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WEAKLY HYPERCENTRAL SUBGROUPS OF FINITE GROUPS

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In this article the study of generalized Frattini subgroups of finite groups, developed by J. C. Beidleman and T. K. Seo, is continued. We call a proper normal subgroup H of a finite group G, a special generalized Frattini subgroup of G provided that $G = N_G(A)$ for each normal subgroup L of G and each Hall subgroup A of L such that $G = HN_G(A)$. Z. Janko proved that a subnormal subgroup K of a finite group G is π-closed, π is a set of primes, whenever $K/(K \cap \phi(G))$ is π-closed, where $\phi(G)$ denotes the Frattini subgroup of G. We prove that a subnormal subgroup K of a finite group G is π-closed whenever $K/(K \cap H)$ is π-closed where H is a special generalized Frattini subgroup of G. From this result we prove that a proper normal subgroup H of a finite group G is a special generalized Frattini subgroup of G if and only if H is a weakly hypercentral subgroup of G.

The properties of weakly hypercentral subgroups were developed by R. Baer. We obtain some of Baer's results in a different manner by using special generalized Frattini subgroups, and we also extend some of the properties of $\phi(G)$ to the class of generalized Frattini subgroups.

Some examples of special generalized Frattini subgroups of a finite group G are the Frattini subgroup $\phi(G)$, the center $Z(G)$ of a nonabelian group G, and the intersection $L(G)$ of all self-normalizing maximal subgroups of a nonnilpotent group G.

2. Notation.

G will always denote a finite group.

$|G|$ denotes the order of G.

$|G\colon H|$ is the index of the subgroup H in G.

$H^x = x^{-1}Hx$ where $x \in G$ and $H \leqq G$.

$Z(G)$ is the center of G.

$Z^*(G)$ is the hypercenter of G (i.e., the terminal member of the upper central series of G).

G' denotes the commutator subgroup of G.

$D(G)$ denotes the hypercommutator of G (i.e., the terminal member of the lower central series of G).

$N_G(H)$ denotes the normalizer of H in G.

$\phi(G)$ is the Frattini subgroup of G.

If π is a set of primes, then π' is the set of primes not in π.

If n is a positive integer and if π is a set of primes, then n is a π-number provided that each prime divisor of n is in π.

The element x of G is a π-element if and only if $|x|$ is a π-number.

G is a π-group provided that each of its elements is a π-element.

G is π-closed if the totality of π-elements in G forms a subgroup of G.

If $H \leqq G$, then H is a Hall π-subgroup of $G(H \in \mathrm{Hall}_\pi(G))$ if H is a π-group and if $|H|$ and $|G: H|$ are relatively prime.

If $H \leqq G$, then H is a Hall subgroup of G provided that $|H|$ and $|G: H|$ are relatively prime.

$L(G)$ is the intersection of all self-normalizing maximal subgroups of G; set $L(G) = G$ if G is nilpotent.

3. Special generalized Frattini subgroups.

The concept of a generalized Frattini subgroup was developed in [3]. Recall that a proper normal subgroup H of a group G is called a generalized Frattini subgroup of G if and only if $G = N_G(P)$ for each normal subgroup L of G and each Sylow p-subgroup P of L, p is a prime, such that $G = HN_G(P)$. We denote the collection of all generalized Frattini subgroups of G by g.f. (G).

DEFINITION 3.1. A proper normal subgroup H of a group G is called a special generalized Frattini subgroup of G provided that $G = N_G(A)$ for each normal subgroup L of G and each Hall subgroup A of L such that $G = HN_G(A)$. We denote the collection of all special generalized Frattini subgroups of G by s.g.f. (G).

Note that it follows immediately that if $H \in$ s.g.f. (G), then $H \in$ g.f. (G). If $H \in$ g.f. (G), then it is not necessary for H to be in s.g.f. (G). In order to see this we give the following example which will also prove useful to us in other contexts.

EXAMPLE 3.2. (See 9.2.14 of [6] and Example 3.3 of [3]). Let $H = P \times K$ where P is the Klein four-group and K is cyclic of order 7. Let G be the relative holomorph of H by an automorphism of H of order 3 which acts as an automorphism of order 3 on P and on K. Then $|G| = 84$, and $G = PKQ$ where $Q \in \mathrm{Syl}_3(G)$ is such that $N_G(Q) = Q$. Also P and K are normal in G. We note that both P and K are maximal generalized Frattini subgroups of G. Their product PK is not a generalized Frattini subgroup of G. Since KQ is a Hall subgroup of G which is not normal in G, it follows that P is not a special generalized Frattini subgroup of G.

