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THE CONTENT OF SOME EXTREME SIMPLEXES

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Formulae are presented that give the content of a simplex in Euclidean n-space: (i) in terms of the lengths of and the angles between the vectors from a fixed point to the vertices of the simplex; (ii) in terms of the lengths of and the angles between the perpendiculars from a fixed point to the bounding faces of the simplex. We then determine the largest simplex whose vertices are given distances from a fixed point and we determine the smallest simplex whose faces are given distances from a fixed point. As special cases we find that the regular simplex is the largest simplex contained in a given sphere and is also the smallest simplex containing a given sphere.

1. Introduction and results. The n-dimensional simplex S_n in Euclidean n-space is the general term in the sequence of figures S_0, $S_1, S_2, S_3 \cdots$ known respectively otherwise as point, line segment, triangle, tetrahedron, $\cdots$. S_n is determined by $n+1$ points, P_1, $P_2, \cdots, P_{n+1}$, — its vertices —, which we assume do not lie in any $(n-1)$-dimensional hyperplane. Taken n at a time, these vertices determine $(n-1)$-dimensional hyperplanes $H_1, H_2, \cdots, H_{n+1}$, where H_i contains all vertices except P_i. We choose the normal of H_i so that P_i lies on the negative side of H_i. S_n can be regarded as the intersection of these $n+1$ nonpositive half spaces; it can also be regarded as the convex hull of its vertices.

Let Q be an arbitrary point. For $i = 1, 2, \cdots, n+1$, let $d_i > 0$ be the distance from Q to P_i and let $e_i > 0$ be the distance from Q to H_i. Let $\boldsymbol{a}_i$ be the unit vector in the direction from Q to P_i and let $\boldsymbol{b}_i$ be the unit vector from Q along the perpendicular to H_i. Let $r_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$, $s_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$, $i, j = 1, 2, \cdots, n+1$.

In this paper, we first show that the content, V_n, of S_n is given by

$$n!\, V_n = \left| \sum_{i,j} R_{ij} \frac{1}{d_i} \frac{1}{d_j} \right|^{1/2} \prod_1^{n+1} d_i \tag{1}$$

$$= \left| \sum_{i,j} S_{ij} e_i e_j \right|^{n/2} \Big/ \prod_1^{n+1} S_{ii}^{1/2} \tag{2}$$

for $n = 1, 2, \cdots$, where R_{ij} is the cofactor of r_{ij} in the $(n+1) \times (n+1)$ matrix $r = (r_{ij})$ and S_{ij} is the cofactor of s_{ij} in $s = (s_{ij})$. Next we determine the largest simplex with given d values and the smallest simplex containing Q with given e values. We find

$$n!\, V_{\max} = \theta^{-1/2} \prod_1^{n+1} (\theta + d_i^2)^{1/2}, \quad r'_{ij} = -\frac{\theta}{d_i d_j}\,, \tag{3}$$

$$(4) \qquad n!\, V_{\min} = n^n \psi^{-1/2} \prod_1^{n+1} (\psi + e_i^2)^{1/2},\ s'_{ij} = -\frac{\psi}{e_i e_j},$$

$i, j = 1, 2, \cdots, n+1$ where θ and ψ are respectively the unique positive roots of

$$(5) \qquad \theta \prod_1^{n+1} \frac{1}{\theta + d_i^2} = 1$$

and

$$(6) \qquad \psi \sum_1^{n+1} \frac{1}{\psi + e_i^2} = 1$$

and where the r'_{ij} are the maximizing values of r_{ij} and the s'_{ij} are the minimizing values of s_{ij}. Q lies inside the simplex given by (3).

If not all the d_i are the same, (5) has a negative real root of smallest absolute value. The simplex (3) corresponding to this root is the largest simplex with given d values having Q on the negative side of exactly one bounding face. Similarly if not all the e_i are the same, (6) has a negative real root of smallest absolute value. The simplex given by (4) corresponding to this root is the smallest simplex with given e values having Q on the positive side of exactly one bounding face.

A special case of these results states that: (a) the largest simplex contained in a given sphere is a regular simplex; (b) the smallest simplex containing a given sphere is a regular simplex.

