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GLEASON PARTS AND CHOQUET BOUNDARY POINTS IN CONVOLUTION MEASURE ALGEBRAS

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Let M be a semisimple convolution measure algebra with structure semigroup S. Then each complex homomorphism of M is given by integrating a semicharacter on S. Gleason parts can be defined on $\hat{S}$, the set of semicharacters on S, by considering the function algebra obtained from the transforms of elements of M. We give a partial characterization of the parts of $\hat{S}$ utilizing only the functional values of the elements of $\hat{S}$. We then completely characterize the one point parts of $\hat{S}$ utilizing only the functional values of elements of $\hat{S}$.

If S is a locally compact topological semigroup, then the measure algebra $M(S)$ is a member of an abstract class of Banach algebras called convolution measure algebras by Taylor in [12]. Other examples include $L'(G)$ for a locally compact group G and the Arens-Singer algebras introduced in [1]. The convolution measure algebras form an extremely large and diverse class of algebras. In fact, a large number of interesting function algebras can be described as completions, in the spectral norm, of convolution measure algebras.

Taylor's main theorem in [12] is the following: if M is a commutative, semisimple convolution measure algebra, then M may be embedded in the measure algebra $M(S)$ of a certain canonical compact semigroup S, in such a way that every complex homomorphism of M is determined by a continuous semicharacter on S.

Taylor's theorem identifies the maximal ideal space Δ of M as the set $\hat{S}$ of all semicharacters on a compact semigroup S. This gives Δ a considerable amount of structure not generally enjoyed by maximal ideal spaces. It is natural to try to use this additional structure to help identify such standard objects as the Shilov and Choquet boundaries and the Gleason parts of Δ.

If $H = \{f \in \hat{S} = \Delta : |f|^2 = |f|\}$, then Taylor showed that every element of $\hat{S} \backslash H$ lies in an analytic disc in $\hat{S}$. Hence the Choquet boundary points and the one point parts all lie in H and the closure, $\bar{H}$, of H contains the Shilov boundary (c.f. [7, 11, 12]). However, these results are not sharp since there are trivial examples where H contains points which are neither one point parts, Choquet boundary points, nor Shilov boundary points. In fact, $\mathscr{L}_1(J)$, where J is the additive semigroup of nonnegative integers, is such an example. Here $\hat{S}$ is the unit disc in the complex plane C and $H = \{z : |z| = 1 \text{ or } 0\}$. The point 0 is in H and also is in a nontrivial part, $(\{z : |z| < 1\})$, of $\hat{S}$.

In this paper we propose to sharpen Taylor's results through a detailed study of the Gleason parts of $\hat{S}$. Our major contribution is the following: if $f, g \in \hat{S}$ and $f \geqq 0, g \geqq 0$ then f and g are in the same part if and only if $\{x \in S: f(x) = 1\} = \{x \in S: g(x) = 1\}$. (Theorems 2.3 and 3.1). For arbitrary $f, g \in \hat{S}$ we conjecture that f and g are in the same part if and only if

$$(*)\ \{x: |f(x)| = 1\} = \{x: |g(x)| = 1\} \subseteqq \{x: f(x) = g(x)\}\ .$$

It is easy to prove that (*) is a necessary condition for f and g to be in the same part (Theorem 2.3). However, we only have the sufficiency in certain special cases.

Our results on parts are sufficient to completely characterize the one point parts. Since the Choquet boundary is known to be contained in the set of one point parts, this is progress towards a complete characterization of the Choquet boundary and the Shilov boundary of a convolution measure algebra. The result is this: $\{f\} \subseteqq \hat{S}$ is a one point part if and only if $f \in H$ and there is no $g \in \hat{S}$, with $g \neq f$, which satisfies (*). (Theorem 4.5).

In § 1 we briefly outline certain background information—largely from Taylor's structure theory. In § 2 and § 3 we solve the problem of deciding when f and g are in the same part (for $f, g \geqq 0$), and then in § 4 we characterize the set of one point parts of $\hat{S}$. Also in § 4 we relate the parts structure of $\hat{S}$ to the idempotents in S.

Section 2 and § 3 each deal with the same problem. However, the proof of our main result, Theorem 3.1, is so drawn out that we have devoted an entire section to it.

We prove Theorem 3.1 in § 3. In that section we show that the problem of deciding when f and g are in the same part can be reduced to showing the equivalence of two special functions on a specific measure algebra situated in the first quadrant of the plane. We then show the equivalence of these special functions through the use of positive definite functions on the plane.

Finally, in § 5 we give some examples.

1. Preliminaries. We will be working in the setting described by Taylor in [12]; so, to simplify things, we will outline the parts of that paper that are most pertinent for what we do here.

In [12], Taylor defines convolution measure algebra and obtains a representation theorem for such algebras. The following is the main result of that paper.

THEOREM. *Let M be a commutative semisimple convolution measure algebra with identity. Then there exists a compact abelian* *semigroup S, called the structure semigroup of M, and an order preserving isomorphism-isometry $\nu \to \nu_s$ of M onto a weak-star dense subalgebra M_s of $M(S)$. Furthermore, each complex homomorphism h_γ of M is obtained by integrating a unique semicharacter γ on S by means of the formula*

$$h_\gamma(\nu) = \hat{\nu}(\gamma) = \int_S \gamma(x) d\nu_s(x) .$$

By a semicharacter on a topological semigroup T we will always mean a continuous homomorphism of T into the unit disc. We do not include the zero homomorphism in this definition. By $M(S)$ we mean the algebra of all finite regular Borel measures on S under convolution.

We will always let M denote a commutative semisimple convolution measure algebra with identity and we will let S denote its structure semigroup. Furthermore, we will identify M with M_s and drop the subscripts. Finally, we let $\hat{S}$ denote the set of all semicharacters on S and we identify $\hat{S}$ with the maximal ideal space of M. We note that as a set of functions, $\hat{S}$ separates the points of S [12].

