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CONTRACTIONS OF FUNCTIONS AND THEIR FOURIER SERIES

To Professor U. N. Singh on his 49th birthday

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The object of the present paper is to define a new type of contraction, called 'Shrivel', of a function and to prove a theorem on the absolute convergence of its Fourier series. Our theorem is similar to a theorem of M. Kinukawa, but as it is shown in the end the two results are essentially different. The original results in this direction are due to A. Beurling and R. P. Boas.

According to Beurling [1] a function g is said to be a contraction of function f if $|g(x) - g(y)| \leqq A|f(x) - f(y)|$, for all x, y, where A is an absolute constant. We shall assume throughout that the functions f and g are each L – integrable in $(-\pi, \pi)$ and periodic with the period 2π. We shall further let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

and

$$g(x) \sim \frac{1}{2} c_0 + \sum_{k=}^{\infty} (c_k \cos kx + d_k \sin kx) .$$

Kinukawa [3] has proved the following

THEOREM 1. *Let f and g be each continuous and g be a contraction of f, or more generally let f, $g \in L_2$ and*

$$\int_{-\pi}^{\pi} |g(x+h) - g(x)|^2 \, dx \leqq \int_{-\pi}^{\pi} |f(x+h) - f(x)|^2 \, dx, \text{ for all } h .$$

If

$$\sum_{n=1}^{\infty} n^{-3\alpha/2} \Big(\sum_{k=1}^{n} k^2 p_k^2 \Big)^{\alpha/2} < \infty \tag{1}$$

and

$$\sum_{n=1}^{\infty} n^{-\alpha/2} \Big(\sum_{k=n+1}^{\infty} p_k^2 \Big)^{\alpha/2} < \infty , \tag{2}$$

where $p_k^\alpha = |a_k|^\alpha + |b_k|^\alpha$ and $0 < \alpha \leqq 2$, then

$$\sum_{k=1}^{\infty} (|c_k|^\alpha + |d_k|^\alpha) < \infty . \tag{3}$$

The case $\alpha = 1$ in this theorem is the theorem of Boas [2], which in turn is a generalization of a theorem of Beurling [1] on the absolute convergence of Fourier series.

However, it has been shown by Leindler [4] and Sunouchi [5][1] that the conditions (1) and (2) are equivalent. In fact, a more general result has been proved by R. Askey in his not yet published work which we shall need later and hence state here in the form of the following

LEMMA[1]. *Let $s > 0$, $0 < p < 1$, $a_k > 0$. There are constants $K = K(p, r, s) > 0$ such that*

$$\sum_{n=1}^{\infty} n^r\left[n^{-s}\sum_{k=1}^{n} k^s a_k\right]^p \geqq K\sum_{n=1}^{\infty} n^r\left(\sum_{k=n}^{\infty} a_k\right)^p, \qquad r > -1\ ;$$

and

$$\sum_{n=1}^{\infty} n^r\left(\sum_{k=n}^{\infty} a_k\right)^p \geqq K\sum_{k=1}^{\infty} n^r\left[n^{-s}\sum_{k=1}^{n} k^s a_k\right]^p, \qquad r < ps - 1\ .$$

The object of this paper is to prove Theorem 2 which is similar to Theorem 1; but as we shall see in the end the two theorems are independent of each other. Before we can state our result precisely, we need introduce a couple of notations and a definition.

We put

$$\Delta_t^m f(x) = \sum_{k=0}^{m} (-1)^k C_m^k f[x + (m - 2k)t]$$

and

$$L^{(m)}(h, x, f) = \frac{1}{h}\int_0^h \Delta_t^m f(x)dt\ .$$

DEFINITION. We shall call a function g a 'shrivel' of order m of a function f if

$$|\,L^{(m)}(h, x, g)\,| \leqq A\,|\,L^{(m)}(h, x, f)\,|\ ,$$

for all x and for all $h > 0$.