2. Derivation of volume formula (1). Let $(x_1, x_2, \cdots, x_n)$ be the coordinates of a general point in Euclidean n-space referred to rectangular coordinate axes. We denote by $\boldsymbol{r}$ the vector from the origin to this general point. Consider the simplex whose vertices are the origin and the termini of the n vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n$ from the origin. The simplex is described by

$$(7) \qquad \boldsymbol{r} = \sum_1^n \xi_i \boldsymbol{y}_i$$

$$(8) \qquad \sum_1^n \xi_i \leqq 1, \quad \xi_1 \geqq 0,\ \xi_2 \geqq 0,\ \cdots,\ \xi_n \geqq 0\,.$$

The volume of the simplex is given by

$$(9) \qquad V = \int_{S_n} dx_1 \cdots \int dx_n = \int_R d\xi_1 \cdots \int d\xi_n \,|J|$$

where R is the ξ-region defined by (8) and J is the Jacobian of the transformation (7). If $\boldsymbol{y}_i = (y_{i1}, y_{i2}, \cdots, y_{in})$, $i = 1, 2, \cdots, n$, then (7) is explicitly $x_i = \sum \xi_j y_{ji}$, whence

$$J = \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix}$$

which is independent of the ξ's. The integral in (9) is readily evaluated to give the formula

$$n!\ V = |J| .$$

To obtain the content of a simplex not located at the origin, we translate the coordinates along the vector $\boldsymbol{x}_{n+1}$. Set $\boldsymbol{y}_i = \boldsymbol{x}_i - \boldsymbol{x}_{n+1}$, $i = 1, 2, \cdots, n$. Then the content of a simplex with vertices given by the termini of $\boldsymbol{x}_i$, $i = 1, \cdots, n+1$, is

$$\text{(10)}\qquad \begin{aligned} \text{n}!\ V &= \left\| \begin{matrix} x_{11} - x_{n+1\,1} & \cdots & x_{1n} - x_{n+1\,n} \\ \vdots & & \vdots \\ x_{n1} - x_{n+1\,1} & \cdots & x_{nn} - x_{n+1\,n} \end{matrix} \right\| \\ &= \left\| \begin{matrix} x_{11} & \cdots & x_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n+1\,1} & \cdots & x_{n+1\,n} & 1 \end{matrix} \right\| , \end{aligned}$$

a well-known formula [1, p. 124]. Here the double line denotes absolute value of a determinant. The equality shown in (10) can easily be established by subtracting the last row of the second determinant shown from each of the first n rows and evaluating the result by the cofactor expansion of the last column.

Squaring (10) we find $[n!\ V]^2 = \| \boldsymbol{x}_i \cdot \boldsymbol{x}_j + 1 \|$ where the determinant is obtained by multiplying the last matrix of (10) by its transpose and we exhibit the element in the i th row and j th column of the result. Introducing the notation of § 1, we set $\boldsymbol{x}_i \cdot \boldsymbol{x}_j = d_i d_j r_{ij}$ with Q located at the origin. We have then

$$\text{(11)}\qquad n!\ V = \| d_i d_j r_{ij} + 1 \|^{1/2} = \left\| r_{ij} + \frac{1}{d_i d_j} \right\|^{1/2} \prod_1^{n+1} d_i .$$

Now the ith row of $\| r_{ij} + 1/d_i d_j \|$ is the sum of the two rows $(r_{i1}, \cdots, r_{in+1})$ and $1/d_i(1/d_1, 1/d_2, \cdots, 1/d_{n+1})$. The determinant can thus be written as the sum of the 2^{n+1} determinants obtained by taking for each row either a row of the matrix (r_{ij}) or a row of the matrix $(1/d_i d_j)$. But any determinant having two or more rows taken from $(1/d_i d_j)$ vanishes, and $| r_{ij} |$ also vanishes since the $n+1$ n-vectors $\boldsymbol{a}_i$

are linearly dependent. The determinant $|r_{ij} + 1/d_i d_j|$ can therefore be expressed as the sum of $n+1$ determinants, the k th term being $|r_{ij}|$ with row k replaced by $1/d_k(1/d_1, 1/d_2, \cdots, 1/d_{n+1})$. Expanding this determinant by the k th row gives $\sum_j R_{kj}(1/d_k)(1/d_j)$ and formula (1) then follows directly.