The dual $\hat{S}$ of S is rich in algebraic and analytic structure. Specifically, note the following: if $f, g \in \hat{S}$ then

(i) $f \cdot g \in \hat{S}, \bar{f} \in \hat{S}$ and $|f| \in \hat{S}$;

(ii) if $f \geqq 0$ and $\operatorname{Re} z > 0$ then $f^z \in \hat{S}$;

(iii) if $\nu \in M$ and $f_i \geqq 0, 1 \leqq i \leqq n$, then

$$(z_1, \cdots, z_n) \to \hat{\nu}(f_1^{z_1} \cdots f_n^{z_n}) = \int_S f_1(x)^{z_1} f_2(x)^{z_2} \cdots f_n(x)^{z_n} d\nu(x)$$

is an analytic function for $\operatorname{Re} z_i > 0, 1 \leqq i \leqq n$.

In [4], Gleason notes that if A is a function algebra on its maximal ideal space X and if $x, y \in X$, then the relationship

$$\sup_{||f|| \leqq 1} |f(x) - f(y)| < 2$$

is an equivalence relation. Gleason calls the equivalence classes under this relation the "parts" of X. In this paper we use the following facts about parts:

(a) x and y are in the same part if and only if there exist representing measures ν_x for x and ν_y for y which are mutually absolutely continuous with bounded Radon-Nikodym derivatives.

(b) x and y are in the same part if and only if there exist representing measures ν_x for x and ν_y for y which are not mutually singular.

(c) If there exists a sequence $\{f_n\}_{n=1}^\infty$ with $||f_n|| \leqq 1$ for all n such that $\lim_{n \to \infty} f_n(x) = z_1$ and $\lim_{n \to \infty} f_n(y) = z_2$ then x and y are not in the same part provided either $|z_1| < 1$ and $|z_2| = 1$ or $|z_1| = |z_2| = 1$

but $z_1 \neq z_2$.

A map φ from $\{z: |z| < 1\}$ into the maximal ideal space X of a function algebra A on X is said to be analytic if $f \circ \varphi$ is an analytic function for all $f \in A$.

(d) The image of an analytic map is contained in a single part.

(a) and (b) can be found in [2] and [4], (c) follows from the definition after application of linear fractional transformations, and (d) follows from the definition and Schwarz's lemma [4].

Throughout this paper we constantly use results from the theory of compact semigroups. For a discussion of this subject see [6].

2. Parts and structure in $\hat{S}$. As a way of getting at the Choquet boundary of a convolution measure algebra M, we look at the Gleason parts of $\hat{S}$ [4]. We mean by this, the Gleason parts of $\hat{S}$ relative to the function algebra obtained by completing, in the spectral norm, the algebra of Gelfand transforms of elements of M. We will write $f \sim g$ if f and g are in the same part.

We have already noted that $\hat{S}$, in addition to being the maximal ideal space of M, has the algebraic structure of a semigroup with involution (conjugation). Furthermore, it has an order relation and an "analytic structure" (provided there are nonnegative semicharacters $f \in \hat{S}$ with $f^2 \neq f$). Thus, one might expect that this additional structure is related to the decomposition of $\hat{S}$ into parts. This is the case as the following propositions show.

PROPOSITION 2.1. *If f, g and h are in $\hat{S}$ and if f and g are in the same part, then $\bar{f}$ and $\bar{g}$ are in the same part and $f \cdot h$ and $g \cdot h$ are in the same part.*

Proof. Since $\hat{\nu}(f) = \int f(x) d\nu(x)$, we have $\hat{\nu}(\bar{f}) = \overline{\hat{\bar{\nu}}(f)}$, and $\bar{f} \sim \bar{g}$ whenever $f \sim g$.

If $h \in \hat{S}$ and $\nu \in M$ then the measure μ defined by $d\mu = h d\nu$ is in M since M is an L-space [12]. Furthermore,

$$|| \hat{\mu} ||_\infty = \sup_{k \in \hat{S}} \left| \int k(x) h(x) d\nu(x) \right| \leqq || \hat{\nu} ||_\infty .$$

Thus,

$$\begin{aligned} \sup_{||\hat{\nu}||_\infty \leqq 1} | \hat{\nu}(f \cdot h) - \hat{\nu}(g \cdot h) | &= \sup_{||\hat{\nu}||_\infty \leqq 1} \left| \int [f(x) - g(x)] h(x) d\nu(x) \right| \\ &\leqq \sup_{||\hat{\nu}||_\infty \leqq 1} \left| \int [f(x) - g(x)] d\nu(x) \right| < 2 \end{aligned}$$

if $f \sim g$. Thus $f \sim g$ implies $f \cdot h \sim g \cdot h$.

PROPOSITION 2.2. *If $f_1, f_2, \cdots f_n$ are in $\hat{S}$ with $f_i(x) \geqq 0$ for all $x \in S$ and $\operatorname{Re} z_i > 0, 1 \leqq i \leqq n$, then $f_1 \cdot f_2 \cdots f_n$ and $f_1^{z_1} \cdot f_2^{z_2} \cdots f_n^{z_n}$ are in the same part.*

Proof. The proposition follows from the facts that the map $(z_1, \cdots, z_n) \rightarrow \hat{\nu}(f_1^{z_1} \cdots f_n^{z_n})$ is an analytic function for each ν in M and that the image of analytic map is contained in a single part [4].

Proposition 2.2 is usually difficult to apply since it is generally difficult to tell whether or not two given semicharacters can be written in the form of Proposition 2.2. However, if one knows how two semicharacters behave as functions then the following theorem can be of use.

THEOREM 2.3. *If f and g are in $\hat{S}$ and if f and g are in the same part then $f(x) = g(x)$ whenever either $|f(x)| = 1$ or $|g(x)| = 1$.*

Proof. Let f, g be elements of $\hat{S}$ and let $x \in S$ where $|f(x)| = 1$ but $g(x) \neq f(x)$. If $|g(x)| < 1$, then consider the sequence $\{x^n\}_{n=1}^{\infty}$ in S. It must cluster, say to y, as S is compact. Since $|g(x^n)| = |g(x)|^n$ and $|g(x)| < 1$, we have $g(y) = 0$. Clearly $|f(y)| = 1$.

