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APPROXIMATION BY INNER FUNCTIONS

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Let $L^\infty(T)$ denote the complex Banach algebra of (equivalence classes of) bounded measurable functions on the unit circle T, relative to Lebesgue measure m. The norm $\|f\|_\infty$ of an f in $L^\infty(T)$ is the essential supremum of $|f|$ on T. The collection of all bounded holomorphic functions in the open unit disc U forms a Banach algebra which can be identified (via radial limits) with the norm-closed subalgebra H^∞ of $L^\infty(T)$.

A function f in $L^\infty(T)$ is unimodular if $|f| = 1$ a.e., on T. The inner functions are the unimodular members of H^∞. It is well known that they play an important role in the study of H^∞.

The main result (Theorem 1) is that the set of quotients of inner functions is norm-dense in the set of unimodular functions in $L^\infty(T)$. One consequence of this (Theorem 7) is that the set of radial limits of holomorphic functions of bounded characteristic in U is norm-dense in $L^\infty(T)$. It is also shown (Theorem 3, 4) that the Gelfand transforms of the inner functions separate points on the Šilov boundary of H^∞, and this is used to obtain a new proof (and generalization) of a theorem of D. J. Newman (Theorem 4).

Our proof of the main result uses only one nontrivial property of H^∞, beyond the fact that H^∞ is a norm-closed subalgebra of L^∞. It therefore applies, without any extra effort, to a much more general situation which we now describe.

Let now L^∞ denote the Banach algebra of all bounded measurable functions on some measure space X, normed by the essential supremum, and let B be a norm-closed subalgebra of L^∞. We say that B has the *annulus property* if the following is true:

If X is the union of disjoint measurable sets E_1 and E_2 and if $0 < r_1 < r_2 < \infty$, then there exists h in B such that

(1) *$1/h$ is in B, and*

(2) $|h| = r_i$ a.e., *on E_i, for $i = 1, 2$.*

That H^∞ (in the classical setting described above) has the annulus property is well known: to see it, put $u = r_i$ on E_i (now $T = E_1 \cup E_2$), and define

$$h(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u(e^{i\theta}) d\theta\right\} \qquad (z \in U).$$

Then h maps U into the annulus $\{w: r_1 < |w| < r_2\}$, and the radial limits of h have modulus r_i a.e., on E_i.

Furthermore, the H^∞-algebras associated with weak*-Dirichlet algebras also have the annulus property. This is a special case of Lemma 2.4.3 of [11]. We shall have no opportunity to use any other property of these algebras, and will therefore not even define them here. An excellent account of them is given in [11].

In order to avoid repetition we now state what our standing assumptions will be. Theorems 1 to 5 will deal with the general situation just described. H^∞ will simply denote some subalgebra of some L^∞, the only other hypothesis being that H^∞ has the annulus property. The "inner functions" will again be the unimodular members of H^∞. Theorems 6, 7, 8 are more special and deal with the classical situation of the unit circle.

THEOREM 1. *The set of all quotients of inner functions is norm-dense in the set of all unimodular functions in* L^∞.

Proof. Since the measurable unimodular functions taking finitely many values are norm-dense in the set of all unimodular functions in L^∞, and since each function of the latter type is a product of finitely many unimodular functions each taking at most two values, it is sufficient to prove the following.

PROPOSITION. *If* E_1 *and* E_2 *are disjoint measurable subsets of* X *whose union is* X, *if* λ_1 *and* λ_2 *are complex numbers of modulus* 1, *and if* $\varepsilon > 0$, *then there exist inner functions* ϕ_1 *and* ϕ_2 *such that*

$$|\lambda_i - \phi_1(x)/\phi_2(x)| < \varepsilon \qquad \text{a.e., } on\ E_i (i = 1, 2)\ .$$