3. Derivation of volume formula (2). Consider the volume, V_n, of the region S_n defined by

$$\mathbf{r}\cdot\mathbf{b}_i = \sum_{j=1}^{n} b_{ij}x_j \leqq e_i,\ i = 1, 2, \cdots, n+1 . \tag{12}$$

Let

$$b = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \tag{13}$$

and let B_{ij} be the cofactor of b_{ij} in b. Set $b_{ij}^{-1} = B_{ji}/|b|$, $i, j = 1, 2, \cdots, n$. Define new variables $y_1, \cdots, y_n$ by

$$y_i = \sum_{j=1}^{n} b_{ij}x_j,\ i = 1, \cdots, n$$

so that

$$x_i = \sum_{j=1}^{n} b_{ij}^{-1}y_j,\ i = 1, \cdots, n .$$

In the new variables, the inequalities (12) are

$$\begin{gathered} y_i \leqq e_i,\ i = 1, 2, \cdots, n \\ \sum_k \Big(\sum_j b_{n+1\,j}b_{jk}^{-1}\Big)y_k \leqq e_{n+1} . \end{gathered} \tag{14}$$

If we now regard the y's as rectangular coordinates, we see that (14) defines a simplex S_n' in this new space. If S_n' has y-volume V_n', then

$$V_n = V_n'/|b| \tag{15}$$

since $dx_1 \cdots dx_n = dy_1 \cdots dy_n/|b|$. We proceed by finding V_n'.

The bounding hyperplanes of S_n' are

$$\begin{aligned} H_1:&\quad y_1 = e_1 \\ \vdots\ &\quad\ \vdots\qquad \vdots \\ H_n:&\quad y_n = e_n \end{aligned} \tag{16}$$

$$H_{n+1}: \sum_k \Big(\sum_l b_{n+1\,l}b_{la}^{-1}\Big)y_k = e_{n+1} . \tag{17}$$

The vertex P_{n+1} of this simplex, given by $H_1 \cap H_2 \cap \cdots \cap H_n$, has coordinates

$$P_{n+1}\colon (e_1, e_2, \cdots, e_n) . \tag{18}$$

Consider the vertex P_i given by $H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_{n+1}$, $i = 1, 2, \cdots, n$. For the j th coordinate of P_i we find

$$y_{ij} = e_j,\ j \neq i,\ i, j, = 1, 2, \cdots, n \tag{19}$$

from (16). The i th coordinate y_{ii} is found from (17) as the solution of

$$y_{ii} \sum_l b_{n+1\,l} b_{li}^{-1} + \sum_{k\neq i} \sum_l b_{n+1\,l} b_{lk}^{-1} e_k = e_{n+1}$$

or

$$y_{ii} \sum_l b_{n+1\,l} B_{il} + \sum_{k\neq i} \sum_l b_{n+1\,l} B_{kl} e_k = |\, b \,|\, e_{n+1} . \tag{20}$$

Now let

$$c = \begin{pmatrix} b_{11} & & b_{1n} & e_1 \\ \vdots & & \vdots & \vdots \\ b_{n+1\,1} & \cdots & b_{n+1\,n} & e_{n+1} \end{pmatrix} = (c_{ij}) \tag{21}$$

and write C_{ij} for the cofactor of c_{ij} in c. Equation (20) now becomes

$$-y_{ii} C_{i\,n+1} - \sum_{k\neq i} C_{k\,n+1} e_k = C_{n+1\,n+1} e_{n+1}$$

or

$$-y_{ii} C_{i\,n+1} = \sum_{k\neq i}^{n+1} C_{k\,n+1} e_k = |\, c \,| - C_{i\,n+1} e_i .$$

Thus

$$y_{ii} = e_i - \frac{|\, c \,|}{C_{i\,n+1}},\ i = 1, 2, \cdots, n . \tag{22}$$

Formulas (18), (19) and (22) provide us with the coordinates of the vertices of S_n'. Using the first equality of (10) with the substitution $x_{ij} = y_{ij}$, $i, j = 1, 2, \cdots, n$, $x_{n+1\,j} = e_j$, $j = 1, 2, \cdots, n$, we find

$$n!\, V_n' = \left| \operatorname{diag}\left(-\frac{|\, c \,|}{C_{1\,n+1}}, -\frac{|\, c \,|}{C_{2\,n+1}}, \cdots, \frac{-|\, c \,|}{C_{nn+1}} \right) \right| = \frac{||\, c \,||^n}{\prod_1^n C_{i\,n+1}} .$$

