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A RATE OF GROWTH CRITERION FOR UNIVERSALITY OF REGRESSIVE ISOLS

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Let α be an infinite retraceable set having the property that if a_n is the retraceable function ranging over α, then for each partial recursive function $p(x)$, there is a number m such that $p(a_n) < a_{n+1}$ whenever $n \geqq m$ and $p(a_n)$ is defined. Recently, T. G. McLaughlin proved the existence of retraceable sets having this property and also of such sets having recursively enumerable complements. In addition, he showed that sets of this kind will be immune and that each of their regressive subsets will be retraceable. The main result of this paper states that (infinite) regressive isols that contain a retraceable set with this property will be universal. As corollary to this result we obtain the existence of cosimple universal regressive isols.

We will assume that the reader is familiar with the terminology and main results of the papers listed in the references. We let E denote the collection of all nonnegative integers, Λ the collection of all isols, and Λ_R the collection of all regressive isols. If $f: E \rightarrow E$ is a recursive and combinatorial function, then we let C_f denote its canonical extension to Λ. If $\alpha \subseteqq E$, then we say that α is *cofinite* if the complement of α is a finite set, i.e., if there is a number m such that $n \geqq m \Rightarrow n \in \alpha$. If f is a partial function (from a subset of E into E) then we denote the domain and range of f by δf and ρf, respectively. If f is a partial function and x and y any numbers, then we write "$f(x) < y$" to mean either that $f(x)$ is undefined or else $f(x)$ is defined and $f(x) < y$; we interpret "$f(x) \leqq y$" in a similar manner. We recall from [7] that an infinite isol A is *universal* if for each pair of recursive combinatorial functions f and g, one has

$$C_f(A) = C_g(A) \Longrightarrow \{n \mid f(n) = g(n)\} \text{ is cofinite .}$$

2. T-regressive isols. We call a retraceable function a_n *T-retraceable* if it has the property that for each partial recursive function $p(x)$, there is a number m such that

$$n \geqq m \Longrightarrow p(a_n) < a_{n+1} \text{ .}$$

We call an infinite retraceable set *T-retraceable* if it is the range of a T-retraceable function. A useful result of T. G. McLaughlin, [8], is

(1) cosimple T-retraceable sets exist. McLaughlin also observed that

(2) T-retraceable sets are immune.
We call an infinite regressive isol *T-regressive* if it contains a T-retraceable set. By (1) and (2) it follows that both T-regressive isols and cosimple T-regressive isols exist. We let Λ_{TR} denote the collection of all T-regressive isols. Let α be a T-retraceable set and δ any finite set. Then it can be easily shown that the set $\alpha \cup \delta$ is also T-retraceable. It follows from this property that

$$(3) \qquad T \in \Lambda_{TR} \text{ and } n \in E \Longrightarrow T + n \in \Lambda_{TR} .$$

REMARK. We wish to give next an example of a T-retraceable set. In a proof not yet published, T. G. McLaughlin used movable markers to obtain the existence of a cosimple T-retraceable set. Our proof here will be a little easier because we do not require that the T-retraceable set that we construct be cosimple.

Let $\{p_i(x)\}$ be an enumeration of all partial recursive functions of one variable such that each partial recursive function appears exactly once in the enumeration. Let the function u_n be defined by

$$u_n = \sum_{0 \leqq i, x \leqq n} p_i(x) ,$$

where we set $p_i(x) = 0$ if $p_i(x)$ is undefined. By [3, Lemma 2], there is a retraceable function t_n^* such that

$$t_n^* > u_n , \qquad \text{for each } n \in E .$$

Let $t_n = 2t_n^*$. Then t_n is also a retraceable function, and ranges over a set of even numbers. In addition, for each partial recursive function $p_e(x)$ (e denoting the index in the enumeration), we see that

$$(4) \qquad n \geqq e \Longrightarrow p_e(n) \leqq u_n < t_n .$$

Let the function a_n be defined by

$$\begin{cases} a_0 = t_1 = t(1) , \\ a_n = t^{n+1}(2n+1) , \qquad \text{for } n \geqq 1 . \end{cases}$$