It involves no loss of generality to assume that $\lambda_1 \neq \lambda_2$. Let α_1 and α_2 be closed disjoint subarcs of T, of length less than ε, containing λ_1 and λ_2, respectively. Let Ω be the complement of $\alpha_1 \cup \alpha_2$ in the Riemann sphere. Then there is an annulus

$$D = \{z : r_1 < |z| < r_2\}$$

and a continuous function Φ on its closure $\bar{D}$ whose restriction to D is a one-to-one conformal map of D onto Ω ([2], p. 247). If $|z| = r_i$ then $\Phi(z)$ is in $\alpha_i (i = 1, 2)$. The reflection principle shows that Φ is holomorphic on $\bar{D}$, except for a simple pole at some point z_0 in D. By a theorem of Ahlfors [1] there exists a function Φ_2, holomorphic on $\bar{D}$, such that Φ_2 has a zero at z_0 and $|\Phi_2(z)| = 1$ on ∂D. Define $\Phi_1 = \Phi \cdot \Phi_2$. Then Φ_1 is holomorphic on $\bar{D}$, $|\Phi_1(z)| = 1$ on ∂D, and $\Phi = \Phi_1/\Phi_2$.

By the annulus property which H^∞ satisfies, there exists h in H^∞ such that $|h| = r_i$ a.e., on E_i, and $1/h$ is in H^∞. Thus h maps X into ∂D, $\|h\|_\infty = r_2$ and $\|1/h\|_\infty = 1/r_1$. Since Φ_1 and Φ_2 are holomorphic on $\bar{D}$, their Laurent expansions converge uniformly on $\bar{D}$. Since H^∞ is norm-closed, this implies that the compositions $\phi_1 = \Phi_1 \circ h$ and $\phi_2 = \Phi_2 \circ h$ are in H^∞. Clearly, they are also inner. Finally, $\phi_1/\phi_2 = \Phi \circ h$, and $(\Phi \circ h)(x)$ is in α_i for almost every x in $E_i (i = 1, 2)$.

This proves the proposition, and hence Theorem 1.

THEOREM 2. *Let Q be the set of all functions of the form $\psi\bar{\phi}$, where ψ is a finite linear combination of inner functions and ϕ is inner. Then Q is norm-dense in L^∞.*

Proof. By Theorem 1, the norm-closure $\bar{Q}$ of Q contains all unimodular functions in L^∞. Let χ_E be the characteristic function of a measurable set $E \subset X$. Note that $2\chi_E - 1$ is unimodular, and hence is in $\bar{Q}$. Since $\bar{Q}$ is a linear space, it follows that χ_E is in $\bar{Q}$ for every measurable $E \subset X$, and hence $\bar{Q} = L^\infty$.

Since Q is the algebra generated by the inner functions and their complex conjugates, Theorem 2 may be restated as follows:

COROLLARY. *The self-adjoint algebra generated by the inner functions is norm-dense in L^∞.*

REMARK. The subgroup G consisting of those unimodular functions which are quotients of inner functions has already occurred in certain studies ([5], [7, p. 12]). Theorem 1 shows how delicate the question of membership in G is. Note that $G \subset Q$ (see Theorem 2) and that $Q \subset \tilde{Q}$, where $\tilde{Q}$ denotes the set of those functions in L^∞ which are of the form $\phi\bar{\psi}$, where ϕ and ψ are in H^∞. In the classical situation, every nonconstant f in $\tilde{Q}$ satisfies

$$\int_T \log |f| \, dm > -\infty .$$

We doubt that this necessary condition is also sufficient (even for unimodular f) but we have no counterexample.

In connection with Theorem 2, we recall that it is still an open question whether the closure J of the set of finite linear combinations of inner functions is H^∞ (cf. [3], p. 348). Actually, J is a subalgebra of H^∞ which in the classical case of the circle has the same maximal ideal space and Šilov boundary as H^∞ (see the footnote to Theorem 3 and the proof of Theorem 4).