From (15), then

$$n!\, V_n = \frac{||\, c \,||^n}{\left| \prod^{n+1} C_{j\,n+1} \right|} . \tag{23}$$

Next we note from (21), by multiplying c by its transpose, that $|c|^2 = |s_{ij} + e_i e_j|$ where as before $s_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$. An argument analogous to that given after equation (11) then shows that $|c|^2 = \sum S_{ij} e_i e_j$ with S_{ij} the cofactor of s_{ij} in (s_{ij}). Finally, we see from (21) that $|C_{j\,n+1}|$ is (apart from sign) the determinant of the $n \times n$ matrix whose rows are the $\boldsymbol{b}$ vectors, $\boldsymbol{b}_j$ being omitted. Multiplying this matrix by its transpose gives $|C_{j\,n+1}|^2 = S_{jj}$. This quantity is positive since we assume every n of the $\boldsymbol{b}$'s are independent and hence, as a matrix S_{jj} is positive definite. Formula (2) then follows by substitution in (23).

4. The largest simplex whose ith vertex is distant d_i from a given point. We choose the origin as the special point Q and denote by $\boldsymbol{a}_i d_i$ the vector from Q to the vertex P_i. Here $\boldsymbol{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ is a unit vector. Equation (10) then gives

$$n!\, V = \prod_1^{n+1} d_i \left\| \begin{matrix} a_{11} \cdots a_{1n} & \frac{1}{d_1} \\ \vdots \quad\quad \vdots & \vdots \\ a_{n+1\,1} a_{n+1\,n} & \frac{1}{d_{n+1}} \end{matrix} \right\| . \tag{24}$$

The vectors $\boldsymbol{a}_i$ are linearly dependent. We write

$$\boldsymbol{a}_{n+1} = \sum_1^n \alpha_j \boldsymbol{a}_j . \tag{25}$$

The determinant D displayed in (24) can now be expressed easily in other terms. Multiply the jth row of D by α_j and subtract from the last row, $j = 1, 2, \cdots, n$. Because of (25), all elements of the last row except the diagonal entry are zero. On expanding by this last row, we then find

$$D = |a| \left[\frac{1}{d_{n+1}} - \sum_1^n \frac{\alpha_j}{d_j} \right] \tag{26}$$

where

$$a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} .$$

We have also $|a|^2 = |\rho|$ where

$$\rho = (\rho_{ij}) = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \tag{27}$$

and as before $\boldsymbol{a}_i \cdot \boldsymbol{a}_j = r_{ij}$. Finally, defining

$$x_i = d_{n+1}/d_i,\ i = 1, 2, \cdots, n \tag{28}$$

equation (24) becomes

$$\frac{n!\ V_n}{\prod_1^n d_j} = |\rho|^{1/2}\left|\left[1 - \sum_1^n \alpha_j x_j\right]\right| . \tag{29}$$

The condition that $\boldsymbol{a}_{n+1}$ is a unit vector becomes from (25)

$$\prod_1^n \rho_{ij}\alpha_i\alpha_j = 1 . \tag{30}$$

We now seek to maximize (29), subject to (30), over all values of $\alpha_1, \alpha_2, \cdots, \alpha_n$ and over all symmetric $n \times n$ nonsingular matrices ρ having

$$\rho_{ii} = 1,\ i = 1, 2, \cdots, n . \tag{31}$$

Introducing the Lagrange multiplier λ, we seek the stationary values of

$$J = |\rho|^{1/2}\left[1 - \sum_1^n \alpha_j x_j\right] - \lambda \sum_{i,j} \rho_{ij}\alpha_i\alpha_j .$$

We have

$$\frac{\partial J}{\partial \alpha_i} = -|\rho|^{1/2} x_i - 2\lambda \sum_j \rho_{ij}\alpha_j = 0,\ i = 1, 2, \cdots, n \tag{32}$$

$$\begin{aligned}\frac{\partial J}{\partial \rho_{ij}} &= \frac{1}{2}\rho_{ji}^{-1}|\rho|^{1/2}[1 - \sum \alpha_l x_l] - \lambda\alpha_i\alpha_j = 0,\\ &i \neq j,\ i, j = 1, 2, \cdots, n .\end{aligned} \tag{33}$$

