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THE SCHWARZIAN DERIVATIVE AND MULTIVALENCE

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A generalization of the Schwarzian derivative and a sufficient condition for disconjugacy of the nth-order differential equation with analytic coefficients are obtained. These results are then used to establish a multivalence criterion for a certain family of analytic functions.

Let y_1 and y_2 be linearly independent solutions of the differential equation

$$y'' + p(z)y = 0 \tag{1.1}$$

and let

$$w = \frac{y_2}{y_1}\,. \tag{1.2}$$

Then, by a classical formula,

$$p = \frac{1}{2}\{w, z\} \tag{1.3}$$

where $\{w, z\}$ is the Schwarzian derivative of w, i.e.,

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 .$$

Conversely, the general solution w of (1.3) is of the form (1.2).

Utilizing the above relations, Nehari [5] proved that for an analytic function f to be univalent in the unit disk $D = \{z: |z| < 1\}$ it is necessary that

$$|\{f, z\}| \leqq \frac{6}{(1 - |z|^2)^2}\,, \qquad z \in D\,,$$

and sufficient that

$$|\{f, z\}| \leqq \frac{2}{(1 - |z|^2)^2}\,, \qquad z \in D\,.$$

Generalizations of formula (1.3) for higher-order differential equations have recently been obtained. Vodička [9] considered the nth-order equation of the type

$$y^{(n)} + p(z)y = 0 \tag{1.4}$$

and derived a relation between the coefficient p and the function $w =$

y_2/y_1, where y_1 and y_2 are any two linearly independent solutions of (1.4). In a recent paper, Lavie [4] established relations between the coefficients of the differential equation

$$y^{(n)} + p_{n-1}(z)y^{(n-1)} + \cdots + p_0(z)y = 0 \tag{1.5}$$

and the function $w = y_2/y_1$, where y_1 and y_2 are certain linearly independent solutions of (1.5).

In § 2 we shall consider the nth-order differential equation (1.5) and derive relations in which each coefficient p_i is expressed as a function of the ratios y_i/y_n, $i = 1, 2, \cdots, n-1$, where $y_1, y_2, \cdots, y_n$ are linearly independent solutions of (1.5).

In § 3, using the relations derived in § 2, we establish a sufficient condition for p-valence of a p-parameter family of analytic functions.

2. In this section we shall obtain some invariants which play a role in the study of differential equation

$$y^{(n)} + p_{n-2}(z)y^{(n-2)} + \cdots + p_0(z)y = 0 \tag{2.1}$$

which is analogous to that played by (1.3) in the study of (1.1). We remark that there is no loss of generality in considering (2.1) because any homogeneous nth-order linear differential equation can be put into the form (2.1) by a standard transformation.

Let y_i, $i = 1, 2, \cdots, n$, be linearly independent solutions of (2.1) and set

$$f_1 = \frac{y_1}{y_n}, \cdots, f_{n-1} = \frac{y_{n-1}}{y_n} .$$

We seek relations of the type

$$p_i = \Phi_i(f_1, f_2, \cdots, f_{n-1}),\ i = 0, 1, \cdots, n-2 . \tag{2.2}$$

Since the left-hand side in (2.2) is independent of the particular choice of n linearly independent solutions, the right-hand side must remain invariant under the transformation

$$f_i \longrightarrow \frac{a_{i0} + a_{i1}f_1 + \cdots + a_{in-1}f_{n-1}}{b_0 + b_1f_1 + \cdots + b_{n-1}f_{n-1}},\ i = 1, 2, \cdots, n-1 ,$$

where the a's and b's are constants.

THEOREM 2.1. *Let y_i, $i = 1, 2, \cdots, n$, be linearly independent solutions of* (2.1), *let*

$$f_1 = \frac{y_1}{y_n}, \cdots, f_{n-1} = \frac{y_{n-1}}{y_n} \tag{2.3}$$

and let W_i be the determinant defined by

$$W_i = \begin{vmatrix} f_1' & f_2' & \cdots f_{n-1}' \\ & \cdots & \\ f_1^{(i-1)} & f_2^{(i-1)} & \cdots f_{n-1}^{(i-1)} \\ f_1^{(i+1)} & f_2^{(i+1)} & \cdots f_{n-1}^{(i+1)} \\ & \cdots & \\ f_1^{(n)} & f_2^{(n)} & \cdots f_{n-1}^{(n)} \end{vmatrix},$$

$i = 1, 2, \cdots, n$. *Then we have*

$$p_i = \frac{1}{W_n \sqrt[n]{W_n}} \left[\sum_{j=0}^{n-i} (-1)^{2n-j} (1 - \delta_{nj}) \binom{n-j}{n-j-i} W_{n-j} (\sqrt[n]{W_n})^{(n-j-i)} \right], \tag{2.4}$$

$i = 0, 1, \cdots, n-2$, *where $\delta_{nn} = 1$ and $\delta_{nj} = 0$ otherwise.*

Conversely, the general solution $(f_1, f_2, \cdots, f_{n-1})$ of the system (2.4) of differential equations is of the form (2.3).

