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ON THE SUM $\sum \langle n\alpha \rangle^{-t}$ AND NUMERICAL INTEGRATION

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Let "$\langle x \rangle$" denote the distance of the real number x from the nearest integer. If α is an irrational number, the growth of the sum

$$\sum_{K<n\leqq AK} \langle n\alpha \rangle^{-t}$$

(A is a fixed number, > 1) as $K \to \infty$ depends on the nature of the rational approximations to α. We shall find estimates of this sum, for certain classes of irrational numbers. Part of the motivation for these estimates is an application to Korobov's theory of numerical evaluation of multiple integrals.

A few years ago N. M. Korobov [8], [9] (and independently E. Hlawka [4]) invented a number-theoretical method for the numerical integration of periodic functions of several variables. Let $E_s^n(C)$, $n > 1$, be the set of all functions f of s real variables having period 1 in each variable, and whose Fourier expansion

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m}} C(\boldsymbol{m}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{m}} \tag{1}$$

(here $\boldsymbol{x}$ and $\boldsymbol{m}$ are s-tuples, of real numbers and of integers respectively, and the sum is over all possible $\boldsymbol{m}$) satisfies the condition

$$| C(\boldsymbol{m}) | \leqq C\left(\prod_{i=1}^{s} \max(1, |m_i|)\right)^{n} . \tag{2}$$

We shall denote the product inside the parentheses in (2) by "$\|\boldsymbol{m}\|$". (It is not a norm in the usual sense.)

Let G_s be the unit cube in s-space. Korobov considered the approximation of

$$I = I(f) = \int_{G_s} f(\boldsymbol{x}) d\boldsymbol{x}$$

by the sum

$$Q(f) = Q(f, N, \boldsymbol{a}) = \frac{1}{N} \sum_{r=1}^{N} f(r\boldsymbol{a}) ; \tag{3}$$

the problem is to choose $\boldsymbol{a} = \boldsymbol{a}(N) = (a_1(N), \cdots, a_s(N))$ so that $| Q - I |$ will go to zero rapidly as N increases. He made the following definition: ([9], p. 96; we have modified the form slightly).

DEFINITION. Let $N_1, N_2, \cdots$ be an increasing sequence of positive integers. Then a sequence $\boldsymbol{a}(N_1), \boldsymbol{a}(N_2), \cdots$ of s-tuples of integers is called an "s-dimensional optimal coefficient sequence" (and each $\boldsymbol{a}(N_i)$ is called a "set of optimal coefficients mod N_i") if:

(1) for $i = 1, 2, \cdots$, each component of $\boldsymbol{a}(N_i)$ is relatively prime to N_i

$$\text{(4) (2)} \qquad \sum_{m_1, \cdots, m_s = -(N_i - 1)}^{N_i - 1}{}' \frac{\delta(\boldsymbol{m} \cdot \boldsymbol{a}(N_i), N_i)}{\| \boldsymbol{m} \|} = O\left(\frac{\log^{\beta} N_i}{N_i}\right)$$

as $N_i \to \infty$, for some fixed number β; where $\delta(p, q) = 1$ if q divides p, and is 0 otherwise. (The prime on the sum indicates that the term with $\boldsymbol{m} = (0, \cdots, 0)$ is omitted.)

The lowest β for which (4) holds is called the "index" of the optimal coefficient sequence.

Korobov then proves ([9], p. 101):

THEOREM A. *Let $\boldsymbol{a}(N_1), \boldsymbol{a}(N_2), \cdots$ be an optimal coefficient sequence of index β. Then for any $f \in E_s^n(C)$,*

$$(5) \qquad | I(f) - Q(f, N_i, \boldsymbol{a}(N_i)) | \leqq C'C \frac{\log^{\beta n} N_i}{N_i^n}$$

where C' is a constant depending on s, n, and the sequence.