If θ is a homomorphism of G onto PQ with kernel K such that θ restricted to P is the identity on P, then $P \in$ g.f. (G), but $P\theta \notin$ g.f. $(G\theta)$. This shows that a homomorphism does not necessarily preserve generalized Frattini subgroups even if its kernel is a generalized Frattini subgroup.

We note that if G is a nilpotent group, then every proper normal subgroup of G is a special generalized Frattini subgroup of G.

THEOREM 3.3. *Let $H \in$ g.f. (G), let L be a normal subgroup of G, and A be a nilpotent Hall π-subgroup of L such that $G = HN_G(A)$. Then A is a normal subgroup of G.*

Proof. Let $p \in \pi$ and P be the unique Sylow p-subgroup of A. Since A is a Hall subgroup of L, it follows that $P \in \mathrm{Syl}_p(L)$. Since $N_G(A) \leqq N_G(P)$, we see that $G = HN_G(P)$. Thus P is normal in G. Therefore, each Sylow subgroup of A is normal in G, and so A is normal in G.

Theorem 3.3 cannot be improved by requiring A to be only supersolvable instead of nilpotent. To see this let G be the group of order 84 of Example 3.2. Then P is a generalized Frattini subgroup of G, and KQ is a supersolvable Hall subgroup of G such that $G = PN_G(KQ)$, but $G \neq N_G(KQ)$.

We now prove a few basic theorems about special generalized Frattini subgroups.

THEOREM 3.4. *Let $H \in$ s.g.f. (G). Then*

(i) *H is nilpotent,*

(ii) *If K is a normal subgroup of G that is contained in H, then $K \in$ s.g.f. (G),*

(iii) *$H\phi(G) \in$ s.g.f. (G),*

(iv) *If $HZ(G) < G$, then $HZ(G) \in$ s.g.f. (G).*

Proof. (i) Let $P \in \mathrm{Syl}_p(H)$ where p is a prime. Then $G = HN_G(P)$, so P is normal in G. Thus H is nilpotent.

(ii) Let K be a normal subgroup of G contained in H. Let L be a normal subgroup of G and $A \in \mathrm{Hall}_\pi(L)$ such that $G = KN_G(A)$. Then $G = HN_G(A)$ which implies $G = N_G(A)$. Thus $K \in$ s.g.f. (G).

(iii) $\phi(G)$ is the set of nongenerators of G, hence $H\phi(G) \in$ s.g.f. (G).

(iv) Since $Z(G) \leqq N_G(A)$ for any subgroup A of G, (iv) follows easily.

COROLLARY 3.5. (i) *$\phi(G) \in$ s.g.f. (G).*

(ii) *If G is nonabelian, then $Z(G) \in$ s.g.f. (G).*

Recall that a group G is said to be π-closed, π is a set of primes, provided that the set of π-elements in G forms a subgroup of G. The following properties are equivalent for the group G: the group G is π-closed; G has a normal Hall π-subgroup; G has a unique maximal π-subgroup. It is easy to show that subgroups and homomorphic images of π-closed groups are also π-closed.

If H is a normal subgroup of G such that H is a π-group and G/H is π-closed, then clearly G is π-closed. If, however, H and G/H are π-closed, then it is not necessary for G to be π-closed. Even if $H \in \text{g.f.}(G)$ and G/H is π-closed, G may not be π-closed. For let G be the group of order 84 of Example 3.2. If $\pi = \{3, 7\}$ and $P \in \text{Syl}_2(G)$, then $P \in \text{g.f.}(G)$ and G/P is a π-group. However, G is not π-closed. We are thus led to make the following definition.

DEFINITION 3.6. The proper normal subgroup H of G is said to satisfy property (N_π), π is a set of primes, if and only if for any normal subgroup K of G containing H such that K/H is π-closed, then K is π-closed.

THEOREM 3.7. *If $H \in \text{s.g.f.}(G)$, then H satisfies property (N_π) for any set of primes π.*

Proof. Let $H \in \text{s.g.f.}(G)$, and let K be a normal subgroup of G that contains H such that K/H is π-closed, where π is a set of primes. Then K/H has a unique Hall π-subgroup L/H which implies that L is a normal subgroup of G. Since H is nilpotent, it has a unique Hall π'-subgroup A. Then A is a normal subgroup of G and $A \in \text{Hall}_{\pi'}(L)$ since $|L: A| = |L: H|\,|H: A|$ is a π-number. Thus A has a complement B in L by the Schur-Zessenhaus Theorem (see Theorem 9.3 of [6]).