We now consider the maximal ideal space M of H^∞. The annulus property implies that 1 is in H^∞, so M is compact. The Gelfand transform $\hat{f}$ of an f in H^∞ is a continuous function of M, such that $\|\hat{f}\| = \|f\|_\infty$, where $\|\hat{f}\|$ denotes the maximum of $|\hat{f}|$ on M, and $\|f\|_\infty$ is the essential supremum of $|f|$ on X. We shall use the following notations:

If ϕ is inner, then

$$K_\phi = \{\gamma \in M: |\hat{\phi}(\gamma)| = 1\} .$$

If Σ is a set of inner functions, then

$$K_\Sigma = \bigcap_{\phi \in \Sigma} K_\phi .$$

If Σ is the set of *all* inner functions in H^∞, we write K in place of K_Σ.

The Šilov boundary of H^∞ will be denoted by ∂.

THEOREM 3. *The Gelfand transforms of the inner functions separate points on K.*[1]

Proof. Let γ_0 and γ_1 be distinct points of K. There exists f in H^∞ with $\hat{f}(\gamma_0) = 0$ and $\hat{f}(\gamma_1) = 1$. By Theorem 2, one can find ϕ and ψ such that ϕ is inner, ψ is a finite linear combination of inner functions, and $\|\phi f - \psi\|_\infty < 1/3$. Hence

$$|\hat{\phi}(\gamma)\hat{f}(\gamma) - \hat{\psi}(\gamma)| < \frac{1}{3}$$

for every $\gamma \in M$, in particular for γ_0 and γ_1. So $|\hat{\psi}(\gamma_0)| < 1/3$, and $|\hat{\psi}(\gamma_1)| > 2/3$ since $|\hat{\phi}(\gamma_1)| = 1$. This shows that $\hat{\psi}$ separates γ_0 and γ_1.

Theorem 3 leads directly to a generalization of a theorem which D. J. Newmann proved in the classical case [9] and which characterizes the Šilov boundary ∂ of H^∞ in terms of inner functions:

THEOREM 4. $\partial = K$.

Proof. Let ϕ be inner. Choose f in H^∞, not identically 0. Since $|\phi| = 1$ on X, $\|f\phi\|_\infty = \|f\|_\infty = \|\hat{f}\|$. There exists γ_0 in M at which $|\hat{f}\hat{\phi}|$ attains its maximum, $\|f\phi\|_\infty$, so that

$$\|\hat{f}\| = |\hat{f}(\gamma_0)\hat{\phi}(\gamma_0)| \leqq \|\hat{f}\| \circ |\hat{\phi}(\gamma_0)| \leqq \|\hat{f}\| .$$

This implies that $|\hat{\phi}(\gamma_0)| = 1$ (i.e., γ_0 is in K_ϕ) and that $|\hat{f}(\gamma_0)| = \|\hat{f}\|$. Thus every $|\hat{f}|$ attains its maximum (relative to M) at some point of K_ϕ. This says: $\partial \subset K_\phi$. Since K is the intersection of all K_ϕ, we have $\partial \subset K$.

To prove that ∂ fills all K, let E be a proper compact subset of K, choose γ_1 in K but not in E. It then follows from Theorem 3 that

[1] Kenneth Hoffman has communicated to us a proof which together with Theorem 3 shows that in the classical case of the circle the inner functions separate points on all of M.

there exist finitely many inner functions, say $\phi_1, \cdots, \phi_n$, such that $\phi_i(\gamma_1) = 1$ for $1 \leqq i \leqq n$, but

$$\inf_i \operatorname{Re} \hat{\phi}_i(\gamma) < 1 \text{ for every } \gamma \text{ in } E .$$

Then $f = 1 + \phi_1 + \cdots + \phi_n$ is in H^∞, $\hat{f}(\gamma_1) = n + 1 = \|\hat{f}\|$, but $|\hat{f}(\gamma)| < n + 1$ for every γ in E. Hence E does not contain ∂. This completes the proof.