He further proved that if $N_1, N_2, \cdots$ is the sequence of prime numbers, then there does in fact exist an optimal coefficient sequence, of index at most equal to s; thus quadrature formulas Q of the form (3) exist for which

$$(6) \qquad | Q - I | = O\left(\frac{\log^{ns} N}{N^n}\right)$$

for the function class $E_s^n(C)$.

N. S. Bahvalov [1] showed that the exponent ns in (6) can be improved to $n(s-1)$; I. F. Sharygin [10] showed that it cannot be lowered beyond $s - 1$. The gap between $n(s-1)$ and $s-1$ has been closed only in the case $s = 2$ (and the case $s = 1$, which is trivial):

Using the expansion (1) in (3), we obtain

$$(7) \qquad Q(f) = C(0, \cdots, 0) + \sum{}' C(\boldsymbol{m}) \delta(\boldsymbol{m} \cdot \boldsymbol{a}, N) .$$

Since $C(0, \cdots, 0) = I(f)$, we have, for $f \in E_s^n$

$$(8) \qquad | Q - I | \leqq \sum{}' | C(\boldsymbol{m}) | \delta(\boldsymbol{m} \cdot \boldsymbol{a}, N) < C \sum{}' \frac{\delta(\boldsymbol{m} \cdot \boldsymbol{a}, N)}{\| \boldsymbol{m} \|^n} .$$

It's easy to show that

$$(9)\qquad \sum{}' \frac{\delta(\boldsymbol{m}\cdot\boldsymbol{a}, N)}{\|\boldsymbol{m}\|^n} = \sum_{\substack{|m_i|\leq(N-1/2)\\ i=1,\cdots,s}}{}' \frac{\delta(\boldsymbol{m}\cdot\boldsymbol{a}, N)}{\|\boldsymbol{m}\|^n} + 0\left(\frac{\log^{s-1} N}{N^n}\right)$$

so that only the finite sum in (9) need be considered. Furthermore, if we let b be the integer such that $ba_1 \equiv 1(\bmod N)$, it is clear from (3) that Q is unchanged if $\boldsymbol{a}$ is multiplied by b and then each of its components is reduced mod N. Thus we may take $a_1 = 1$.

Thus in the case $s = 2$, we have to estimate the sum

$$(10)\qquad \sum_{|m_1|,|m_2|\leq(N-1/2)}{}' \frac{\delta(m_1 + a_2m_2, N)}{\|\boldsymbol{m}\|^n}.$$

Now N divides $m_1 + a_2m_2$ if and only if (m_1/N) and (m_2a_2/N) sum to an integer; and then (since each m is smaller than $N/2$),

$$\left\langle m_2\frac{a_2}{N}\right\rangle = \frac{m_1}{N}$$

(where "$\langle x\rangle$" denotes the distance from x to the nearest integer). For each $m_2 \neq 0$ there is exactly one m_1 such that $\delta(m_1 + a_2m_2, N) = 1$.

Thus (10) can be rewritten as

$$(11)\qquad N^{-n}\sum_{m=1}^{N-1}\left(m\left\langle m\frac{a_2}{N}\right\rangle\right)^{-n}.$$

To estimate this we use the following result of Hardy and Littlewood ([2] – [3]):

THEOREM. *If α is an irrational number, or a rational number whose denominator (when α is expressed in lowest terms) is greater than K, and the partial quotients of the continued fraction expansion of α are bounded by a fixed number M, then*

$$(12)\qquad \left.\begin{matrix} C_1K\log K, & t = 1\\ C_2K^t & , t > 1\end{matrix}\right\} < \sum_{n=1}^{K}\frac{1}{|\sin 2\pi n\alpha|^t} < \left\{\begin{matrix} C_3K\log K, & t = 1\\ C_4K^t & , t > 1\end{matrix}\right.$$

where the C's depend only on M and on t.