Proof. It is easily confirmed that $1, f_1, \cdots, f_{n-1}$ are linearly independent solutions of the differential equation

$$y^{(n)} - \frac{W_{n-1}}{W_n} y^{(n-1)} + \cdots + (-1)^{n+1} \frac{W_1}{W_n} y' = 0$$

and that $W_{n-1} = W_n'$. Put

$$y = Y \cdot \exp\left(\frac{1}{n} \int \frac{W_{n-1}}{W_n} dz \right) = Y \cdot \sqrt[n]{W_n}\ .$$

Then the function Y satisfies the differential equation

$$Y^{(n)} + q_{n-2}(z) Y^{(n-2)} + \cdots + q_0(z) Y = 0 \tag{2.5}$$

where

$$q_i = \frac{1}{W_n \sqrt[n]{W_n}} \left[\sum_{j=0}^{n-i} (-1)^{2n-j} (1 - \delta_{nj}) \binom{n-j}{n-j-i} W_{n-j} (\sqrt[n]{W_n})^{(n-j-i)} \right],$$

$i = 0, 1, \cdots, n-2$. Furthermore, it is evident that

$$\frac{f_1}{\sqrt[n]{W_n}}, \cdots, \frac{f_{n-1}}{\sqrt[n]{W_n}}, \frac{1}{\sqrt[n]{W_n}}$$

are linearly independent solutions of (2.5).

We now assert that

$$\frac{f_1}{\sqrt[n]{W_n}} = K y_1, \cdots, \frac{f_{n-1}}{\sqrt[n]{W_n}} = K y_{n-1}, \frac{1}{\sqrt[n]{W_n}} = K y_n \tag{2.6}$$

for some constant K. But, if this assertion is true, it would imply that the differential equations (2.1) and (2.5) have the same set of linearly independent solutions $y_1, \cdots, y_n$. In other words, (2.1) and (2.5) are identical, i.e., $p_i = q_i$, $i = 0, 1, \cdots, n-2$, which proves the theorem. To prove the equalities in (2.6), it suffices to prove only the last equality. It is easily confirmed that

$$(-1)^{n-1} W_n = \frac{W}{y_n^n},$$

where W is the Wronskian of $y_1, \cdots, y_n$ (see, e.g., [7]). Since the Wronskian W is constant, we may set $K = -1/\sqrt[n]{W}$ to obtain the last equality in (2.6).

The converse is easy to prove; it follows from the fact that

$$\frac{f_1}{\sqrt[n]{W_n}}, \cdots, \frac{f_{n-1}}{\sqrt[n]{W_n}}, \frac{1}{\sqrt[n]{W_n}}$$

are linearly independent solutions of (2.1).

For the second-order equation (1.1), the formulas in (2.4) yield the familiar relation (1.3); and for the third-order equation $y''' + p_1(z)y' + p_0(z)y = 0$,

$$p_0 = \frac{-1}{3}\left[\frac{2}{9}\left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)^3 - \left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)'' \right.$$
$$\left. - \left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)\left(\frac{f_1''f_2''' - f_1'''f_2''}{f_1'f_2'' - f_1''f_2'}\right)\right],$$

$$p_1 = \frac{f_1''f_2''' - f_1'''f_2''}{f_1'f_2'' - f_1''f_2'} + \left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)' - \frac{1}{3}\left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)^2.$$

3. Let $p_0, \cdots, p_{n-2}$ in (2.1) be analytic functions which are regular in a domain D of the complex plane. The differential equation (2.1) is said to be disconjugate in D if no nontrivial solution of (2.1) has more than $n-1$ zeros (where the zeros are counted with their multiplicities) in D. We now state an elementary principle which relates disconjugacy with a certain function-theoretic aspect of (2.1), as a theorem for convenient reference.

THEOREM 3.1. *Let $y_1, y_2, \cdots, y_n$ be linearly independent solutions of* (2.1), *and let $f_i = y_i/y_n$, $i = 1, 2, \cdots, n-1$. Then the differential equation* (2.1) *is disconjugate in D if and only if every nontrivial linear combination of $f_1, f_2, \cdots, f_{n-1}$ is $(n-1)$-valent in D, i.e., it does not take on any one value more than $n-1$ times in D.*

Proof. If (2.1) is not disconjugate in D, then there exists a nontrivial solution $y = \sum_{i=1}^{n} a_i y_i$, for some constants $a_i \neq 0$, $i = 1, 2, \cdots, n$, which has more than $n - 1$ zeros in D. Without loss of generality, we may assume that none of the zeros of y_n coincide with the zeros of y. Thus, we find that $a_n + \sum_{i=1}^{n-1} a_i f_i$ has more than $n - 1$ zeros in D, i.e., the linear combination $\sum_{i=1}^{n-1} a_i f_i$ assumes the value $-a_n$ more than $n - 1$ times in D. Conversely, if some nontrivial linear combination $\sum_{i=1}^{n-1} a_i f_i$ takes on the value $-a_n$ more than $n - 1$ times in D, the nontrivial solution $y = \sum_{i=1}^{n} a_i y_i$ has more than $n - 1$ zeros in D.