Multiply (32) by α_i and sum. By (30) one finds

$$2\lambda = +|\rho|^{1/2}/u \tag{34}$$

where we have written

$$u = -\frac{1}{\sum_1^n \alpha_j x_j} . \tag{35}$$

Equations (32) and (33) then become

$$\sum_{j=1}^n \rho_{ij}\alpha_j = -u x_i,\ i = 1, 2, \cdots, n \tag{36}$$

$$\rho_{ij}^{-1} = \frac{1}{1 + u}\alpha_i\alpha_j,\ i \neq j,\ i, j = 1, 2, \cdots, n . \tag{37}$$

Our task now is to solve the non-linear system (31), (35), (36), (37) for the α's and ρ_{ij}.

Multiply (36) by α_i to obtain

$$\begin{aligned} -u\alpha_i x_i &= \sum_j \rho_{ij}\alpha_i\alpha_j \\ &= \alpha_i^2 + \sum_{j\neq i} \rho_{ij}\alpha_i\alpha_j \\ &= \alpha_i^2 + (1+u)\sum_{j\neq i} \rho_{ij}\rho_{ji}^{-1} \\ &= \alpha_i^2 + (1+u)[1-\rho_{ii}^{-1}] \,. \end{aligned}$$

Here (31) was used to obtain the second line and (37) was used to obtain the third line. We have then

$$\rho_{ii}^{-1} = 1 + \frac{1}{1+u}[\alpha_i^2 + u\alpha_i x_i] \,. \tag{38}$$

From (36) we also have

$$\alpha_i = -u\sum_j \rho_{ij}^{-1}x_j,\; i = 1, 2, \cdots, n \,.$$

We now use (37) and (38) to replace ρ_{ij}^{-1} in this sum. There results

$$\begin{aligned} -\alpha_i/u &= \rho_{ii}^{-1}x_i + \sum_{j\neq i}\rho_{ij}^{-1}x_j \\ &= x_i + \frac{x_i}{1+u}[\alpha_i^2 + u\alpha_i x_i] + \frac{\alpha_i}{1+u}\sum_{j\neq i}\alpha_j x_j \\ &= x_i + \frac{x_i}{1+u}[\alpha_i^2 + u\alpha_i x_i] + \frac{\alpha_i}{1+u}\left[-\frac{1}{u} - \alpha_i x_i\right] . \end{aligned} \tag{39}$$

To obtain the last line we have employed (35). The quadratic terms in α_i cancel in (39) and the equation yields

$$\alpha_i = -\frac{(1+u)x_i}{1+ux_i^2},\; i = 1, 2, \cdots, n \,. \tag{40}$$

Therefore

$$\begin{aligned} \sum_1^n \alpha_i x_i &= -(1+u)\sum_1^n \frac{x_i^2}{1+ux_i^2} \\ &= -\frac{1}{u} \end{aligned}$$

by (35). The parameter u must therefore satisfy

$$\sum_1^n \frac{x_i^2}{1+ux_i^2} = \frac{1}{u(1+u)} \,. \tag{41}$$

We now write (38) in the form

$$\rho_{ii}^{-1} = \frac{1}{1+u}[\alpha_i^2 + q_i],\ i = 1, 2, \cdots, n \tag{42}$$

where

$$q_i = 1 + u + u\alpha_i x_i,\ i = 1, 2, \cdots, n\ . \tag{43}$$

It is easy to invert the matrix ρ^{-1} whose elements are given by (37) and (42). One finds

$$|\rho^{-1}| = \frac{1}{(1+u)^n}[1 + \sum \alpha_i^2/q_i] \prod q_i \tag{44}$$

$$\rho_{ii} = \frac{(1+u)\left[1 + \sum_{j\neq i} \alpha_j^2/q_j\right]}{q_i[1 + \sum \alpha_j^2/q_j]}\ ,\ i = 1, 2, \cdots, n \tag{45}$$

$$\rho_{ij} = -\frac{(1+u)}{1 + \sum \alpha_j^2/q_j} \cdot \frac{\alpha_i\alpha_j}{q_i q_j},\ i \neq j,\ i, j = 1, 2, \cdots, n\ . \tag{46}$$