The left-hand inequality is stated without proof by Hardy and Littlewood, and is in fact true without any hypothesis on the partial quotients of α. For completeness we include a proof here (the scheme of this proof will be used again in this paper):

Since $|\sin 2\pi x|/\langle x\rangle$ is bounded away from zero and from infinity, the sum in (12) may be replaced by

$$(13)\qquad \sum_{n=1}^{K}\langle n\alpha\rangle^{-t}.$$

Let $B \in (0, 1)$ be a real number to be specified later. Partition $[0, 1]$ into $[BK]$ equal subintervals (where "$[x]$" is the greatest integer less than or equal to x). Some one subinterval must contain $\{\alpha n\}$ -the fractional part of αn- for at least $[1/B]$ distinct values of n between 1 and K. It follows by substraction that for at least $[1/B] - 1$ values of n,

$$\langle n\alpha \rangle < 1/[BK] .$$

Choose $[1/B] - 1$ such values, and note that each of them contributes at least $[BK]^t$ to the sum in (13). Now partition $[0, 1]$ into $[BK/2]$ equal subintervals. We see, as before, that there are at least $[2/B] - 1$ values of n for which

$$\langle n\alpha \rangle < 2/[BK] ,$$

and that at least $[2/B] - [1/B] \geqq [1/B]$ of them are distinct from the n's previously chosen. We now choose $[1/B]$ of these new n's; the resulting set contributes least

$$[1/B]\left(\frac{[BK]}{2}\right)^t$$

to the sum in (13).

Repeating this process with $[BK/4]$ subintervals, we find a second group of n's, distinct from the previous ones, which contributes at least

$$[2/B]\left(\frac{[BK]}{4}\right)^t .$$

Continuing in this manner for $[\log BK]$ steps, we see that

$$\sum_{n=1}^{K} \langle \alpha n \rangle^{-t} \geqq [BK]^t \sum_{s=1}^{[\log BK]} [2^{s-1}/B] 2^{-st} .$$

Taking $B = 1/2$, the sum on the right becomes

$$\sum_{s=1}^{[\log K/2]} \left(\frac{1}{2^{t-1}}\right)^s$$

which is bounded below for any $t > 1$, and is of the order of $\log K$ for $t = 1$ and the inequality follows.

Returning to (11), we rewrite the sum as

$$\sum_{m=1}^{2} + \sum_{m=3}^{4} + \sum_{m=5}^{8} + \cdots + \sum_{m=p+1}^{N-1} \tag{14}$$

where p is the highest power of 2 below $N - 1$. If we now assume that $a_2 = a_2(N)$ is such that the partial quotients of the continued fraction expansion of a_2/N are bounded by some number M independent of N, we can apply (12) to these sums and conclude that each one is

$$\leqq \begin{cases} 2C_3 \log N, & n = 1 \\ 2^n C_4 \quad , & n > 1 \end{cases}.$$

Since the number of these sums is $< 2 \log N$, we conclude that

$$\sum_{m=1}^{N=1} \left(m\left\langle m\frac{a_2}{N}\right\rangle\right)^{-n} \leqq \begin{cases} C' \log^2 N, & n = 1 \\ C'' \log N, & n > 1 \end{cases}. \tag{15}$$

By (8)-(11), the case $n > 1$ implies:

THEOREM B. (*N.S. Bahvalov; L.K. Hua and Y. Wang* [5]). *If $N_1, N_2, \cdots$ is an increasing sequence of positive integers, and $a(N_1)$, $a(N_2), \cdots$ are integers relatively prime to $N_1, N_2, \cdots$ respectively and such that the partial quotients of the simple continued fraction of $a(N_i)/N_i$ are bounded uniformly for all i, then there is a constant C' such that if $f \in E_2^n(C)$,*

$$| I(f) - Q(f, N_i, (1, a(N_i))) | \leqq C'C \frac{\log N}{N^n}. \tag{16}$$

In particular, if α is an irrational number having bounded partial quotients and p_i/q_i is the i'th convergent to α, then (16) *holds with $N_i = q_i$, $a(N_i) = p_i$.*