Let δ_y denote the unit point mass at y. We have $|\hat{\delta}_y(f)| = |f(y)| = 1$ but $\hat{\delta}_y(g) = 0$. Since M is weak-star dense in $M(S)$ and since M is an L-space, there must exist a net $\{\nu_\alpha\}$ in M, with $\nu_\alpha \geqq 0$, $||\nu_\alpha|| \leqq 1$ for all α, that converges weak-star to δ_y [12]. Thus we have $\lim_\alpha \hat{\nu}_\alpha(f) = 1$ and $\lim_\alpha \hat{\nu}_\alpha(g) = 0$. Since $||\hat{\nu}_\alpha||_\infty \leqq ||\nu_\alpha|| \leqq 1$ we conclude that f and g are not in the same part.

If $|g(x)| = 1$ then as above there is a net $\{\nu_\alpha\}$ in M satisfying $\nu_\alpha \geqq 0$, $||\nu_\alpha|| \leqq 1$ for all α and converging weak-star to δ_x. Since $\{\hat{\nu}_\alpha(f)\}$ and $\{\hat{\nu}_\alpha(g)\}$ converge to different points on the unit circle, we conclude that f and g are in different parts.

The above arguments are symmetric in f and g so we have the conclusion of the theorem.

We conjecture that the converse of the above theorem is also true. That is, we conjecture that f and g are in the same part if and only if $f(x) = g(x)$ whenever $|f(x)| = 1$ or $|g(x)| = 1$. The complete solution to this problem is still open, but we give a partial converse in Proposition 2.7. Then, in Theorem 3.1, we prove the converse for nonnegative semicharacters.

DEFINITION 2.4. If $f \in \hat{S}$ define f_0' by

$$f_0'(x) = \begin{cases} 0 & \text{if} \quad f(x) = 0 \\ \dfrac{f(x)}{|f(x)|} & \text{if} \quad f(x) \neq 0 \,. \end{cases}$$

Note that f_0' is a multiplicative Borel function and that $f = f_0'|f|$. Also note that if $f, g \in \hat{S}$ and if $f(x) = 0$ implies $g(x) = 0$ then $f_0' \cdot g$ is in $\hat{S}$. In fact, Taylor shows that if f is in $\hat{S}$ then f_0' is equal almost everywhere with respect to each ν in M to an element f_0 in $\hat{S}$. Furthermore, $f = f_0|f|$. This is Taylor's polar decomposition theorem [12]. In what immediately follows, it will not matter whether f_0 or f_0' is used since $f_0(s) = f_0'(s)$ whenever $f(s) \neq 0$. At all other points we will be multiplying by functions that are zero. However, in § 3 there will be one point where the full force of the polar decomposition theorem will be used.

LEMMA 2.5. *Let $f, g \in \hat{S}$ and suppose that there exists an r, $0 \leqq r < 1$ such that $f(s) = g(s)$ whenever $|f(s)| > r$ or $|g(s)| > r$. Then there exists a $k > 0$ such that $||f(s)|^z f_0(s) - |g(s)|^z g_0(s)| \leqq 2e^{-k\operatorname{Re} z}$ for all $s \in S$ and $\operatorname{Re} z > 0$.*

Proof. If $r = 0$ then the inequality holds for any $k > 0$ so let $r > 0$.

If either $|f(s)| > r$ or $|g(s)| > r$ then $g(s) = f(s)$ so $f_0(s) = g_0(s)$ and $|f(s)| = |g(s)|$. Hence $|f(s)|^z f_0(s) - |g(s)| g_0(s) = 0$ and the inequality holds for any $k > 0$. So, let $|f(s)| \leqq r$ and $|g(s)| \leqq r$. We then have $||f(s)|^z f_0(s) - |g(s)|^z g_0(s)| \leqq |f(s)|^{\operatorname{Re} z} + |g(s)|^{\operatorname{Re} z} \leqq 2r^{\operatorname{Re} z}$. If we set $k = -\ln(r)$ then we have the conclusion of the lemma.

The next lemma is an immediate consequence of the maximum modulus theorem.

LEMMA 2.6. *Let $k > 0$ and let h be an analytic function on $\{z \in C: \operatorname{Re} z > 0\}$. Furthermore, let $|h(z)| \leqq De^{-k\operatorname{Re} z}$ for some $D > 0$. If $|h(z)| \leqq 2$ then in fact $|h(z)| \leqq 2e^{-k\operatorname{Re} z}$.*

PROPOSITION 2.7. *Let $f, g \in \hat{S}$ and suppose that there exists an r, $0 \leqq r < 1$, such that $f(s) = g(s)$ whenever either $|f(s)| > r$ or $|g(s)| > r$. Then f and g are in the same part.*

Proof. Note that $h(z) = \int (|f(s)|^z f_0(s) - |g(s)|^z g_0(s)) d\nu(s)$ is analytic for $\operatorname{Re} z > 0$ and $\nu \in M$. If $\|\hat{\nu}\|_\infty \leqq 1$ then $|h(z)| \leqq 2$. Also, by 2.5 there exists a $k > 0$ such that

$$|h(z)| \leqq \int ||f(s)|^z f_0(s) - |g(s)|^z g_0(s)| d|\nu|(s) \leqq 2\|\nu\| e^{-k\operatorname{Re} z} .$$

So, by 2.6, $|h(z)| \leqq 2e^{-k\operatorname{Re} z}$ whenever $\|\hat{\nu}\|_\infty \leqq 1$. Thus,

$$\sup_{\|\hat{\nu}\|_\infty \leqq 1} |\hat{\nu}(|f|^z f_0) - \hat{\nu}(|g|^z g_0)| \leqq 2e^{-k\operatorname{Re} z} .$$

At $z = 1$ this gives $f \sim g$.

3. **More parts.** The object of this section will be to prove the following theorem.

THEOREM 3.1. *If f and g are in $\hat{S}$ with $f \geqq 0$ and $g \geqq 0$ and if $\{x \in S: f(x) = 1\} = \{x \in S: g(x) = 1\}$ then f and g are in the same part.*

We will prove Theorem 3.1 by first introducing a specific example. The truth of Theorem 3.1 for this example implies its truth in general.

DEFINITION 3.2. If $f, g \in \hat{S}$ set $N(f, g) = \{s \in S: f(s) \neq 0, g(s) \neq 0\}$, set $0(f) = \{s \in S: |f(s)| = 1\}$, and set $Z(f) = \{s \in S: f(s) = 0\}$.

Each of $Z(f)$ and $Z(g)$ is a closed prime ideal [12] in S. Since the union of two closed prime ideals is a prime ideal, we have the following lemma.