The following result about function algebras was stated without proof in [4] by the first author. We point out that it does not depend on the annulus property.

LEMMA. *Let Σ be a multiplicative semigroup of inner functions. Let $\mathfrak{A}_\Sigma$ be the norm-closed subalgebra of L^∞ which is generated by H^∞ and the complex conjugates of the members of $\sum$. Then the maximal ideal space M_Σ of $\mathfrak{A}_\Sigma$ can be identified with the set $K_\Sigma \subset M$.*

Proof. Let Γ be a multiplicative linear functional on $\mathfrak{A}_\Sigma$. Restricting Γ to H^∞, we see that to each such Γ corresponds a unique γ in M, denoted by $\tau(\Gamma)$, such that $\Gamma(f) = \hat{f}(\gamma)$ for all f in H^∞.

Suppose $\gamma = \tau(\Gamma)$ and ϕ is in Σ. Since $\phi\bar{\phi} = 1$, we have

$$\Gamma(\bar{\phi}) = \Gamma(\phi^{-1}) = 1/\Gamma(\phi) = 1/\hat{\phi}(\gamma) .$$

This shows that Γ is determined by γ, so $\tau\colon M_\Sigma \to M$ is one-to-one. It is easy to see that τ is continuous. Since both spaces are compact and Hausdorff, τ is a homeomorphism. Furthermore, $\tau(M_\Sigma) \subset K_\Sigma$, for if $\gamma = \tau(\Gamma)$, then $|\hat{\phi}(\gamma)| \leqq \|\phi\|_\infty = 1$, and also

$$|1/\hat{\phi}(\gamma)| = |\Gamma(\bar{\phi})| \leqq \|\bar{\phi}\|_\infty = 1 ,$$

so that $|\hat{\phi}(\gamma)| = 1$ for every ϕ in Σ and every γ in $\tau(M_\Sigma)$.

We want to prove that $\tau(M_\Sigma) = K_\Sigma$. To do this, we fix γ in K_Σ, and show that γ is in $\tau(M_\Sigma)$.

For ψ in H^∞ and ϕ in Σ, define

$$\Gamma_0(\psi\bar{\phi}) = \hat{\psi}(\gamma)/\hat{\phi}(\gamma) .$$

If $\psi_1\bar{\phi}_1 = \psi_2\bar{\phi}_2$, then $\psi_1\phi_2 = \psi_2\phi_1$, which implies $\hat{\psi}_1(\gamma)\hat{\phi}_2(\gamma) = \hat{\psi}_2(\gamma)\hat{\phi}_1(\gamma)$, and since γ is in K_Σ, it follows that $\Gamma_0(\psi_1\bar{\phi}_1) = \Gamma_0(\psi_2\bar{\phi}_2)$. In otherwords, Γ_0 is well defined on a dense subalgebra of $\mathfrak{A}_\Sigma$. It is easy to check that Γ_0 is linear and multiplicative on this subalgebra. Finally (using the fact that γ is in K_Σ once more),

$$|\Gamma_0(\psi\bar{\phi})| = |\hat{\psi}(\gamma)/\hat{\phi}(\gamma)| = |\hat{\psi}(\gamma)| \leqq \|\psi\|_\infty = \|\psi\bar{\phi}\|_\infty ,$$

so that Γ_0 is bounded and can therefore be extended to a multiplicative linear functional Γ on $\mathfrak{A}_\Sigma$. It is clear that $\tau(\Gamma) = \gamma$, and the proof is complete.

As a consequence, we obtain a theorem of I. J. Schark ([10], [8, p. 174]) which Srinivasan and Wang [11, p. 232] have extended to the context of Weak*-Dirichlet algebras:

THEOREM 5. *The Šilov boundary ∂ of H^∞ can be identified with the maximal ideal space M_∞ of L^∞.*

Proof. Let Σ be the set of all inner functions. Then

$$\partial = K = K_\Sigma = M_\Sigma = M_\infty .$$

The first of these equalities is Theorem 4, the second is the definition of K, the third is the preceding lemma, and the fourth follows from Theorem 2, since the latter asserts that $\mathfrak{A}_\Sigma = L^\infty$.