Using (40), (41) and (43) in these expressions, one verifies that $\rho_{ii} = 1$ and finds

$$\rho_{ij} = -u x_i x_j,\ i \neq j,\ i, j = 1, 2, \cdots, n\ . \tag{47}$$

We note that from (25)

$$\begin{aligned} r'_{n+1\,i} &= \boldsymbol{a}_{n+1} \cdot \boldsymbol{a}_i = \sum_1^n \alpha_j \rho_{ij} \\ &= \alpha_i + \sum_{j\neq 1} \alpha_j(-u x_i x_j) \\ &= \alpha_i - u x_i\left(-\frac{1}{u} - \alpha_i x_i\right) \\ &= -u x_i\ . \end{aligned} \tag{48}$$

Here we have used (47) to obtain the second line, (35) to obtain the third line and (40) to obtain the final line. From (44), using (40), (41) and (43), one finds

$$|\rho| = \frac{u}{1+u} \prod_1^n (1 + u x_i^2)\ . \tag{49}$$

We now symmetrize the formulae thus far obtained by introducing

$$\theta = u d_{n+1}^2\ . \tag{50}$$

With the help of (28), (41) becomes

$$\theta \sum_1^{n+1} \frac{1}{\theta + d_j^2} = 1\ . \tag{51}$$

Equations (47) and (48) can be written jointly as

$$r'_{ij} = -\frac{\theta}{d_i d_j}\ , \ i \neq j,\ i, j = 1, 2, \cdots, n+1\ . \tag{52}$$

Finally, (29), (35) and (49) give us

$$n!\ V = |\,\theta\,|^{-1/2} \prod_1^{n+1} |\,\theta + d_i^2\,|^{1/2}\ .$$

To complete our demonstration of (3) and (5), we must show that θ must be chosen as the unique positive root of (51).

Let us suppose that the distances d_i are all distinct and that $0 < d_1 < d_2 < \cdots < d_{n+1}$. The modifications of our argument necessary when several d's are identical are easily made. It is readily seen from (51) that θ is the root of a polynomial of degree $n+1$ whose $n+1$ roots are real and can be labelled so that

$$\theta_1 > 0 > -d_1^2 > \theta_2 > -d_2^2 > \cdots > \theta_{n+1} > -d_{n+1}^2\ .$$

We shall show that the roots $\theta_3, \theta_4, \cdots, \theta_{n+1}$ do not correspond to a realizable simplex. Let $H(\theta) = \theta^{-1} \prod^{n+1} (\theta + d_i^2)$ so that $n!\ V = |\,H(\theta)\,|^{1/2}$. We shall also show that $H(\theta_1) > H(\theta_2) > 0$ which will then complete the proof.

Consider the $(n+1) \times (n+1)$ matrix r whose elements are given by (52) and $r_{ii} = 1$, $i = 1, 2, \cdots, n+1$. The elements $r_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$ of this matrix are scalar products of the optimal $\boldsymbol{a}$'s and since for arbitrary real numbers γ_i,

$$|\sum \gamma_i \boldsymbol{a}_i|^2 = \sum \gamma_i \boldsymbol{a}_i \cdot \sum \gamma_j \boldsymbol{a}_j = \sum_{i,j} r_{ij}\gamma_i\gamma_j \geqq 0$$

it follows that r must be nonnegative definite. The determinant of r and all the principal minors of r must then also be nonnegative. One readily finds

$$|\,r\,| = \left[1 - \theta \sum_1^{n+1} \frac{1}{\theta + d_j^2}\right] \prod_1^{n+1} \frac{\theta + d_j^2}{d_j^2}\ . \tag{53}$$

An expression for the principal minor of r obtained by deleting rows and columns $j_1, j_2, \cdots, j_l$ is given by (53) by omitting the terms and factors involving $d_{j_1}, d_{j_2}, \cdots, d_{j_l}$.

