\bar{X}}$ and $\underline{\bar{Y}}$, respectively, is said to be totally positive of order r (denoted by TP_r) if for all $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, with $x_i \in \underline{\bar{X}}$, $y_j \in \underline{\bar{Y}}$, and $m = 1, 2, \cdots, r$, the determinant of the mxm matrix with $(i, j)^{\text{th}}$ element $g(x_i, y_j)$ is nonnegative. If $g(x, y)$ is TP_r for all r, this fact is indicated by saying $g(x, y)$ is TP. Also, a nonnegative function $k(x)$, defined for all real x, is said to be a Polya frequency function of order r (PF_r) if $g(x, y) = k(x - y)$ is TP_r. In statistical applications, y is usually a parameter and $g(x, y)$ is a probability density function (p.d.f.) in x for each fixed y.

Following [1] we shall say that a d.f. F has increasing hazard rate (IHR) if $ln[1 - F(x)]$ is concave (in which case the support I of F is an interval). The following results are well-known (see [1], [4], [5], and [6]). (6.1)

(i) F has IHR if and onlf if $1 - F$ is PF_2.

(ii) If F is absolutely continuous with p.d.f. f, then F has IHR if and only if the *hazard* function $q(x) = f(x)/[1 - F(x)]$ is nondecreasing in $x \in I$.

(iii) The class $\mathscr{H}$ of all d.f.'s which have IHR is closed under convolution: $F, G \in \mathscr{H} \Rightarrow H = F*G \in \mathscr{H}$.

(iv) Each member of the exponential family, a class of d.f.'s having p.d.f.'s of the form $k(x, y) = \beta(y)e^{xy}$ with respect to a σ-finite measure μ on $(-\infty, \infty)$, is TP. Here, $\underline{\bar{X}} = (-\infty, \infty)$ and

$$\underline{\bar{Y}} = \left\{y: \beta(y) = \int e^{xy} d\mu(x) < \infty\right\}$$

is an interval. The family includes the binomial, Poisson, gamma, and normal d.f.'s, for example.

(v) If F has IHR, then $1 - F(x)$ tends to zero as $x \to \infty$ exponentially fast. Thus, if F has IHR or $1 - F$ is PF_2, all moments of F exist and are finite; in fact, F has an analytic c.f.

(vi) F and/or $1 - F$ may be PF_2 while the p.d.f. f (if it exists) is *not* PF_2. This is so even if F has IHR. However, if F is an absolutely continuous d.f. such that the p.d.f. f is also PF_2, then F has IHR.

(vii) If $G(y) = 1 - e^{-y}$ for $y \geqq 0$ and zero elsewhere, and F is a d.f. of a nonnegative r.v., then F has IHR if and only if there exists a nonnegative convex increasing function h such that $F(x) = G[h(x)]$.

In [1], p. 384, it is shown that if F (1) is absolutely continuous, (2) is the d.f. of a nonnegative r.v., and (3) has IHR, then for all real numbers $t > s > 0$,

$$(\lambda_{i+t}/\lambda_i)^s \leqq (\lambda_{i+s}/\lambda_i)^t \tag{6.2}$$

where $\lambda_r = \mu_r/\Gamma(r+1)$ with $\mu_r = \int_0^\infty x^r dF(x)$. The inequality (6.2) was proven in [6] (cf., equation (16), p. 1032) for the special case $i = 0$, assuming that F has a *continuous* p.d.f. $f(x)$ which is PF_2. From (6.2), it follows immediately, on putting $i = n - 1$, $t = 2$, and $s = 1$, that

$$[\mu_{n+2}/(n+1)(n+2)\mu_n] \leqq [\mu_{n+1}/(n+1)\mu_n]^2 , \tag{6.3}$$

i.e.,

$$\mu_{n+2}\mu_n/\mu_{n+1}^2 \leqq 1 + (n+1)^{-1} . \tag{6.4}$$

Since $\mu_{n+2}\mu_n/\mu_{n+1}^2 \geqq 1$ always, we obtain the following theorem.

THEOREM 6.2. *Let F be the* d.f. *of a nonnegative* r.v. *and assume that F has* IHR *and is absolutely continuous. Then*

$$\delta_n = \mu_n\mu_{n+2}/\mu_{n+1}^2 \to 1 , \qquad as \qquad n \to \infty .$$

We remark that if F is a finite d.f. on $[a, b]$, $0 \leqq a < b < \infty$, then by Lemma 3.3, $\delta_n \to 1$ as $n \to \infty$; no other assumptions are needed. Furthermore, all d.f.'s satisfying the hypotheses of Theorem 6.2 have *analytic c.f.'s*; an example in the following section shows that this is *not* necessary, so that Theorem 6.2 provides only sufficient conditions for δ_n to converge to unity.