We now show that B is a normal Hall π-subgroup of K. Since $A \in \text{Hall}_{\pi'}(L)$, we see that $B \in \text{Hall}_\pi(L)$. Now A is solvable, so any two complements of A in L are conjugate in L (see Theorem 9.3.9 of [6]). It follows that $G = LN_G(B)$. Therefore, $G = ABN_G(B) = AN_G(B) = N_G(B)$, since $A \in \text{s.g.f.}(G)$. Now $|K: B| = |K: L|\,|L: B|$ is a π'-number. Thus B is a normal Hall π-subgroup of K which implies that K is π-closed. Therefore, H satisfies property (N_π) for each set of primes π. This completes the proof.

THEOREM 3.8. *Let $H \in \text{s.g.f.}(G)$ and K be a subnormal subgroup of G such that $K/(H \cap K)$ is π-closed. Then K is π-closed.*

Proof. HK/H is a π-closed subnormal subgroup of G/H. Let M/H be a maximal π-closed subnormal subgroup of G/H which contains HK/H. Z. Janko proves in [5, p. 247] that a maximal π-closed subnormal subgroup of G is normal in G. Thus M is a normal subgroup of G which is π-closed by Theorem 3.7. Since K is a subgroup of M, K is π-closed.

DEFINITION 3.9. The proper normal subgroup H of a group G is said to satisfy property (N'_π), π is a set of primes, if and only if, for each subnormal subgroup K of G containing H such that K/H is π-closed, then K is π-closed.

DEFINITION 3.10. (See Baer, [1].) The normal subgroup H of a finite group G is a weakly hypercentral subgroup of G, if it has the following property:

(W) If H is contained in the normal subgroup K of G, if x and y are elements of H and K respectively, and if $|x|$ is relatively prime to $|y|$ and to $|K: H|$, then $xy = yx$.

THEOREM 3.11. *The following statements are equivalent for the proper normal subgroup H of the group G:*

(i) $H \in$ s.g.f. (G).

(ii) *H satisfies property (N'_π) for any set of primes π.*

(iii) *H satisfies property (N_π) for any set of primes π.*

(iv) *H is a weakly hypercentral subgroup of G.*

Proof. (i) implies (ii) by Theorem 3.8. It is clear that (ii) implies (iii). We now show that (iii) implies (i). Let L be a normal subgroup of G, and let $A \in \mathrm{Hall}_\pi(L)$, π is a set of primes, such that $G = HN_G(A)$. Then HA is a normal subgroup of G and HA/H is a π-group. Since H satisfies property (N_π), HA is π-closed. Then $HA \cap L$ is π-closed and $A \in \mathrm{Hall}_\pi(HA \cap L)$. Therefore, A is a normal subgroup of G and so $H \in$ s.g.f. (G). Thus (i), (ii), and (iii) are equivalent statements.

If H is a weakly hypercentral subgroup of G, then H satisfies property (N_π) for each set of primes π by Corollary 1 of [1, p. 637].

Now suppose that H satisfies the first three equivalent statements. R. Baer has shown in [1, p. 636] that H is weakly hypercentral in G if and only if H is nilpotent and if H is contained in the normal subgroup K of G, then the totality of π-elements in K is a subgroup of K where π is the set of prime divisors of $|K: H|$. Since $H \in$ s.g.f. (G), we have that H is nilpotent. Now let K be a normal subgroup of G containing H and let π denote the set of primes dividing $|K: H|$. Then K/H is a π-group and so K is π-closed. Hence by Baer's characterization, we see that H is a weakly hypercentral subgroup of G. This completes the proof.

COROLLARY 3.12. *Let* $H \in$ s.g.f. (G) *and let* K *be a proper normal subgroup of* G *which contains* H. *Then* $K \in$ s.g.f. (G) *if and only if* $K/H \in$ s.g.f. (G/H).

Proof. Suppose that $K \in$ s.g.f. (G), and L/H is a normal subgroup of G/H which contains K/H such that $L/H/K/H$ is π-closed, where π is a set of primes. Then L is a normal subgroup of G such that L/K is π-closed. By Theorem 3.11, L is π-closed and so L/H is π-closed. Thus $K/H \in$ s.g.f. (G/H) by Theorem 3.11.

Conversely, suppose that $K/H \in$ s.g.f. (G/H). Let L be a normal subgroup of G which contains K such that L/K is π-closed, where π is a set of primes. Then $L/H/K/H$ is π-closed which implies that L/H is π-closed. Thus L is π-closed and so $K \in$ s.g.f. (G) by Theorem 3.11. This coupletes the proof.

The previous corollary was proved in a different manner in [1, p. 638].

COROLLARY 3.13. *If* G *is a nonnilpotent group, then* $L(G)$ *and the hypercenter* $Z^*(G)$ *are special generalized Frattini subgroups of* G.