Suppose now $\theta = \theta_3$. Since θ_3 is a root of (51),

$$0 = 1 - \theta_3 \sum_1^{n+1} \frac{1}{\theta_3 + d_j^2} < 1 - \theta_3 \sum_2^{n+1} \frac{1}{\theta_3 + d_j^2}$$

since $\theta_3/(\theta_3 + d_j^2) > 0$. The principal minor of r obtained by deleting the first row and column has value

$$R_{11} = \left[1 - \theta_3 \sum_2^{n+1} \frac{1}{\theta_3 + d_j^2}\right] \prod_2^{n+1} \frac{\theta_3 + d_j^2}{d_j^2} .$$

We have seen that the bracketed expression is positive. Of the factors, $\theta_3 + d_2^2$ is negative, and all others positive. R_{11} is therefore negative and we must reject the root θ_3.

In a similar manner one sees that for $\theta = \theta_k$, $k > 2$ the principal minor obtained by deleting rows and columns 1, 2, $\cdots$, $k - 2$ is negative. We complete the proof by showing $H(\theta_1) > H(\theta_2) > 0$. Since $\theta_1 > 0$ while $0 > -d_1^2 > \theta_2 > -d_2^2 \cdots$

$$\frac{d_1^2}{\theta_1} > \frac{d_1^2}{\theta_2}$$

so

$$1 + \frac{d_1^2}{\theta_1} > 1 + \frac{d_1^2}{\theta_2}$$

or

$$\frac{\theta_1 + d_1^2}{\theta_1} > \frac{\theta_2 + d_1^2}{\theta_2} > 0 .$$

Now

$$\theta_1 + d_j^2 > \theta_2 + d_j^2 > 0 \quad \text{for} \quad j \geqq 2$$

so that

$$H(\theta_1) = \frac{\theta_1 + d_1^2}{\theta_1} \prod_2^{n-1} (\theta_1 + d_j^2) > \frac{\theta_2 + d_1^2}{\theta_2} \prod_2^{n+1} (\theta_2 + d_j^2) = H(\theta_2) > 0 .$$

We close this section with the remark that the origin and P_j lie on the same side of H_j if and only if $(\theta + d_j^2)/\theta$ is positive. We omit the direct demonstration of this fact here. Corresponding to the root $\theta_1 > 0$ of (51) we obtain a simplex containing the special point Q. For the root θ_2, satisfying $-d_1^2 > \theta_2 > -d_2^2$, we see that Q lies outside the simplex, since $(\theta_2 + d_2^2)/\theta_2 < 0$ for example.

5. The smallest simplex whose i th bounding plane is distant e_i from a given interior point. We choose the origin as the given interior point. Let $\boldsymbol{b}_i$ be the unit vector from the origin along the perpendicular to boundary H_i, $i = 1, 2, \cdots, n + 1$. The volume of the simplex is given by (23) with c defined in (21). Now the vectors $\boldsymbol{b}_i$ are linearly dependent. We write

(54) $$\boldsymbol{b}_{n+1} = \sum_{j=1}^{n} \beta_j \boldsymbol{b}_j$$

in analogy with (25). Making an obvious association between $|c|$ and the determinant in (24), we find from (26) that

$$|c| = |b|\left[e_{n+1} - \sum_1^n \beta_j e_j\right]$$

where b is the $n \times n$ matrix given in (13). We note that $|C_{j\,n+1}| = |\alpha_j|b||$, $j = 1, \cdots, n$ while $C_{n+1\,n+1} = |b|$. Equation (23) then gives us

$$(55) \qquad n!\, V_n = \left| \frac{|b|^n \left[e_{n+1} - \sum_1^n \beta_j e_j\right]^n}{|b|^{n+1} \prod_1^n \beta_j} \right| = \frac{\left|e_{n+1} - \sum_1^n \beta_j e_j\right|^n}{|\sigma|^{1/2} \left|\prod_1^n \beta_j\right|}$$

where

$$\sigma = (\sigma_{ij}) = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}$$

where as before $s_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$. Finally, defining

$$(56) \qquad y_i = e_i/e_{n+1},\ i = 1, \cdots, n$$

(55) becomes

$$(57) \qquad \frac{n!\, V_n}{e_{n+1}^n} = \frac{|1 - \sum \beta_j y_j|^n}{|\sigma|^{1/2} \left|\prod_1^n \beta_j\right|}\ .$$