LEMMA 3.3. *If $f, g \in \hat{S}$ then $S \backslash N(f, g)$ is a closed prime ideal. In particular, $N(f, g)$ is an open sub-semigroup of S.*

In α is an angle satisfying $0 \leqq \alpha \leqq \pi/4$ and a is a nonnegative real number, set $X_{a,\alpha}$ equal to $\{(x, y) \in R^2$: either $x = y = 0$, or $x > 0$, $y > 0$ and $\tan \alpha \leqq y/x \leqq \tan(\pi/2 - \alpha)$, or $x \geqq a, y \geqq a\}$. Note that $X_{a,\alpha}$ is a subsemigroup of R^2 under coordinatewise addition.

LEMMA 3.4. *Let f and g be elements of $\hat{S}$ satisfying:*
(i) $f \geqq 0, g \geqq 0$
(ii) $f(x) = 1$ *if and only if* $g(x) = 1$.
Define $F: N(f, g) \rightarrow R^2$ by $F(s) = (-\ln f(s), -\ln g(s))$. Then F is a continuous homomorphism of $N(f, g)$ into R^2. Furthermore, there exists an angle α, $0 < \alpha \leqq \pi/4$ and an $a > 0$ such that $F(N(f, g)) \subseteqq X_{a,\alpha}$.

Proof. F is clearly a continuous homomorphism of $N(f, g)$ into $X_{0,0} \subseteqq R^2$. Let $B = \{s \in S: f(s) \leqq 1/e\}$ and $b = \inf_{s \in B}[-\ln g(s)]$. We claim that $b > 0$. To show this, note that B is a compact sub-semigroup of S. Thus, if it were the case that $b = 0$, then there would be an element $s \in B$ satisfying $g(s) = 1$ but $f(s) < 1$. This cannot happen so $b > 0$.

Similarly, if $B = \{s \in S: g(s) \leqq 1/e\}$ then $0 < c = \inf_{s \in B}[-\ln f(s)]$. If we let $a = \min(1, b, c)$ then for $s \in S$, either $-\ln f(s) \geqq a$ and $-\ln g(s) \geqq a$ or else $F(s) \in [0, 1] \times [0, 1]$.

There cannot exist an $s \in S$ satisfying $0 < -\ln f(s) \leqq 1$ and

$$\frac{-\ln g(s)}{-\ln f(s)} < \frac{a}{2} ;$$

otherwise, $a/2 > -\ln g(s^n)/-\ln f(s^n)$ for all n. Thus we could choose an n such that $1 < -\ln f(s^n) \leqq 2$ but $-\ln g(s^n) < a$, a situation that cannot occur. Similarly, there is no $s \in S$ for which $0 < -\ln g(s) \leqq 1$ and $-\ln f(s)/-\ln g(s) < a/2$. Thus, if $\alpha = \arctan(a/2)$ we have

$$F(N(f, g)) \subseteqq X_{a,\alpha} .$$

The above lemma can be used to map the algebra of Theorem 3.1 into $M(X_{a,\alpha})$, the algebra under convolution of all finite regular Borel measures on $X_{a,\alpha}$. Once in $M(X_{a,\alpha})$ we note that integration of the function $e^{-|x|}$ and $e^{-|y|}$ define complex homomorphisms of $M(X_{a,\alpha})$. We will show that $e^{-|x|}$ and $e^{-|y|}$ are equivalent relative to $M(X_{a,\alpha})$. The equivalence of f and g in Theorem 3.1 will then follow.

Our method of attack in showing that $e^{-|x|}$ and $e^{-|y|}$ define equivalent homomorphism of $M(X_{a,\alpha})$ will be to construct nonmutually singular representing measures for $e^{-|x|}$ and $e^{-|y|}$. To do this we need some specific positive definite functions on R^2. (For a discussion of positive definite functions on a group see [9].)

Let $T(x)$ be the function defined by $T(x) = 1 - |x|$ if $|x| \leqq 1$ and $T(x) = 0$ if $|x| > 1$.

LEMMA 3.5. *For sufficiently large $k > 0$, the functions*
(1) $f(x) = e^{-|x|} - 1/ke^{-b|x|}T(x)$ *on* R,
(2) $f(x, y) = e^{-|x|} - 1/ke^{-b(x)}T(x)$ *on* R^2,
(3) $g(x, y) = 1/ke^{-|bx-y|}e^{-|y|}T(x)$ *on* R^2,
(4) $h(x, y) = e^{-|x|} - 1/ke^{-b|x|}T(x) + 1/ke^{-|bx-y|}e^{-|y|}T(x)$ *on* R^2,
where $b \geqq 1$ is fixed, are positive definite.

Proof. If $\hat{a}$ denotes the Fourier transform of a, (either on R or R^2, whichever is appropriate), then for $E(x) = e^{-|x|}$ we have

$$(1) \qquad \hat{E}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-iwx}dx = \frac{1}{\pi(1 + w^2)} ,$$

$$(2) \qquad \begin{aligned} \widehat{E^b T}(w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwx}e^{-b|x|}T(x)dx \\ &= \frac{b[b^2 + w^2] + [w^2 - b^2] + e^{-b}[(b^2 - w^2)\cos w - 2bw \sin w]}{\pi[b^2 + w^2]^2} \end{aligned}$$

and since $f = E - 1/kE^bT$

(3)
$$\hat{f}(w) = \frac{\dfrac{k[b^2+w^2]^2}{1+w^2} - \{b(b^2+w^2)+(w^2-b^2)+e^{-b}[(b^2-w^2)\cos w - 2bw\sin w]\}}{k\pi(b^2+w^2)^2},$$

By Bochner's theorem [9], an integrable function is positive definite if its Fourier transform is nonnegative. Hence, E is positive definite. To show that f is positive definite, consider

(4)
$$\begin{aligned} &\frac{k(b^2+w^2)^2}{1+w^2} - \{b(b^2+w^2)+(w^2-b^2)+e^{-b}[(b^2-w^2)\cos w - 2bw\sin w]\} \\ &\geqq k(b^2+w^2) - \{b(b^2+w^2)+(w^2-b^2)+e^{-b}[(b^2-w^2)\cos w - 2bw\sin w]\} \\ &= (k+1)b^2 - b^3 + (k-b-1)w^2 - e^{-b}(b^2-w^2)\cos w + 2bwe^{-b}\sin w\ . \end{aligned}$$

If $|w| \leqq b$ then (4) becomes greater than or equal to

(5)
$$(k+1)b^2 - b^3 + (k-b-1)w^2 - b^2e^{-b} - 2b^2e^{-b}$$

which is strictly positive if $k > b + 2$.