Proof. It can be shown that $L(G)/\phi(G) = Z(G/\phi(G))$. Thus $L(G)/\phi(G) \in$ s.g.f. $(G/\phi(G))$ and $\phi(G) \in$ s.g.f. (G). Hence $L(G) \in$ s.g.f. (G) by Corollary 3.12. Since $Z^*(G) \leqq L(G)$, it follows that $Z^*(G) \in$ s.g.f. (G).

COROLLARY 3.14. *Let* $H \in$ s.g.f. (G) *and let* K *be a subnormal subgroup of* G *which properly contains* H. *Then* $H \in$ s.g.f. (K).

Proof. Let L be a normal subgroup of K which contains H such that L/H is π-closed, where π is a set of primes. Then L is a subnormal subgroup of G such that L/H is π-closed. Since H satisfies property (N_π') with respect to G, we see that L is π-closed. Thus H satisfies property (N_π) with respect to K, and so $H \in$ s.g.f. (K).

DEFINITION 3.15. A group G is said to be a π-dissolved group, π is a set of prime, if and only if the set of all π-elements of G is a solvable subgroup of G.

THEOREM 3.16. *If* $H \in$ s.g.f. (G) *and* K *is a subnormal subgroup of* G *containing* H *such that* K/H *is* π*-dissolved, then* K *is* π*-dissolved.*

Proof. Let $H \in$ s.g.f. (G) and let K be a subnormal subgroup of

G such that K/H is π-dissolved. By Theorem 3.8, K is π-closed. Let A be the subgroup consisting of all π-elements in K. Then $A \cap H$ and AH/H are solvable; whence A is solvable. Thus K is π-dissolved, and the proof is complete.

Since $\phi(G) \in$ s.g.f. (G), we get the following result due to R. Baer [2, p. 135].

COROLLARY 3.17. *A group G is π-dissolved if and only if $G/\phi(G)$ is π-dissolved.*

We note that the results of this section enable us to prove in a different manner some of the results of R. Baer in [1]. For example, Baer proves that if H is a weakly hypercentral subgroup of G, then $H\phi(G)$ is also weakly hypercentral (see p. 644 of [1]). This follows by our methods from Theorem 3.4 (iii) and Theorem 3.11. From this we see that $\phi(G) \leqq H$ for each maximal weakly hypercentral subgroup H of G. Baer defines the weak hypercenter $H_W(G)$ to be the intersection of all maximal weakly hypercentral subgroups of G. Thus $\phi(G) \leqq H_W(G)$ by the above remarks. A slightly stronger result is the following.

THEOREM 3.18. *If G is a group, then $L(G) \leqq H_W(G)$.*

Proof. If G is nilpotent, then the result is clear. If G is nonnilpotent, then $L(G)/\phi(G) = Z(G/\phi(G)) \leqq K/\phi(G)$ for any maximal weakly hypercentral subgroup K of G. This follows from Theorem 3.4 and Corollary 3.12. Thus $L(G) \leqq K$ which implies $L(G) \leqq H_W(G)$.

4. Products of weakly hypercentral subgroups. We recall that the product of two generalized Frattini subgroups may not be a generalized Frattini subgroup, even if their orders are relatively prime (see Example 3.2). If, however, we require one of the subgroups to be weakly hypercentral and the other to be generalized Frattini of relatively prime order, then the product is a generalized Frattini subgroup. We first prove the following lemma.

LEMMA 4.1. *Let K be a weakly hypercentral subgroup of G, and let H be a normal subgroup of G. If $|H|$ and $|K|$ are relatively prime, then KH/H is a weakly hypercentral subgroup of G/H.*

Proof. We show that KH/H satisfies the definition of weakly hypercentral subgroup. Let M/H be a normal subgroup of G/H which contains KH/H. Let $kH \in KH/H$ and $mH \in M/H$ where $k \in K$ and $m \in M$, be such that their orders are relatively prime and $|kH|$

is relatively prime to $|M/H: KH/H|$. It suffices to show that kH and mH commute.

Now there exists $m_1 \in M$ such that $|m_1|$ is divisible only by the primes that divide $|mH|$ and such that $m_1H = mH$. Hence $|m_1|$ and $|k|$ are relative prime. It also follows that $|k|$ is relatively prime to $|M: K|$. Since K is a weakly hypercentral subgroup of G, we see that $m_1k = km_1$. Therefore mH and kH commute and the proof is complete.