Because the retraceable function t_n assumes only even values, it is readily seen that a_n is a retraceable function. Also, for each partial recursive function $p_e(x)$,

$$n \geqq e \Longrightarrow t^{n+1}(2n+1) \geqq e ,$$

and therefore by (4) we have, for $n \geqq e$

$$\begin{aligned} p_e(a_n) &= p_e(t^{n+1}(2n+1)) \\ &< t^{n+2}(2n+1) \\ &< t^{n+2}(2n+3) = a_{n+1} . \end{aligned}$$

Hence a_n is a T-retraceable function, and its range will be a T-retraceable set.

3. The main result. The main result that we wish to prove is that T-regressive isols are universal. For this purpose we will need two lemmas, each of which involves a relation $T \leqq^* u_n$ between infinite regressive isols T and functions u_n; the relation was introduced in [3] and we now recall how it is defined. If T is an infinite regressive isol and u_n any function from E into E, then $T \leqq^* u_n$ if there is a regressive function t_n that ranges over a set in T such that $t_n \leqq^* u_n$; here $t_n \leqq^* u_n$ means that the mapping $t_n \rightarrow u_n$ has a partial recursive extension. It can be shown that if $T \leqq^* u_n$, then $t_n \leqq^* u_n$ for every regressive function t_n that ranges over a set in T [3]. Also if T is any infinite regressive isol and u_n any recursive function, then $T \leqq^* u_n$. The first lemma we will state without proof because it can be readily obtained from results in [3].

LEMMA 1. *Let T be an infinite regressive isol and let u_n be any function such that $T \leqq^* u_n$. Let $u_n \geqq 1$, for each number $n \in E$. Then $\sum_T u_n \in \Lambda_R$ and if t_n is any regressive function that ranges over a set in T, then*

$$j(t_0, 0), \cdots, j(t_0, u_0 - 1), j(t_1, 0), \cdots, j(t_1, u_1 - 1), \cdots ,$$

represents a regressive enumeration of a set belonging to $\sum_T u_n$.

LEMMA 2. *Let T be a T-regressive isol and let u_n and $\tilde{u}_n$ be any functions such that both $T \leqq^* u_n$ and $T \leqq^* \tilde{u}_n$. Let $u_n \geqq 1$ and $\tilde{u}_n \geqq 1$, for each number $n \in E$. Let*

$$\sum_T u_n = \sum_T \tilde{u}_n \ . \tag{5}$$

Then the set $\{n \mid u_n = \tilde{u}_n\}$ is cofinite.

Proof. Let t_n be a T-retraceable function that ranges over a set in T, and let $\tau = \rho t_n$. By Lemma 1,

$$j(t_0, 0), \cdots, j(t_0, u_0 - 1), j(t_1, 0), \cdots, j(t_1, u_1 - 1), \cdots ,$$
$$j(t_0, 0), \cdots, j(t_0, \tilde{u}_0 - 1), j(t_1, 0), \cdots, j(t_1, \tilde{u}_1 - 1), \cdots ,$$

represent regressive enumerations of sets belonging to $\sum_T u_n$ and $\sum_T \tilde{u}_n$ respectively. Let the regressive functions determined by these two enumerations be given by g_n and $\tilde{g}_n$ respectively. In light of (5) we see that g_n and $\tilde{g}_n$ will be regressive functions that range over sets in the same isol, and therefore by results in [5] it follows that $g_n \simeq \tilde{g}_n$, i.e., there is a one-to-one partial recursive function $p(x)$ such that