If $|w| \geqq b$ then (4) is greater than or equal to

(6)
$$(k+1)b^2 - b^3 + (k-b-1)w^2 - w^2e^{-b} - 2w^2e^{-b}$$

which is strictly positive if $k > b + 4$.

Thus, for sufficiently large k, (3) is always positive and hence f is positive definite. Furthermore, by Bochner's theorem, if $d\nu = \hat{f}(w)dw$ then ν is a positive measure and the inverse Fourier transform of ν is f.

Let δ_0 be the point mass at zero in R. The positive measure $\nu \times \delta_0$ on R^2 has inverse Fourier transform equal to $f(x, y)$ on R^2. Again by Bochner's theorem, this makes $f(x, y)$ positive definite on R^2.

We have shown that $x \to e^{-|x|}$ is positive definite. Likewise, T is positive definite as can be seen by computing its Fourier transform. By using product measures as before, we can show that $(x, y) \to e^{-|x|}$ and $(x, y) \to T(x)$ are positive definite on R^2. Furthermore, since $x \to e^{-|x|}$ is positive definite, it follows from the definition of positive definite that $(x, y) \to e^{-|bx-y|}$ is positive definite on R^2.

We note that the convolution product of positive measures yields a positive measure. We also note that the inverse Fourier transform of a convolution product of measures gives a pointwise product of functions. Hence, by Bochner's theorem, $g(x, y)$ is positive definite. Finally, by taking a positive sum of positive definite functions, we get $h(x, y)$ positive definite.

Let $C^+ = \{z \in C\colon \mathrm{Re}\, z \geqq 0\}$. Then for $(z_1, z_2) \in C^+ \times C^+$, define $f_{(z_1,z_2)}(x, y) = e^{-(z_1x+z_2y)}$. The formula $\hat{\nu}(z_1, z_2) = \int e^{-(z_1x+z_2y)}d\nu(x, y)$ defines a complex homomorphism of $M(X_{a,\alpha})$ for fixed (z_1, z_2). Thus, we may consider $C^+ \times C^+$ a subset of the maximal ideal space of $M(X_{a,\alpha})$. This subset is, in fact, the maximal ideal space of the ideal of $M(X_{a,\alpha})$ consisting of absolutely continuous measures. Using this fact together with the maximum modulus theorem one can easily show that the product of the imaginary axes in $C^+ \times C^+$ is contained in the Shilov boundary of $M(X_{a,\alpha})$. We state this as a lemma.

LEMMA 3.6. *The product of the imaginary axes in $C^+ \times C^+$ is contained in the Shilov boundary of $M(X_{a,\alpha})$.*

Note that that the Gelfand transform of $\nu \in M(X_{a,\alpha})$, when restricted to the product of the imaginary axes in $C^+ \times C^+$, agrees with the ordinary Fourier transform $\hat{\nu}$. Thus, if η is a positive measure concentrated on the product of the imaginary axes in $C^+ \times C^+$ we have $\int \hat{\nu} d\eta = \int \hat{\eta} d\nu$ where $\hat{\eta}$ is the positive definite function obtained from η through the inverse Fourier transform. With this in mind we prove the following lemma.

LEMMA 3.7. *If $a > 0$ and $0 < \alpha \leqq \pi/4$ then the complex homomorphisms of $M(X_{a,\alpha})$ obtained by integrating the functions $(x, y) \rightarrow e^{-|x|}$ and $(x, y) \rightarrow e^{-|y|}$ are in the same part.*

Proof. We will let $f(x, y)$, $g(x, y)$ and $h(x, y)$ be as in 3.5. We also assume that $a \geqq 1$. [If $a \geqq 1$, redefine $T(x)$ so that the support of $T(x)$ lies in $(-a, a)$.] We choose $b > \max(1, \cot(\alpha))$ so that $h(x, y)$ agrees with $(x, y) \rightarrow e^{-|x|}$ on $X_{a,\alpha}$. Finally, we choose k large enough so that all of the functions of 3.5 are positive definite.

Let ν_1 be the positive measure on R^2 such that the inverse Fourier transform of ν_1 if $h(x, y)$. Furthermore, identify R^2 with the product of the imaginary axes in $C^+ \times C^+$. By 3.6, this gives a positive regular Borel measure on the Shilov boundary of $M(X_{a,\alpha})$. Since $h(x, y)$ agrees with $e^{-|x|}$ on $M(X_{a,\alpha})$ and since $h(0, 0) = 1$, this gives a representing measure for $e^{-|x|}$.

If we interchange x and y in the above argument then we obtain a representing measure ν_2 for $e^{-|y|}$. Each of ν_1 and ν_2 has an absolutely continuous part with an analytic Radon-Nikodym derivative (due to the term $g(x, y)$ in $h(x, y)$). Hence, ν_1 and ν_2 are not mutually singular. Thus $e^{-|x|}$ and $e^{-|y|}$ are in the same part relative to $M(X_{a,\alpha})$.

PROPOSITION 3.8. *Let $f, g \in \hat{S}$ satisfy $f(s) = g(s)$ whenever either $|f(s)| = 1$ or $|g(s)| = 1$. Also, suppose that each of $\{s \in S: f(s) \neq 0, g(s) = 0\}$ and $\{s \in S: f(s) = 0, g(s) \neq 0\}$ is a set of measure zero for* *each $\nu \in M$. Then f and g are in the same part.*

Proof. Define $F: N(f, g) \rightarrow R^2$ by $F(s) = (-\ln f(s), -\ln g(s)))$. Then by 3.4, F is a continuous homomorphism of $N(f, g)$ into $X_{a,\alpha}$ for some α, $0 < \alpha \leqq \pi/4$, and $a > 0$. If $\nu \in M$ define $\tilde{\nu}$ on $X_{a,\alpha}$ by $\tilde{\nu}(E) = \nu(F^{-1}(E))$. Since $S \backslash N(f, g)$ is a prime ideal, the restriction of measures on S to $N(f, g)$ is a homomorphism. Since F is also a continuous homomorphism, it follows that the map $\nu \rightarrow \tilde{\nu}$ is a homomorphism of M into $M(X_{a,\alpha})$. By composing homomorphisms we see that $\|\hat{\tilde{\nu}}\|_\infty \leqq \|\hat{\nu}\|_\infty$ so $\nu \rightarrow \tilde{\nu}$ is spectral norm decreasing.