Although Sharygin's theorem shows that (16) is best possible, it is desirable to have a direct proof that (15) cannot be improved. This will have implications for the "index" of optimal coefficient sequences. To do this it is sufficient to get lower bounds on sums of the form occurring in (14). We thus show

THEOREM 1. *If $t \geqq 1$ and $A > 1$, and M and r are fixed positive numbers, then*

$$\sum_{n=k+1}^{[AK]} \langle n\alpha\rangle^{-t} > \begin{cases} CK \log K, & t = 1 \\ CK^{1+(t-1)/r}, & t > 1 \end{cases} \tag{17}$$

if the convergents $p_1/q_1, p_2/q_2, \cdots$ of α satisfy

$$q_{i+1} < Mq_i^r \tag{18}$$

and α is either irrational or is a rational number whose denominator (when α is expressed in lowest terms) is greater than AK. $C =$

$C(t, A, M, r)$ is independent of α and of K.

Proof. Set $D = (A-1)/2$. Following the proof of the left half of (12), we see that

$$\sum_{n=1}^{[DK]} \langle n\alpha\rangle^{-t} \geqq [BDK]^t \sum_{s=1}^{[\log BDK]} 2^{-st}\left[\frac{2^{s-1}}{B}\right].$$

The s'th term in the sum on the right arose from the consideration of $[2^s/B]n$'s-all between 1 and DK-each of which satisfies the condition

$$\langle n\alpha\rangle < \frac{2^s}{[BDK]}.$$

we shall show that to each such n there corresponds a distinct n', with $K < n' \leqq AK$, such that

$$\langle n'\alpha\rangle < 2\langle n\alpha\rangle; \tag{19}$$

and it will follow that

$$\sum_{n=K+1}^{[AK]} \langle \alpha n\rangle^{-t} > \frac{1}{2}[BDK]^t \sum_{s=1}^{[\log BDK]} 2^{-st}\left[\frac{2^{s-1}}{B}\right]. \tag{20}$$

To define n', we let q_i be the greatest denominator of a convergent of α which is less than DK. Then

$$\langle q_i\alpha\rangle < \frac{1}{q_{i+1}} \leqq \frac{1}{DK},$$

and by our hypothesis on the q's, there is a constant E such that $q_i > EK^{1/r}$. There is therefore a number $N < E^{-1}K^{1-1/r}$ such that for every one of the n's under consideration $n + Nq_i$ is between $K+1$ and $[AK]$. We set $n' = n + Nq_i$; then

$$\langle n'\alpha\rangle \leqq \langle n\alpha\rangle + N\langle q_i\alpha\rangle \leqq \frac{2^s}{[BDK]} + \frac{1}{EDK^{1/r}}.$$

If we now choose B to satisfy $BDK = EDK^{1/r}$, (19) will hold, and (20) becomes

$$\sum_{n=K+1}^{[AK]} \langle \alpha n\rangle^{-t} > \frac{1}{2}(ED)^t K^{t/r} \sum_{s=1}^{M} 2^{-st}\left[\frac{2^{s-1}}{B}\right],$$

where $M = [1/r \log K + \log ED]$. Since

$$\sum_{s=1}^{M} 2^{-st}\left[\frac{2^{s-1}}{B}\right] = \sum_{s=1}^{M} 2^{-st}\left(\frac{2^{s-1}}{B}\right) + O(1),$$

the theorem follows.

COROLLARY. *If* $N_1, N_2, \cdots$ *and* $a(N_1), a(N_2), \cdots$ *are sequences satisfying the hypotheses of Theorem B, then* $(1, a(N_1)), (1, a(N_2)), \cdots$ *is an optimal coefficient sequence of index* 2.