(6) $$\rho g \subset \delta p \text{ and } (\forall n)[p(g_n) = \tilde{g}_n] .$$

In addition, because $T \leqq^* u_n$ and $T \leqq^* \tilde{u}_n$ there will also be partial recursive functions f_1 and f_2 such that

$$\tau \subset \delta f_1 \text{ and } (\forall n)[f_1(t_n) = u_n] ,$$
$$\tau \subset \delta f_2 \text{ and } (\forall n)[f_2(t_n) = \tilde{u}_n] .$$

Define the four functions,

$$p_1(x) = kpj(x, 0) ,$$
$$p_2(x) = kpj(x, f_1(x) \dot{-} 1) ,$$
$$q_1(x) = kp^{-1}j(x, 0) ,$$
$$q_2(x) = kp^{-1}j(x, f_2(x) \dot{-} 1) ,$$

where k denotes the familiar recursive function having the property that $kj(x, y) = x$. Then each of these functions is partial recursive and will map τ into τ in the following way:

if $pj(t_n, 0) = j(t_k, y)$ then $p_1(t_n) = t_k$,
if $pj(t_n, u_n - 1) = j(t_k, y)$ then $p_2(t_n) = t_k$,
if $p^{-1}j(t_n, 0) = j(t_k, y)$ then $q_1(t_n) = t_k$, and
if $p^{-1}j(t_n, \tilde{u}_n - 1) = j(t_k, y)$ then $q_2(t_n) = t_k$.

For each number $n \in E$, let

$$p_1(t_n) = t_{n'} , \qquad p_2(t_n) = t_{n''} ,$$
$$q_1(t_n) = t_{n^*} , \qquad q_2(t_n) = t_{n^{**}} .$$

Because t_n is a T-retraceable function and $p_1(x)$ is a partial recursive function, there will exist a number m_1 such that

$$n \geqq m_1 \Longrightarrow p_1(t_n) < t_{n+1} .$$

Combining this with the property that t_n is a retraceable function and hence strictly increasing, we see that

$$n \geqq m_1 \Longrightarrow p_1(t_n) < t_{n+1}$$
$$\Longrightarrow t_{n'} < t_{n+1}$$
$$\Longrightarrow n' \leqq n .$$

Therefore,

(A) $$n \geqq m_1 \Longrightarrow n' \leqq n .$$

In a similar fashion it can be shown that there are numbers m_2, k_1 and k_2 such that

(B) $$n \geqq m_2 \Longrightarrow n'' \leqq n \ ,$$

(C) $$n \geqq k_1 \Longrightarrow n^* \leqq n \ ,$$

(D) $$n \geqq k_2 \Longrightarrow n^{**} \leqq n \ .$$

Let

$$\bar{m} = \max \{m_1, m_2, k_1, k_2\} \ ,$$

and let m be a number chosen such that

(7) $$\begin{cases} m \geqq \bar{m} \text{ and} \\ (\forall n)[n \geqq m \Longrightarrow n' \geqq \bar{m} \text{ and } n^* \geqq \bar{m}] \ . \end{cases}$$

To complete the proof, we now verify that

(8) $$n \geqq m \Longrightarrow u_n = \tilde{u}_n \ .$$

In view of the definition of the functions g_n and $\tilde{g}_n$ and the relation (6), we see that to verify (8), it suffices to prove

(*) $$n \geqq m \Longrightarrow pj(t_n, 0) = j(t_n, 0) \ ;$$

and this will be our approach here. To prove the relation (*), assume that $n \geqq m$ and let

(9) $$pj(t_n, 0) = j(t_r, x) \ .$$

Then we wish to verify

(a) $r = n$ and

(b) $x = 0$.

For (a). We first note that $r = n'$ and therefore by (7) and (A), we have

(10) $$r \leqq n \ .$$

If $x = 0$, then $r^* = n$. In this event we have by (7) and $n \geqq m$ that $n' = r \geqq k_1$ so by (C) it follows that $r^* \leqq r$, and hence also that $n \leqq r$. Combining this with (10), we see that if $x = 0$ then $r = n$, and we are done.