THEOREM 4.2. *Let K be a weakly hypercentral subgroup of G and H be a subgroup of G such that $|H|$ and $|K|$ are relatively prime and HK is a proper subgroup of G. If $H \in$ g.f. (G), then $HK \in$ g.f. (G). If H is weakly hypercentral, then HK is weakly hypercentral.*

Proof. If $H \in$ g.f. (G), then HK/H is a weakly hypercentral subgroup of G/H by Lemma 4.1. Thus $HK/H \in$ g.f. (G/H). By Theorem 3.4 of [3], $HK \in$ g.f. (G). If H is weakly hypercentral, HK is weakly hypercentral by Lemma 4.1 and Corollary 3.12.

EXAMPLE 4.3. (Baer [1, p. 640].) In the previous theorem, we cannot delete the requirement that $|H|$ and $|K|$ are relatively prime. Let p be an odd prime and J_p denote the cyclic group of order p. Let T be the automorphism of $J_p \times J_p$ that sends each element into its inverse. Let G be the relative holomorph of $J_p \times J_p$ by $\langle T \rangle$. Then every cyclic subgroup of $J_p \times J_p$ is a weakly hypercentral subgroup of G, but $J_p \times J_p$ is not generalized Frattini.

THEOREM 4.4. *Let K be a weakly hypercentral subgroup of G and $H \in$ g.f. (G). If either H or K is a Hall subgroup of G, and if HK is proper in G, then $HK \in$ g.f. (G).*

Proof. Let $p_1, \cdots, p_n$ be the primes that divide $|H|$, and let $q_1, \cdots, q_r$ be the primes that divide $|K|$. If $|H|$ is relatively prime to $|K|$, then the result follows from Theorem 4.2. Hence we may suppose that $p_1 = q_1, \cdots, p_k = q_k$ are the only primes that divide both $|H|$ and $|K|$. For each i between 1 and k, let P_i be the Sylow p_i-subgroup of H and Q_i be the Sylow q_i-subgroup of K.

Now suppose that H is a Hall subgroup of G. Then $Q_i \leqq P_i$ for $i = 1, \cdots, k$. Now let Q_j be the Sylow q_j-subgroup of K for $j = k + 1, \cdots, r$. Since K is nilpotent, $Q_{k+1} \cdots Q_r$ is normal in G and is thus a weakly hypercentral subgroup of G. Hence $HK = H(Q_{k+1} \cdots Q_r)$ is a generalized Frattini subgroup of G by Theorem 4.2.

If K is a Hall subgroup of G, then we can prove in a similar manner that $HK \in$ g.f. (G). This completes the proof.

We note that if H and K are weakly hypercentral subgroups of G, one of which is a Hall subgroup of G, then it can be shown that HK is a weakly hypercentral subgroup of G. The proof is analogous to the proof of Theorem 4.4.

If the group G possesses a unique minimal normal subgroup M, and if G/M possesses a normal subgroup, not 1, whose order is prime to the order of M, then R. Baer [2, p. 118] proved that $\phi(G) = 1$. We now show that in such a group, the only weakly hypercentral subgroup is the identity.

THEOREM 4.5. *If the group G possesses a unique minimal normal subgroup M and if G/M possesses a normal subgroup, not* 1, *whose order is prime to the order of M, then the only weakly hypercentral subgroup of G is the identity subgroup. In particular,* $\phi(G) = 1$.

Proof. Suppose that H is a nontrivial weakly hypercentral subgroup of G. Since M is the unique minimal normal subgroup of G, we see that $M \leqq H$. Let K/M be a nontrivial normal subgroup of G/M whose order is prime to the order of M. Then M is a normal Hall subgroup of K, and M is solvable. Thus M has a complement S in K by the Schur-Zassenhaus theorem (see Theorem 9.3.6 of [6]). Also any two complements of M in K are conjugate in K. It follows that $G = MN_G(S)$. Since $M \leqq H$, we see that $M \in$ s.g.f. (G), and so $G = N_G(S)$. Thus $K = M \times S$ and $M \leqq S$, whence $M = 1$ which is impossible. This completes the proof.

It is clear that by replacing π by $\{p\}$ in the theorems of §'s 3 and 4, we get corresponding theorems for generalized Frattini subgroups. For example, N. Ito in [4] proves that a subnormal subgroup K of G is nilpotent if and only if its commutator subgroup K' is contained in $\phi(G)$. By Theorem 3.8, one can show that a subnormal subgroup K of G is nilpotent provided that $K' \leqq H$ for some $H \in$ g.f. (G). Also one can easily see that a subgroup H is a generalized Frattini subgroup of G if and only if H is a normal nilpotent subgroup of G with the property that $F(G/H) = F(G)/H$, where $F(G)$ denotes the Fitting subgroup of G.

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