Proof. By (9), and the equality of (10) and (11),

$$\sum_{m_1, m_2=1-N}^{N-1}{}' \frac{\delta(m_1 + m_2 a(N_i), N_i)}{\| m \|} = \frac{1}{N} \sum_{m=1}^{N-1} \left(m \left\langle m \frac{a(N_i)}{N_i} \right\rangle \right)^{-1} + O\left(\frac{\log N_i}{N_i} \right). \tag{21}$$

If b_j is the j'th partial quotient in the continued fraction of $a(N_i)/N_i$, and p_j/q_j the r'th convergent, then

$$q_{j+1} = b_j q_j + q_{j-1} \leqq (b_j + 1) q_j < M q_j$$

for some constant M, by the assumptions on the $a(N_i)$. Thus the $a(N_i)/N_i$ satisfy the hypothesis of Theorem 1., with $r = 1$. Breaking up the sum on the right of (21) as in (14) and using (17) (with $t = 1$), we see that

$$\sum_{m=1}^{N_i-1} \left(m \left\langle m \frac{a(N_i)}{N_i} \right\rangle \right)^{-1} > C\left(\frac{\log 2}{2} + \frac{\log 4}{2} + \cdots + \frac{\log p}{2} \right) > C_1 \log^2 N_i \tag{22}$$

for some C_1 independent of i. Thus

$$\sum_{m_1, m_2=1-N}^{N-1} \frac{\delta(m_1 + m_2 a(N_i), (N_i))}{\| m \|} > \frac{C_1}{2} \frac{\log^2 N_i}{N_i}. \tag{23}$$

The case $n = 1$ of (15) is a reverse of (22), and (23) can similarly be reversed by using (15) in place of (22); so that $(1, a(N_1)), (1, a(N_2)), \cdots$ is an optimal coefficient sequence of index $\leqq 2$. By (23), its index is also $\geqq 2$.

It follows that for these sequences, Korobov's Theorem A proves much less than Theorem B. Korobov's proof of Theorem A seems to leave no opening for reducing the exponent on the right side of (5) below β. It thus seems that the concept of "index" for optimal coefficients does not seem helpful for indicating the accuracy of the optimal coefficient sequence in evaluation of integrals.

(It appears likely that any 2-dimensional optimal coefficient sequence is of index 2 or higher.)

Theorem 1 suggests further consideration of sums of the form

$$\sum_{n=K+1}^{[AK]} \langle n\alpha \rangle^{-t}.$$

In the following theorems A is any fixed number greater than 1.

THEOREM 2. *If α is any irrational number, then*

$$\frac{1}{K}\sum_{n=K+1}^{[AK]}\frac{1}{\langle n\alpha\rangle}\longrightarrow\infty$$

as $K\to\infty$; but if f is any (however slowly) increasing function such that $\lim_{x\to\infty} f(x)=\infty$, then there is an α such that

$$\liminf_{K\to\infty}\frac{1}{Kf(K)}\sum_{n=K+1}^{[AK]}\frac{1}{\langle n\alpha\rangle}=0\ .$$

Proof. For the 1'st part, let p_i/q_i be the i'th convergent of α, and let g be a monotonic increasing function such that $q_{i+1}<g(q_i)$, $i=1,2,\cdots$. Then the proof of Theorem 1 (in the case $t=1$) can be carried through with $s^{-1}(K)$ in place of $K^{1/r}$ until

$$\sum_{n=K+1}^{[AK]}\langle\alpha n\rangle^{-1}>\frac{1}{2}ED\,g^{-1}(K)\sum_{s=1}^{M}2^{-s}\left[\frac{2^{s-1}}{B}\right]$$

is obtained, with $M=[ED\log g^{-1}(K)]$; and it follows that

$$\sum_{n=K+1}^{AK}\langle\alpha n\rangle^{-1}>CK\log g^{-1}(K)$$

for some constant C.