Next we shall establish a sufficient condition for disconjugacy of (2.1). We first require the following lemma.

LEMMA 3.1. *Let y be analytic in a region R. If $y(a_i) = 0$, $a_i \in R$, $i = 1, 2, \cdots, n$, then*

$$y^{(k)}(z) = \sum_{j=1}^{k+1} \binom{k}{j-1} P_{n-j}^{(k+1-j)}(z) I_j(z)(a_j - z)^{-j+1} , \tag{3.1}$$

$k = 0, 1, \cdots, n - 1$, *where*

$$I_n(z) = \int_{a_n}^{z} (a_n - \zeta)^{n-1} y^{(n)}(\zeta) d\zeta ,$$

$$I_j(z) = \int_{a_j}^{z} \frac{(a_j - \zeta)^{j-1}}{(a_{j+1} - \zeta)^{j+1}} I_{j+1}(\zeta) d\zeta,\ j = 1, 2, \cdots, n - 1 ,$$

and

$$P_{n-j}(z) = \prod_{i=j+1}^{n} (a_i - z) .$$

Proof. It is easily confirmed that $y = P_{n-1} I_1$, which proves (3.1) for $k = 0$[1, 3]. The rest follows from induction on k.

We remark that the a_i's in the above lemma are not necessarily distinct; we may put $a_k = a_{k+1} = \cdots = a_{k+m-1}$ if the y has a zero of order m at a_k.

THEOREM 3.2. *Let $p_0, \cdots, p_{n-1}$ be analytic in the unit disk $D = \{z : |z| < 1\}$. If*

$$\sum_{k=1}^{n-1} \frac{(1 + |z|)^{n-k}}{(n - k)!} |p_k(z)| + \frac{(1 - |z|)(1 + |z|)^{n-1}}{n!} |p_0(z)| \leqq 1 , \tag{3.2}$$

then the differential equation

$$y^{(n)} + p_{n-1}(z) y^{(n-1)} + \cdots + p_0(z) y = 0 \tag{3.3}$$

is disconjugate in D.

Proof. Suppose that (3.3) has a nontrivial solution y with n zeros, i.e., $y(a_i) = 0$, $a_i \in D$, $i = 1, 2, \cdots, n$. Then from Lemma 3.1 we have

$$(3.4)\quad \begin{aligned} y(z) = (a_n - z) \cdots (a_2 - z)\int_{a_1}^{z} \frac{1}{(a_2 - \zeta_1)^2}\int_{a_2}^{\zeta_1} \frac{a_2 - \zeta_2}{(a_3 - \zeta_2)^3} \\ \cdots \int_{a_{n-1}}^{\zeta_{n-2}} \frac{(a_{n-1} - \zeta_{n-1})^{n-2}}{(a_n - \zeta_{n-1})^n}\int_{a_n}^{\zeta_{n-1}} (a_n - \zeta_n)^{n-1} y^{(n)}(\zeta_n) d\zeta_n \cdots d\zeta_1 . \end{aligned}$$

Let H be the convex hull of $a_1, \cdots, a_n$. Since $|y^{(n)}(z)|$ is continuous in H, it attains its maximum in H at some point $z = z_0 \in H$. Taking the absolute values in (3.4) and performing the n-fold integration along the straight line segments connecting a_k and ζ_{k-1}, we arrive at

$$(3.5)\quad \begin{aligned} |y(z)| &\leqq \frac{1}{n!} |y^{(n)}(z_0)| \prod_{i=1}^{n} |a_i - z| \\ &< \frac{1}{n!} |y^{(n)}(z_0)| (1 + |z|)^n, \ z \in H . \end{aligned}$$

Similarly,

$$(3.6)\quad |y^{(k)}(z)| < \frac{(1 + |z|)^{n-k}}{(n - k)!} |y^{(n)}(z_0)|, \ z \in H ,$$

$k = 1, 2, \cdots, n - 1$. It is easily confirmed that

$$|I_j| \leqq \frac{(j - 1)!}{n!} |y^{(n)}(z_0)| |a_j - z|^j ,$$

and that $P_{n-j}^{(k+1-j)}(z)$ is the sum of $(n - j)!/(n - k - 1)!$ terms of the form $\prod_{l=1}^{n-k-1} (a_{il} - z)$. Therefore, we obtain from (3.1)

$$\begin{aligned} |y^{(k)}(z)| &< |y^{(n)}(z_0)| \sum_{j=1}^{k+1} \binom{k}{j - 1} \frac{(n - j)!}{(n - k - 1)!} \frac{(j - 1)!}{n!} (1 + |z|)^{n-k} \\ &= \frac{(1 + |z|)^{n-k}}{(n - k)!} |y^{(n)}(z_0)|, \ z \in H , \end{aligned}$$

which proves (3.6).