The condition that $\boldsymbol{b}_{n+1}$ is a unit vector becomes from (54)

$$(58) \qquad \sum_1^n \sigma_{ij}\beta_i\beta_j = 1\ .$$

We now seek to minimize (57), subject to (58), over all values of $\beta_1, \cdots, \beta_n$ and all symmetric $n \times n$ nonsingular matrices σ having

$$(59) \qquad \sigma_{ii} = 1,\ i = 1, 2, \cdots, n\ .$$

Introducing the Lagrange multiplier μ, we seek the stationary values of

$$K = n \log\,[1 - \sum \beta_j y_j] - \frac{1}{2} \log|\sigma| - \sum \log \beta_j - \mu \sum \sigma_{ij}\beta_i\beta_j\ .$$

We have

$$(60) \qquad \frac{\partial K}{\partial \beta_i} = \frac{-n y_i}{1 - \sum \beta_j y_j} - \frac{1}{\beta_i} - 2\mu \sum \sigma_{ij}\beta_j = 0,\ i = 1, 2, \cdots, n\ ,$$

$$(61) \qquad \frac{\partial K}{\partial \sigma_{ij}} = -\frac{1}{2}\sigma_{ji}^{-1} - \mu\beta_i\beta_j = 0,\ i \neq j,\ i, j = 1, 2, \cdots, n\ .$$

Multiply (60) by β_i and sum. By (58) one finds

(62) $$2\mu = -\frac{n}{1 - \sum \beta_j y_j} = -\frac{1}{1+v}$$

where we have set

(63) $$v = -\frac{1}{n}\left[n - 1 + \sum_1^n \beta_j y_j\right].$$

Equations (60) and (61) then become

(64) $$\sum_j \sigma_{ij}\beta_j = y_i + \frac{1+v}{\beta_i},\ i = 1, 2, \cdots, n$$

(65) $$\sigma_{ij}^{-1} = \frac{1}{1+v}\beta_i\beta_j,\ i \neq j,\ i, j = 1, 2, \cdots, n.$$

Our task now is to solve the nonlinear system (59), (63), (64), (65) for the β's and σ_{ij}.

Multiply (64) by β_i to obtain

$$\begin{aligned}\beta_i y_i + 1 + v &= \beta_i^2 + \sum_{j\neq i} \sigma_{ij}\beta_i\beta_j \\ &= \beta_i^2 + (1+v)\sum_{j\neq i}\sigma_{ij}\sigma_{ji}^{-1} \\ &= \beta_i^2 + (1+v)(1 - \sigma_{ii}^{-1})\end{aligned}$$

whence

(66) $$\sigma_{ii}^{-1} = \frac{1}{1+v}[\beta_i^2 - \beta_i y_i].$$

From (64)

$$\beta_i = \sum_{j=1}^n \sigma_{ij}^{-1}\left[y_j + \frac{1+v}{\beta_j}\right].$$

Replace σ_{ij}^{-1} by values given in (65) and (66). Use (63). There results

$$\beta_i = -\frac{(1+v)y_i}{v + y_i^2},\ i = 1, 2, \cdots, n.$$

Multiply by y_i and sum. Insert the result in (63). One finds that v must satisfy

$$\sum \frac{1}{v + y_j^2} = \frac{1}{v(1+v)}.$$

The analogy between (37) and (65) and between (42) and (66) permits us to use (44), (45) and (46) directly to obtain

$$\| \sigma \| = \frac{v \prod_1^n (v + y_j^2)}{(v + 1) \prod_1^n y_j^2}$$

$$\sigma_{ij} = -\frac{v}{y_i y_j} .$$

The substitution

$$v = \psi / e_{n+1}^2$$

now yields (4) and (6). We omit the details.

In analogy with (5), the roots of (6) are all real and can be labelled so that $\psi_1 > 0 > -e_1^2 > \psi_2 > -e_2^2 > \cdots > \psi_{n+1} > -e_{n+1}^2$, if $e_1 < e_2 \cdots < e_{n+1}$. Only ψ_1 and ψ_2 correspond to realizable simplexes and the content corresponding to ψ_1 is greater than the content of the simplex corresponding to the root ψ_2. It is not difficult to show that P_j and the origin lie on the same side of H_j if and only if $\psi + e_j^2 > 0$. For the solution corresponding to ψ_1, then, Q lies within the simplex; for the solution corresponding to ψ_2, Q and the simplex lie on opposite sides of H_1.

Reference

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