We shall prove the following

THEOREM 2. *If g is a shrivel of order m of f, or more generally, if*

[1] The author thanks the referee for pointing out to him these references and the unpublished work of R. Askey to show the equivalence of the conditions (1) and (2) which he missed in his original text of the paper.

$$(4) \qquad \int_{-\pi}^{\pi} | L^{(m)}(h, x, g)|^2 \, dx \leqq \int_{-\pi}^{\pi} | L^{(m)}(h, x, f) |^2 dx \ ,$$

and if (2) *holds, then the conclusion* (3) *remains valid.*

Before we proceed for the proof, we mark that for $0 < |h| \leqq \pi$,

$$| L^{(m)}(h, x, f) | \leqq \frac{A(m)}{|h|} \int_{-\pi}^{\pi} |f(t)| \, dt \ ;$$

and hence $L^{(m)}(h, x, f)$ is a bounded function of x for fixed h and f. [$A(m)$ denotes here, as also in the sequel, a constant depending on m but not necessarily the same everywhere.] Therefore $L^{(m)}(h, x, f) \in L_2$. The same is true about $L^{(m)}(h, x, g)$ also. Thus the condition 'g is a shrivel of order m of f' does imply (4). Obviously, the converse is not true.

Again, it follows from the lemma that the condition (2) is equivalent to the condition

$$(5) \qquad \sum_{n=1}^{\infty} n^{-(m+1/2)\alpha} \left(\sum_{k=1}^{n} k^{2m} p_k^2 \right)^{\alpha/2} < \infty \ ,$$

and this is what we shall need in the proof of our theorem .

Proof. We can obtain by simple calculations that, for an even m,

$$\Delta_t^m g(x) \sim (-1)^{m/2} 2^m \sum_{k=1}^{\infty} (c_k \cos kx + d_k \sin kx) \sin^m kt \ ;$$

and hence

$$L^{(m)}(h, x, g) \sim (-1)^{m/2} 2^m \sum_{k=1}^{\infty} (c_k \cos kx + d_k \sin kx) \left(\frac{1}{h} \int_0^h \sin^m kt \, dt \right) .$$

Similarly for an odd m,

$$L^{(m)}(h, x, g) \sim (-1)^{(m-1)/2} 2^m \sum_{k=1}^{\infty} (d_k \cos kx - c_k \sin kx) \left(\frac{1}{h} \int_0^h \sin^m kt \, dt \right) .$$

Therefore by Parseval's theorem

$$(6) \qquad \begin{aligned} \sum_{k=1}^{\infty} q_k^2 \left(\frac{1}{h} \int_0^h \sin^m kt \, dt \right)^2 &= A(m) \int_{-\pi}^{\pi} | L^{(m)}(h, x, g) |^2 \, dx \\ &\leqq A(m) \int_{-\pi}^{\pi} | L^{(m)}(h, x, f) |^2 \, dx \\ &= A(m) \sum_{k=1}^{\infty} p_k^2 \left(\frac{1}{h} \int_{-\pi}^{\pi} \sin^m kt \, dt \right)^2 , \end{aligned}$$

where

$$q_k^2 = c_k^2 + d_k^2 .$$

Now taking $h = \pi/2^{n+1}$ and observing that

(i) $\sin kt > 0$, and is an increasing function of t for $\pi/2^{n+2} \leqq t \leqq \pi/2^{n+1}$; $2^{n-1} < k \leqq 2^n$,

(ii) $\sin^2 k\pi/2^{n+2} > \sin^2 \pi/8$ for $2^{n-1} < k \leqq 2^n$, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} q_k^2\left(\frac{2^{n+1}}{\pi}\int_0^{\pi/2^{n+1}} \sin^m kt\,dt\right)^2 \\
&\geqq \sum_{k=2^{n-1}+1}^{2^n} q_k^2\left(\frac{2^{n+1}}{\pi}\int_0^{\pi/2^{n+1}} \sin^m kt\,dt\right)^2 \\
&\geqq \sum_{k=2^{n-1}+1}^{2^n} q_k^2\left(\frac{2^{n+1}}{\pi}\int_{\pi/2^{n+2}}^{\pi/2^{n+1}} \sin^m kt\,dt\right)^2 \qquad (7)\\
&\geqq \sum_{k=2^{n-1}+1}^{2^n} q_k^2\left(\frac{2^{n+1}}{\pi}\cdot\frac{\pi}{2^{n+2}}\cdot\sin^m \frac{k\pi}{2^{n+1}}\right)^2 \\
&\geqq A(m) \sum_{k=2^{n-1}+1}^{2^n} q_k^2 .
\end{aligned}$$