We remark that the second inequality in (3.5) may be improved; by a result of Schwarz [8],

$$\prod_{i=1}^{n} |a_i - z| < (1 - |z|)(1 + |z|)^{n-1}, \ z \in H ,$$

and therefore

$$(3.7)\quad |y(z)| < \frac{1}{n!}(1 - |z|)(1 + |z|)^{n-1} |y^{(n)}(z_0)|, \ z \in H .$$

Finally, we deduce from (3.3), (3.6), and (3.7) that

$$|y^{(n)}(z)| < |y^{(n)}(z_0)| \Big[\sum_{k=1}^{n-1} \frac{(1+|z|)^{n-k}}{(n-k)!} |p_k(z)| + \frac{1}{n!}(1-|z|)(1+|z|)^{n-1}|p_0(z)|\Big], \ z \in H ,$$

which, for $z = z_0 \in H$, yields

$$1 < \sum_{k=1}^{n-1} \frac{(1+|z_0|)^{n-k}}{(n-k)!} |p_k(z_0)| + \frac{1}{n!}(1-|z_0|)(1+|z_0|)^{n-1}|p_0(z_0)| ,$$

contrary to (3.2). This contradiction proves the theorem.

We add two remarks. A slight modification of the above proof will establish the following statements: Let R be a convex region with diameter δ. If

$$\sum_{k=0}^{n-1} \frac{\delta^{n-k}}{(n-k)!} |p_k(z)| \leqq 1, \ z \in R ,$$

then (3.3) is disconjugate in R. Theorem 3.2 generalizes a result recently obtained by Hadass [2, Th. 2].

There are known to the author a few other disconjugacy criteria for higher-order equations with analytic coefficients [4, 6].

We are now ready to state the disconjugacy condition (Theorem 3.2) as a multivalence criterion. From Theorems 2.1 and 3.1 we see that every nontrivial linear combination of $f_1, f_2, \cdots, f_{n-1}$ is $(n-1)$-valent if the equation

$$y^{(n)} + p_{n-2}(z)y^{(n-2)} + \cdots + p_0(z)y = 0 ,$$

where $p_0, \cdots, p_{n-2}$ are defined as in (2.4), is disconjugate. In view of this relation and Theorem 3.2, we have the following theorem.

THEOREM 3.3. *Let $f_1, f_2, \cdots, f_{n-1}$ be analytic in the unit disk $D = \{z: |z| < 1\}$. Define $p_0, p_1, \cdots, p_{n-2}$ as in (2.4). If* $\det (f_j^{(i)})_{i,j=1}^{n-1}$ *does not vanish in D, and if*

$$\sum_{k=1}^{n-2} \frac{(1+|z|)^{n-k}}{(n-k)!} |p_k(z)| + \frac{1}{n!}(1-|z|)(1+|z|)^{n-1}|p_0(z)| \leqq 1, \ z \in D ,$$

then every nontrivial linear combination of $f_1, f_2, \cdots, f_{n-1}$ is $(n-1)$-valent in D.

REFERENCES

1. G. A. Bessmertnyh and A. Yu. Levin, *Some inequalities satisfied by differentiable functions of one variable*, Dokl. Akad. Nauk SSSR **144** (1962), 471-474. (Translation: Soviet Math. Dokl. **3** (1962), 737-740).
2. R. Hadass, *On the zeros of the solutions of the differential equation* $y^{(n)}(z)+p(z)y(z)=0$, (to appear in the Pacific J. Math.)
3. W. J. Kim, *On a theorem of Pokornyi*, (to appear in the Proc. Amer. Math. Soc.)
4. M. Lavie, *The Schwarzian derivative and disconjugacy of n^{th}-order linear differential equations*, Canad. J. Math. **21** (1969), 235-249.
5. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545-551.
6. ———, *On the zeros of solutions of n-th order linear differential equations* J. London Math. Soc. **39** (1964), 327-332.
7. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. II, Springer-Verlag, Belrin, 1964, p. 113.
8. B. Schwarz, *On the product of the distance of a point from the vertices of a polytope*, Israel J. Math. **3** (1965), 29-38.
9. V. Vodička, *Verallgemeinerung einer Schwarzschen Differentialgleichung*, Monatsh. Math. **67** (1963), 137-141.

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