7. **Examples.** This section contains several examples illustrating the results obtained in this paper. The first example illustrates Theorem 3.1; the second shows that analyticity is not necessary for the existence of normalizing constants c_n such that $G_n(c_n x) \to G(x)$ with $\delta_n \to 1$; and the last few examples illustrate the concepts of § 6 and Theorem 6.2.

EXAMPLE 1. Let $F(x) = 1 - p$ for $0 < x \leqq 1$, 1 for $x > 1$ and zero elsewhere be the d.f. of the Bernoulli distribution $P(X = 0) = 1 - p$, $P(X = 1) = p$, $0 < p < 1$, with mean $\mu_1 = p$. Then $G_n(x) = 1 - (1 - x)^n$ for $0 \leqq x \leqq 1$, 1 for $x > 1$ and zero elsewhere. $G_n(x)$ converges to the d.f. degenerate at the origin, but $\lim_{n\to\infty} G_n(x/n) = G(x) = 1 - e^{-x}$ for $x > 0$ and zero elsewhere.

EXAMPLE 2. $\mu_n = (4n+3)!/6$ is the n^{th} moment of a d.f. F which is not uniquely determined by $\{\mu_n\}$ and hence does not have an analytic c.f. (cf., Widder [11], p. 126). But since

$$\mu_{n+2}\mu_n/\mu_{n+1}^2 \to 1,\ c_n = \mu_{n+1}/(n+1)\mu_n \sim 256n^3$$

are normalizing constants, i.e., $G_n(c_n x) \to G(x)$. Since $l = 1$ the limit d.f. G is exponential. (This can be varified by the moment convergence theorem also).

EXAMPLE 3. Let $F_\beta(x) = 1 - e^{-x^\beta}$, for $x > 0, \beta > 0$, and zero otherwise, be the d.f. of the Weibull distribution. The corresponding p.d.f. is given by $f_\beta(x) = \beta x^{\beta-1}[1 - F_\beta(x)]$, so that the hazard function $q(x) = \beta x^{\beta-1}$ is nondecreasing for $\beta \geqq 1$, so that (using 6.1 (ii)) F has IHR for these values of β. By Theorem 6.2, $\delta_n \to 1$ as $n \to \infty$, for $\beta \geqq 1$. By direct calculations it is easily seen that $\mu_n = \Gamma(n + \beta/\beta)$ and that $\delta_n \to 1$ for *all* $\beta > 0$; clearly, the assumptions of Theorem 6.2 are not necessary.

EXAMPLE 4. Let $G_{\alpha,\beta} = F_\alpha * F_\beta$ be the convolution of two Weibull d.f.'s with parameters α and β. Then for $\alpha > 1, \beta > 1$, we conclude, using 6.1 (iii), that $G_{\alpha,\beta}$ has IHR, so that Theorem 6.2 applies. Verification of the fact that $\delta_n \to 1$ by computing the moments of $G_{\alpha,\beta}$ directly would appear to pose some difficulty.

EXAMPLE 5. For $x > 0$, let $F_1(x) = 1 - e^{-x} = 1 - e^{-h_0(x)}$, $F_2(x) = 1 - e^{-e^x} = 1 - e^{-h_1(x)}, \cdots, F_{k+1}(x) = 1 - e^{-h_k(x)}$, where $h_0(x) = x$, $h_1(x) = e^x, \cdots, h_{k+1}(x) = e^{h_k(x)}$. It is readily seen that $h_1, \cdots, h_k$ are convex increasing functions, so that by 6.1 (vii) $F_1, \cdots, F_{k+1}$ have IHR's, and Theorem 6.2 again is applicable.

EXAMPLE 6. The p.d.f. $f(x, y) = e^{(x-y)-e^{(x-y)}}$, for

$$x > y > 0,\ 0 < y < \infty\ ,$$

belongs to the exponential family of distributions. Since $f(x, y) = k(x - y)$, it is seen that f is *TP*, i.e., f is *PF*; Theorem 6.2 applies.

EXAMPLE 7. It is shown in [4] that

$$f(x) = A \sum_{\nu=-\infty}^{\infty} (-1)^\nu e^{-\nu^2 x},\ x > 0\ ;\ 0 \text{ for } x \leqq 0\ .$$

is *TP*, where A is a normalizing constant. For suitable choice of A, f is PF_r, for all r, i.e., f is *PF*. More generally, each member of the family of p.d.f.'s given by $f(x - y), 0 < y < \infty, x > y$, is *PF*. Therefore, the conclusion of Theorem 6.2 holds.

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