If it were the case that f and g are not in the same part of $\hat{S}$, then there would exist a sequence $\{\nu_n\}_{n=1}^\infty$ in M satisfying $\|\hat{\nu}_n\|_\infty \leqq 1$ for all n and $\lim_{n\rightarrow\infty} |\hat{\nu}_n(f) - \hat{\nu}_n(g)| = 2$.

Note that

$$\hat{\nu}(f) = \int f d\nu = \int e^{-|x|} d\tilde{\nu}(x, y) \qquad \text{and} \qquad \hat{\nu}(g) = \int g d\nu = \int e^{-|y|} d\tilde{\nu}(x, y)$$

since each of $\{s \in S: f(s) \neq 0, g(s) = 0\}$ and $\{s \in S: g(s) \neq 0, f(s) = 0\}$ is a set of measure zero for ν. Thus we have

$$\lim_{n\rightarrow\infty} \left| \int e^{-|x|} d\tilde{\nu}_n(x, y) - \int e^{-|y|} d\tilde{\nu}_n(x, y) \right| = \lim_{n\rightarrow\infty} |\hat{\nu}_n(f) - \hat{\nu}_n(g)| = 2 .$$

But $\|\hat{\tilde{\nu}}_n\|_\infty \leqq 1$ for all n. This says that $e^{-|x|}$ and $e^{-|y|}$ are not equivalent relative to $M(X_{a,\alpha})$, contradicting 3.7. Thus, it must be the case that f and g are in the same part.

We will now prove Theorem 3.1. Let $f, g \in \hat{S}$ satisfy $f, g \geqq 0$ and $f(x) = 1$ if and only if $g(x) = 1$. We then have that $N(f, g)$ is an open subsemigroup of S and $S \backslash N(f, g)$ is a prime ideal. Thus, the characteristic function of $N(f, g)$, χ_N, is a multiplicative Borel function on S. By Taylor's polar decomposition theorem [12], there is an $h \in \hat{S}$ such that $h^2 = h$ and $h = \chi_N$ almost everywhere with respect to each ν in M. Furthermore, $h(x) = 1$ whenever $f(x) = 1$ or $g(x) = 1$.

By 3.8, $f \cdot h$ is equivalent to $g \cdot h$. Thus, it suffices to show that $f \cdot h \sim f$ and $g \cdot h \sim g$.

Note that $Z(h) = \{x \in S: h(x) = 0\}$ is compact, so f must assume a maximum r on $Z(h)$. Clearly r cannot be one so $0 \leqq r < 1$. If $f(x) > r$ then $x \in O(h) = \{x \in S: h(x) = 1\}$ so $f(x)h(x) = f(x)$. If $h(x) > r$ then $h(x) = 1$ so $f(x)h(x) = f(x)$. Thus by 2.7, $f \cdot h \sim f$. Similarly, $g \cdot h \sim g$. Hence $f \sim g$.

4. The one point parts of $\hat{S}$. Even through Theorem 3.1 does not give a complete converse to Theorem 2.3, it is sufficiently general to allow us to characterize the one point parts of $\hat{S}$. The Choquet boundary [8] is the set of points having unique representing measures. Since points in the same part must have mutually absolutely continuous representing measures with bounded Radon-Nikodym derivatives [2], the Choquet boundary is contained in the set of one point parts of a function algebra. Thus, the characterization of the one point parts can be considered as progress toward characterizing the Choquet boundary. With this in mind we proceed.

DEFINITION 4.1. *$H = \{f \in \hat{S}: |f(x)| = 0$ or 1 for all $x \in S\}$.*

PROPOSITION 4.2. *If $\{f\} \subseteqq \hat{S}$ is a one point part, then $f \in H$.*

Proof. If f is not in H then by definition there is an $x \in S$ for which $0 < |f(x)| < 1$. Thus, there exists a z with $\operatorname{Re} z > 0$ such that $|f|^z \neq |f|$. By 2.1 and 2.2, $f = f_0|f| \sim f_0|f|^z \neq f$. Thus, the one point parts of $\hat{S}$ are contained in H.

PROPOSITION 4.3. *If $f \in H$ and $\{f\}$ is not a one point part then f is equivalent to an "analytic half plane" in $\hat{S}$. That is, f is equivalent to elements in the "analytic structure" of $\hat{S}$.*

Proof. Let $f \in H$ and $f \sim g$ where $f \neq g$. By 2.3, $f(x) = g(x)$ whenever $|f(x)| = 1$ or $|g(x)| = 1$ so $f = g_0|f|$. Furthermore, $g \neq f$ gives that g is not in H. In particular, $|g|$ is not equal to $|g|^z$ for all $\operatorname{Re} z > 0$. By 3.1 and 2.2, $|f| \sim |g|$ and $|g| \sim |g|^z$ for all $\operatorname{Re} z > 0$. Thus, $f = g_0|f| \sim g_0|g| = g \sim g_0|g|^z \neq f$ and f is equivalent to the "analytic half-plane" $z \to g_0|g|^z$.

PROPOSITION 4.4. *An element f in H is in the same part as an element g in $\hat{S}$ if and only if $f(x) = g(x)$ whenever either $|f(x)| = 1$ or $|g(x)| = 1$.*

Proof. One direction is just 2.3. To go the other way, let $f \in H$, $g \in \hat{S}$ where $f(x) = g(x)$ whenever either $|f(x)| = 1$ or $|g(x)| = 1$. Then by 3.1, $|f| \sim |g|$ so by 2.1, $f = g_0|f| \sim g_0|g| = g$.