For the second part, we first specify that a_i, the i'th partial quotient of the simple continued fraction of α, be $\geqq 1000\,\mathrm{A}$, $i=1,2\cdots$. For large i we can then choose K so that $q_{i+1}/10<AK<q_{i+1}/5$. Let

$$I_s=\frac{sp_i+p_{i-1}}{sq_i+q_{i-1}}\ ,\ \frac{a_{i+1}}{10A}<s<a_{i+1}$$

be any "intermmediate fraction" (see, e.g., [6], p. 22) whose denominator lies between K and $[AK]+q_i$. Then

$$sq_i+q_{i-1}<\frac{1}{5}(a_{i+1}q_i+q_{i-1})+q_i;$$

so that

$$s+\frac{q_{i-1}}{q_i}-1<\frac{1}{5}\left(a_{i+1}+\frac{q_{i-1}}{q_i}\right)$$

and therefore (since $s>100$)

$$s+\frac{q_{i-1}}{q_i}<\frac{1}{4}\left(a_{i+1}+\frac{q_{i-1}}{q_i}\right)\cdot$$

Now

$$\left|I_s - \frac{p_{i+1}}{q_{i+1}}\right| = \frac{1}{q_i^2}\sum_{r=s}^{a_{i+1}-1}\frac{1}{\left(r+\frac{q_{i-1}}{q_i}\right)\left(r+1+\frac{q_{i-1}}{q_i}\right)}$$
$$= \frac{1}{q_i^2}\left(\frac{1}{s+\frac{q_{i-1}}{q_i}} - \frac{1}{a_{i+1}+\frac{q_{i-1}}{q_i}}\right)$$
$$> \frac{3/4}{q_i^2}\frac{1}{s+\frac{q_{i-1}}{q_i}} > \frac{3/4}{(s+1)q_i^2};$$

and since

$$\left|\frac{p_{i+1}}{q_{i+1}} - \alpha\right| < \frac{1}{q_{i+1}q_{i+2}} < \frac{1}{a_{i+1}^2 a_{i+2} q_i^2} < \frac{1}{100 s q_i^2},$$

we have

$$|I_s - \alpha| > \frac{1}{2sq_i^2}.$$

Now if m/n is any rational number with

$$sq_i + q_{i-1} \leqq n < (s+1)q_i + q_{i-1}$$

then either $|m/n - \alpha| \geqq |I_s - \alpha|$ or $|m/n - \alpha| \geqq |p_i/q_i - \alpha|$. In the first case

$$\langle n\alpha\rangle \geqq \langle (sq_i + q_{i-1})\alpha\rangle > \frac{1}{2q_i} > \frac{1}{20Aq_i}$$

and in the second

$$\langle n\alpha\rangle \geqq n\,|p_i/q_i - \alpha| > \frac{n}{2q_iq_{i+1}} > \frac{q_{i+1}/10A}{2q_iq_{i+1}} = \frac{1}{20Aq_i}.$$

We therefore have

$$\sum_{n=K+1}^{[AK]}\frac{1}{\langle n\alpha\rangle} < 20A(A-1)Kq_i;$$

and we now specify that a_{i+1} be also sufficiently large that

$$q_i < \frac{1}{i}f\left(\frac{a_{i+1}}{10A}\right) < \frac{1}{i}f(K),$$

and the construction is complete.

We conclude by showing that the results of Theorem 1 cannot be improved. If $t = 1$ or $r = 1$, this is clear from the theorem of Hardy and Littlewood. In the remaining case we have:

THEOREM 3. *If t and r are any real numbers greater than 1, then there is a constant C and an irrational number α whose convergents p_i/q_i satisfy*

$$q_{i+1} < Mq_i^r$$

(for some fixed M) such that

$$\sum_{n=K+1}^{[AK]} \langle n\alpha\rangle^{-t} < CK^{1+(t-1)/r}$$

for arbitrarily large values of K.

Proof. As before, we specify that each partial quotient of α be $\geqq 1000A$, and for each of a sequence of numbers $m_1, m_2, \cdots$ (which we shall later construct inductively), we choose K to satisfy

$$q_{m_i+1}/10 < AK < q_{m_i+1}/5 .$$

Then by the previous argument,

$$\langle n\alpha\rangle \geqq \frac{1}{20Aq_{m_i}}$$

for all n between $K + 1$ and $[AK]$.