Also, by Hölder's inequality

$$\sum_{k=2^{n-1}+1}^{2^n} q_k^\alpha \leqq 2^{(n-1)(1-\alpha/2)}\left(\sum_{k=2^{n-1}+1}^{2^n} q_k^2\right)^{\alpha/2} . \qquad (8)$$

Therefore we obtain from (6), (7) and (8)

$$\begin{aligned}
\sum_{n=2}^{\infty} q_n^\alpha &= \sum_{n=1}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} q_k^\alpha \\
&\leqq A(m, \alpha) \sum_{n=1}^{\infty} 2^{n(1-\alpha/2)}\left[\sum_{k=1}^{\infty} p_k^2\left(\frac{2^{n+1}}{\pi}\int_0^{\pi/2^{n+1}} \sin^m kt\,dt\right)^2\right]^{\alpha/2} \\
&\leqq A(m, \alpha)\left[\sum_{n=1}^{\infty} 2^{n(1-\alpha/2)}\left(\sum_{k=1}^{2^n} p_k^2 \sin^{2m} \frac{k\pi}{2^{n+1}}\right)^{\alpha/2}\right. \\
&\qquad \left. + \sum_{n=1}^{\infty} 2^{n(1-\alpha/2)}\left(\sum_{k=2^n+1}^{\infty} p_k^2\right)^{\alpha/2}\right] \\
&\leqq A(m, \alpha)\left[\sum_{n=1}^{\infty} 2^{n(1-\alpha/2-m\alpha)}\left(\sum_{k=1}^{2^n} k^{2m} p_2^2\right)^{\alpha/2}\right. \\
&\qquad \left. + \sum_{n=1}^{\infty} 2^{n(1-\alpha/2)}\left(\sum_{k=2^n+1}^{\infty} p_k^2\right)^{\alpha/2}\right] .
\end{aligned}$$

Now it is not difficult to see that the two series within the square brackets are covergent if and only if (5) and (2) hold respectively. See Szàsz [6], Lemma 2.2. Since

$$|c_k|^\alpha \leqq q_k^\alpha , \qquad |d_k|^\alpha \leqq q_k^\alpha ,$$

(3) follows; and hence the proof.

To prove that the result of Theorem 2 is completely independent of Kinukawa's Theorem 1, we first see that g is a shrivel of order 1 of f if for all x and all $h > 0$,

$$\left|\int_0^h \{g(x+t) - g(x-t)\}dt\right| \leqq A \left|\int_0^h \{f(x+t) - f(x-t)\}dt\right| . \tag{9}$$

Now consider the 2π − periodic function defined by

$$\begin{aligned} f(x) &= 0 \quad \text{for} \quad -\pi < x < 0 , \\ &= \sin 2x \quad \text{for} \ 0 \leqq x \leqq \pi . \end{aligned}$$

Also let

$$\begin{aligned} \phi(x) &= x \quad \text{for} \quad x < 0 \\ &= x - x^2 \quad \text{for} \quad x \geqq 0 , \end{aligned}$$

and let $g(x) = \phi(f(x))$ for all x. The function ϕ is differentiable and has a bounded derivative on the interval $[-1, 1]$. Hence ϕ belongs to the class Lip 1 on this interval, whence, g is a contraction of f in Beurling's sense. On the other hand,

$$\int_0^\pi \{f(t) - f(-t)\}dt = \int_0^\pi \sin 2t dt = 0 ,$$

while

$$\begin{aligned} \int_0^\pi \{g(t) - g(-t)\}dt &= \int_0^\pi g(t)dt \\ &= \int_0^{\pi/2} (\sin 2t - \sin^2 2t)2t + \int_{\pi/2}^\pi \sin 2t dt \\ &= -\int_0^{\pi/2} \sin^2 2t dt \neq 0 , \end{aligned}$$

so that (9) is not satisfied for any A when $x = 0$ and $h = \pi$.

Conversely, since (9) can be satisfied when f, g are merely integrable, (9) does not imply that g is a contraction of f.

Finally we remark that Theorem 2 has its usual dual.

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