THEOREM 4.5. *Let $f \in \hat{S}$. The following statements are equivalent:*

(a) *$\{f\}$ is a one point part;*

(b) *if $g \in S$ and $f(x) = g(x)$ whenever either $|f(x)| = 1$ or $|g(x) = 1$ then $f = g$.*

Proof. The theorem follows immediately from 4.2 and 4.4.

It is well known, [10], that if K is the kernel of S, (minimal ideal of S [6]), e the identity element in K and $\hat{K}$ the dual group of K then every f in $\hat{K}$ can be extended continuously to all of S to give an element $\tilde{f}$ of $\hat{S}$ by means of the formula $\tilde{f}(x) = f(xe)$. It is clear from 4.5 that every such element of $\hat{S}$ must form a one point part. Thus we get the following corollary.

COROLLARY 4.6. *Every* $f \in \hat{S}$ *satisfying* $|f(x)| = 1$ *for every* $x \in S$ *forms a one point part.*

Let J be an open-compact prime ideal in S, let K be the kernel of $S \backslash J$, let e_1 be the identity of K and suppose that J has an identity e_2. If $g \in \hat{K}$ then g can be extended to $\tilde{g} \in \hat{S}$ by defining $\tilde{g}(x) = 0$ for $x \in J$ and $\tilde{g}(x) = g(xe_1)$ for $x \in S \backslash J$.

If $h \in \hat{S}$ then $h(e_2) = 0$ or 1. If $h(e_2) = 1$, then h cannot be equivalent to an element derived from $\hat{K}$ in the above manner. If $h(e_2) = 0$, on the other hand, then h is identically zero on J. Furthermore, if $h(e_1) = 1$ then $h(x) = h(xe_1)$ for all $x \in S \backslash J$ so h is completely determined on $S \backslash J$ by its action on K. Thus we have the following corollary to 4.5.

COROLLARY 4.7. *If J is an open-compact prime ideal in S, if J contains an identity ρ and if K is the kernel of $S \backslash J$ then each element of $\hat{K}$ extends to an element in the set of one point part of $\hat{S}$.*

Corollary 4.7 indicates the important role that idempotents play in determining the parts structure of $\hat{S}$. We will later give an example to show that if the J of 4.7 does not have an identity then the elements of $\hat{K}$ need not extend to elements in the set of one point parts of $\hat{S}$. In the meantime, we will investigate more closely the relationship between idempotents in S and one point parts of $\hat{S}$.

DEFINITION 4.8. Let $s, t \in S$. We will say that s is dominated by t, denoted $s \leqq t$, if $f(s) \leqq f(t)$ for all $f \in \hat{S}$ satisfying $f \geqq 0$.

LEMMA 4.9. *Let* $q, s \in S$ *where* $q^2 = q$ *and* $s \leqq q$. *Then* $q \cdot s = s$.

Proof. If $qs \neq s$ then there is an $f \in \hat{S}$ such that $f(qs) \neq f(s)$ [12]. Since $f(qs) = f(q)f(s)$, this forces $f(q) \neq 1$. Hence $f(q) = 0$. In particular, $|f|(q) = 0$. Since $s \leqq q$ we have $0 \leqq |f|(s) \leqq |f|(q) = 0$ so $f(s) = 0$. This cannot be, so $qs = s$.

The following proposition shows the relationship between one-point parts in $\hat{S}$ and the existence of certain kinds of idempotents in S.

PROPOSITION 4.10. *If $f^2 = f \in \hat{S}$ and p is the idempotent in the kernel of $O(f)$, then $\{f\}$ is a one-point part if and only if whenever $s \leqq p$ and $f(s) = 0$, there exists an idempotent $q \in S$ such that $q \neq p$ and $s \leqq q \leqq p$.*

Proof. Suppose that $\{f\}$ is a one point part. Define $G = \{g \in \hat{S}: g \geqq 0, g(s) > 0\}$. Since $p \geqq s$, $g(p) = 1$ for all $g \in G$. Since $f(s) = 0$ and $\{f\}$ is a one point part, we must have $O(f) \neq O(g)$ for all $g \in G$. Furthermore, since $g(p) = 1$ for all $g \in G$, we must have $g(x) = 1$ for all $x \in O(f)$. Thus, $O(g)$ properly contains $O(f)$ for every $g \in G$.

If $g, h \in G$, then $g(s)h(s) > 0$ so $g \cdot h \in G$. In particular,

$$[O(g) \cap O(h)] \backslash O(f)$$

is not empty. It follows that $\{O(g) \backslash O(f): g \in G\}$ is a family of compact sets having the finite intersection property.

Set $A = \bigcap_{g \in G} [O(g) \backslash O(f)] \neq \phi$ and $B = \bigcap_{g \in G} O(g)$. Note that each of A and B is a compact subsemigroup of S and that $g(x) = 1$ for all $g \in G$, and $x \in B$.

Let K be the kernel of B. Since $A \subseteqq Z(f)$ and $A \subseteqq B$ we have $K \subseteqq A$. Let q be the idempotent in K. Note that $pq = q$ and $p \neq q$. Hence $p \geqq q$. Thus, we need only show that $q \geqq s$.

If $g \in \hat{S}$ with $g \geqq 0$ then either $g(s) = 0$ so $g(q) \geqq g(s)$ or else $g(s) > 0$, in which case $g \in G$ and $g(q) = 1$. Thus, $q \geqq s$.

Now to go the other way, suppose that $\{f\}$ is not a one point part so $f \sim g \in \hat{S}$ where we may choose $g \geqq 0$, $f \neq g$. Since $f = g$ whenever $f = 1$ or $g = 1$, there is an $s \in S$ such that $0 < g(s) < 1$. Clearly $f(s) = 0$. Also, clearly, $ps \leqq p$ and $0 < g(ps) < 1$. If $q = q^2 \in S$ with $ps \leqq q \leqq p$ then we have $g(s) = g(ps) \leqq g(q) \leqq g(p)$. Since $g(q) = 0$ or 1 this forces $g(q) = 1$ and $q \in O(g) = O(f)$. Since, $q \leqq p$ we have $pq = q$ by 4.9. Since p is the idempotent of the kernel of $O(f)$ and $q \in O(f)$, $pq = p$. Thus $p = q$. But by hypothesis, there is a $q \neq p$ with $q^2 = q$ and $ps \leqq q \leqq p$. Thus, $\{f\}$ must be a one point part.