We now return to the classical situation, i.e., to the unit circle. Recall that an inner function in the open unit disc U is said to be *singular* if it has no zero in U.

THEOREM 6. *Suppose f is in $L^\infty(T)$, $|f| = 1$, $0 < \varepsilon < 1$.*

(a) *There exist Blaschke products B_1 and B_2 such that*

$$||f - B_1/B_2||_\infty < \varepsilon .$$

(b) *There exist inner functions ϕ_1 and ϕ_2, with ϕ_2 singular such that*

$$||f - \phi_1/\phi_2||_\infty < \varepsilon .$$

Of course, the expression B_1/B_2 in (a) refers to the radial limit function of the quotient of the two Blaschke products, and the norm is the essential supremum over T.

Proof. (a) is an immediate consequence of Theorem 1, because of Frostman's Theorem ([6, pp. 112–113], [8, p. 175]) which asserts that the Blaschke products are norm-dense in the set of all inner functions.

By Theorem 1, it suffices to prove (b) for the case $f = 1/\psi$, where ψ is inner. Define

$$u(w) = \exp\left\{c\,\frac{w+1}{w-1}\right\}$$

where $c > 0$ is so chosen that $3u(0) < \varepsilon$, and put

$$u_1(w) = \frac{u(w) - u(0)}{w[1 - \overline{u(0)}u(w)]} .$$

Then u_1 is inner, and one checks easily that

$$| u(w) - wu_1(w) | < \varepsilon \qquad (w \in U) .$$

Put $w = \psi(z)$ in this inequality, define $\phi_1 = u_1 \circ \psi$ and $\phi_2 = u \circ \psi$. Then ϕ_1 and ϕ_2 are inner, ϕ_2 has no zero in U, and

$$| \phi_2(z) - \psi(z)\phi_1(z) | < \varepsilon \qquad (z \in U) .$$

To complete the proof, take radial limits in the last inequality and divide by $\psi\phi_2$.

Because of Theorem 6(b), Theorem 2 now takes the following form:

THEOREM 7. *If f is in $L^\infty(T)$ and $\varepsilon > 0$, then there is a singular inner function ϕ and a finite linear combination ψ of inner functions, such that*

$$|| f - \psi/\phi ||_\infty < \varepsilon .$$

Note that ψ/ϕ is a holomorphic function in U, of bounded characteristic (being a quotient of two H^∞-functions). Thus the radial limits of holomorphic functions of bounded characteristic are norm-dense in $L^\infty(T)$.

We conclude with the observation that the set K which was described prior to Theorem 3 can be defined (in the classical case) by means of the *singular* inner functions alone:

THEOREM 8. *If γ in M is such that $| \hat{\psi}(\gamma) | < 1$ for some inner function ψ, then there is a singular inner function ϕ with $| \hat{\phi}(\gamma) | < 1$.*

Proof. By Theorem 6(b), with $\varepsilon = 1 - | \hat{\psi}(\gamma) |$, there are inner functions ϕ_1 and ϕ_2, with ϕ_2 singular, such that

$$| \hat{\phi}_2(\gamma) - \hat{\psi}(\gamma)\hat{\phi}_1(\gamma) | \leqq || \phi_2 - \psi\phi_1 ||_\infty < 1 - | \hat{\psi}(\gamma) | ,$$

which implies that

$$| \hat{\phi}_2(\gamma) | < | \hat{\psi}(\gamma)\hat{\phi}_1(\gamma) | + 1 - | \hat{\psi}(\gamma) | \leqq 1 .$$

Theorem 8 adds an eighth equivalent condition to the seven that are listed on p. 179 of [8].

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