Now if n_1 and n_2, $n_1 > n_2$, both satisfy

$$\langle n\alpha\rangle \leqq \frac{1}{4q_{m_i-1}} . \tag{24}$$

then

$$\langle(n_1 - n_2)\alpha\rangle \leqq \frac{1}{2q_{m_i-1}}$$

This implies that $n_1 - n_2 > q_{m_i} - 2$; for otherwise $\langle(n_1 - n_2)\alpha\rangle \leqq \langle q_{m_i-2}\alpha\rangle$, and $\langle q_{m_i-2}\alpha\rangle > (1/2)q_{m_i-1}$ since

$$\left|\frac{p_{m_i-2}}{q_{m_i-2}} - \frac{p_{m_i-1}}{q_{m_i-1}}\right| = \frac{1}{q_{m_i-2}q_{m_i-1}}$$

while

$$\left|\frac{p_{m_i-1}}{q_{m_i-1}} - \alpha\right| < \frac{1}{q_{m_i-1}q_{m_i}} < \frac{1}{2q_{m_i-2}q_{m_i-1}} .$$

Therefore there are at most $(A - 1)K/q_{m_i-2}$ n's satisfying (24); and their contribution to the sum in (17) is at most

$$\frac{(A-1)K}{q_{m_i-2}}(20Aq_{m_i})^t .$$

Similarly, the n's between $K+1$ and $[AK]$ which satisfy

$$\frac{1}{4q_{m_i-1}} < \langle n\alpha\rangle \leqq \frac{1}{4q_{m_i-2}}$$

contribute at most

$$\frac{(A-1)K}{q_{m_i-3}}(4q_{m_i-1})^t < \frac{(A-1)K}{q_{m_i-3}}(20Aq_{m_i-1})^t ,$$

etc.; therefore

$$(25)\qquad \sum_{n=K+1}^{[AK]}\langle n\alpha\rangle^{-t} \leqq (20A)^t(A-1)K\left(\frac{q^t_{m_i}}{q_{m_i-2}} + \frac{q^t_{m_i-1}}{q_{m_i-3}} + \cdots\right) .$$

Now we suppose that the q's have been determined through $q_{m_{i-1}+1}$. For some constant $C_0 > 1000A$, we determine

$$q_{m_{i-1}+2}, \cdots, q_{m_{i-1}+L+1} = q_{m_i}$$

so that

$$q_{m_i-(s+1)} = \frac{q_{m_i-s}}{C_0+\theta_s} \qquad (0 \leqq s \leqq L-1)$$

where $-1 \leqq \theta_s \leqq 1$ and L is the least positive integer satisfying

$$(C_0-1)^L > (q_{m_i})^{(t+1/2t)} .$$

Then the sum of the first $L+1$ terms in the sum on the right of (25) is no greater than

$$(C_0+1)^2 q^{t-1}_{m_i} + \frac{(C_0+1)^2}{(C_0-1)^{t-1}} q^{t-1}_{m_i} + \frac{(C_0+1)^2}{(C_0-1)^{2t-2}} q^{t-1}_{m_i} + \cdots \leqq C_1 q^{t-1}_{m_i} ;$$

and the sum of the remaining terms is no greater than

$$3q^{(t-1/2)}_{m_i}\log q_{m_i} = o(q^{t-1}_{m_i})$$

(since there are less than $3\log q_{m_i}$ terms). Therefore

$$\sum_{n=K+1}^{[AK]}\langle n\alpha\rangle^{-t} < C_2 K q^{t-1}_{m_i} ;$$

we finally specify that $q^r_{m_i} < q_{m_i+1} < 2q^r_{m_i}$ and $m_{i-1} < m_i - L$ and conclude that

$$q_{m_i} < C_3 K^{1/r} ,$$

so that

$$\sum_{n=K+1}^{[AK]} \langle n\alpha \rangle^{-t} < C_4 K^{1+(t-1)/r} .$$

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