The hypothesis $f^2 = f$ in 4.10 was needed since the converse of the following proposition does not hold. We will later give an example to show this.

PROPOSITION 4.11. *If $\{|f|\}$ is a one point part of $\hat{S}$ then so is $\{f\}$.*

Proof. Let $g \sim f$. Then $O(f) = O(g)$ and $f(x) = g(x)$ for all $x \in O(f)$ by 2.3. If $f \neq g$ then since $|f| \in H$, there must exist an $s \in S$ such that $0 < |g(s)| < 1$. By 3.1, $|f| \sim |g|$. But $|f| \neq |g|$ and $\{|f|\}$ is a one point part. This cannot happen so we must have $f = g$ and $\{f\}$ is a one point part.

5. Examples. In this section we give several examples to illustrate theorems or to serve as counterexamples to questions that arise naturally as a result of this work.

Let M be a convolution measure algebra. Suppose that M can be represented as an algebra of measures on a locally compact semigroup T which is not necessarily the structure semigroup S. Further suppose that integration of the semicharacters on T yields all of the complex homomorphisms of M. Then it follows from the uniqueness of the structure semigroup S that S is just the compactification of T having the property that every semicharacter on T has a unique continuous extension to S (c.f. [12]). We will make use of this fact in the following examples.

Let T_1 be the semigroup under addition given by $T_1 = \{e\} \cup \{x \in R: x \geqq 0\}$ where e is a discrete identity. Let T_2 be the semigroup under addition, $T_2 = \{x \in R: x = 0 \text{ or } x \geqq 1\}$. Let ν_1 be the measure on T_1 consisting of the unit point mass at e plus Lebesgue measure on the nonnegative reals. Let ν_2 be the measure on T_2 consisting of the unit point mass at zero plus Lebesgue measure on the reals greater than or equal to one. Finally, let $M_1 = L'(\nu_1)$ and $M_2 = L'(\nu_2)$ under convolution.

Topologically, T_1 and T_2 are the same, and the maximal ideal space of each of M_1 and M_2 can be identified with the one point compactification of the complex numbers with nonnegative real part. That is, if $\operatorname{Re} z > 0$ define f_z by $f_z(x) = e^{-zx}(f_z(e) = 1)$. Also, on T_1 let f_∞ be the characteristic function of e and on T_2, let f_∞ be the characteristic function of zero. Then integration of each f_z gives a complex homomorphism of M_1 (or M_2), and these are all of the complex homomorphisms.

It follows immediately from 4.10 that $\{f_\infty\}$ is a one point part relative to M_1, but f_∞ is equivalent to $\{f_z: \operatorname{Re} z > 0\}$ relative to M_2. The difference is that T_1 has two idempotents, e and 0, whereas T_2 has only one idempotent, 0.

Hoffman [5] has constructed a type of function algebra having a one point part off of the Shilov boundary. Garnett used a specific one of these in [3]. One might ask whether or not function algebras obtained from convolution measure algebras can have this property. The answer is yes so our characterization of the one point part does not always give a complete characterization of the Choquet boundary.

The version of Hoffman's example used by Garnett can be obtained by completing, in the spectral norm, the set of Gelfand transforms of the convolution measure algebra which we now construct. Let $\alpha > 0$ be an irrational number and let $X = \{(m, n): m, n \text{ are integers}, m + n\alpha \geqq 0\}$ where the operation in X is coordinate-wise addition. Let $M = M(X)$ under convolution.

The semicharacter on X which corresponds to the one point part off of the Shilov boundary in Hoffman's example is the characteristic function of $(0, 0)$ which we denote by f_∞. It follows from what Hoffman did that f_∞ has all of the required properties, but we will show that $\{f_\infty\}$ is a one point part using the results of this paper.

If (a_1, a_2) and (b_1, b_2) are in X with $b_1 + \alpha b_2 > a_1 + \alpha a_2 > 0$ then $(c_1, c_2) = (b_1 - a_1, b_2 - a_2)$ is in X. If $f \in \hat{X}$ and $f = f_\infty$ then there must exist a point $(b_1, b_2) \neq (0, 0)$ in X such that $f(b_1, b_2) \neq 0$. We can clearly pick $(a_1, a_2) \in X$ such that $b_1 + \alpha b_2 > a_1 + \alpha a_2 > 0$ so we can find (c_1, c_2) as above. Note, $|f((c_1, c_2) + (a_1, a_2))| = |f(c_1, c_2)| \cdot |f(a_1, a_2)| = |f(b_1, b_2)| > 0$. Since $|f| \leqq 1$ we must have either $|f(c_1, c_2)| \geqq |f(b_1. b_2)|^{1/2}$ or $|f(a_1, a_2)| \geqq |f(b_1, b_2)|^{1/2}$. By proceeding in this manner and noting that S is a compactification of X, we see that $|f(s)| = 1$ for some $s \in S$ where $s \neq (0, 0)$. Thus, f is not equivalent to f_∞ relative to M. Hence $\{f_\infty\}$ is a one point part.

We would now like to show that the converse to 4.11 is not true. Set $X_1 = \{(z, x) \in C^2 \colon |z| = 1, x \geqq 0\}$ and let $(z_1, x_1) \circ (z_2, x_2) = (z_1 z_2, x_1 + x_2)$ in X_1. Set $X_2 = \{x \colon x \geqq 1\}$ and let $x \circ y = x + y$ in X_2. Finally set $X = X_1 \cup X_2$ where for $(z, x) \in X_1$ and $y \in X_2 (z, x) \circ y = y$, (the operations within X_1 and X_2 remain the same). Then X is a topological semigroup.

Let ν be the measure on X which is obtained as follows: on X_1, ν is Haar measure on the circle group cross Haar measure on the real line restricted to the nonnegative reals, plus the unit point mass at $(0, 0)$; on X_2, ν is Haar measure on the real line restricted to the reals greater than or equal to one. Our algebra M is $L'(\nu)$.

If f is a semicharacter on X then f is either identically zero on X_2 or else it is identically one on X_1. Any semicharacter on X which is absolute value one on X_1, but not identically equal to one on X_1, forms a one point part. However, the characteristic function on X_1 is equivalent to the function which is one on X_1 and e^{-x} on X_2.

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