Assume now that $x > 0$; then $0 < x \leqq \tilde{u}_r - 1$. Consider the diagram,

$$\begin{array}{ccc} j(t_n, 0) & & j(t_s, y) \\ p \big\downarrow & & \big\uparrow p^{-1} \\ j(t_r, x) & \text{———} & j(t_r, \tilde{u}_r - 1) \ . \end{array}$$

We note first that $n \leqq s$ and $s = r^{**}$. Also from (7) and $n \geqq m$ we have $n' = r \geqq k_2$. By (D) it follows then that $s = r^{**} \leqq r$. Hence

$n \leqq r$; and combining this relation with (10) implies $r = n$. This completes the proof of part (a).

For (b). By part (a), we know that

$$pj(t_n, 0) = j(t_n, x) , \tag{11}$$

where $0 \leqq x \leqq \tilde{u}_n - 1$. We wish to show here that $x = 0$. It can be proven by an argument similar to that in part (a), that one will also have

$$p^{-1}j(t_n, 0) = j(t_n, y) , \tag{12}$$

for some $y, 0 \leqq y \leqq u_n - 1$; and we will omit the details. We will therefore have the following diagram,

$$\begin{array}{ccc} j(t_n, y) & & j(t_n, 0) \\ {\scriptstyle p^{-1}}\uparrow & & \downarrow{\scriptstyle p} \\ j(t_n, 0) & \text{———} & j(t_n, x) \end{array}$$

and this array will only be possible if $x = y = 0$. This verifies part (b) and completes the proof.

COROLLARY 1. *Let T be a T-regressive isol and let u_n and $\tilde{u}_n$ be any functions such that both $T \leqq^* u_n$ and $T \leqq^* \tilde{u}_n$. Then*

$$\sum_T u_n = \sum_T \tilde{u}_n \Longrightarrow \{n \mid u_n = \tilde{u}_n\} \textit{ is cofinite} .$$

Proof. Let the symbol 1 denote the recursive function identically equal to 1. Then $T \leqq^* 1$, and by [3, Lemma 3] both $T \leqq^* (u_n + 1)$ and $T \leqq^* (\tilde{u}_n + 1)$. Consider the following implications:

$$\begin{aligned} \sum_T u_n = \sum_T \tilde{u}_n &\Longrightarrow \sum_T u_n + \sum_T 1 = \sum_T \tilde{u}_n + \sum_T 1 \\ &\Longrightarrow \sum_T (u_n + 1) = \sum_T (\tilde{u}_n + 1) \\ &\Longrightarrow \{n \mid u_n + 1 = \tilde{u}_n + 1\} \text{ is cofinite} \\ &\Longrightarrow \{n \mid u_n = \tilde{u}_n\} \text{ is cofinite} . \end{aligned}$$

The first implication is clear, the second follows from results in [3], the third from Lemma 2 and the last one is clear. Together they imply the desired result and this completes the proof.

THEOREM 1. *Let T be a T-regressive isol. Then T is universal.*

Proof. Let f and g be any recursive combinatorial functions. We wish to show that

(13) $$C_f(T) = C_g(T) \Longrightarrow \{x \mid f(x) = g(x)\} \text{ is cofinite} .$$

Let the functions e_n and $\tilde{e}_n$ be defined by

(14) $$\begin{cases} e_0 = f(0) , \\ e_n = f(n) - f(n-1) , & \text{for } n \geqq 1 , \end{cases}$$

(15) $$\begin{cases} \tilde{e}_0 = g(0) , \\ \tilde{e}_n = g(n) - g(n-1) , & \text{for } n \geqq 1 . \end{cases}$$

Clearly e_n and $\tilde{e}_n$ are recursive functions, since combinatorial functions are also increasing. In addition to this, e_n and $\tilde{e}_n$ will be the *e-difference* functions associated with the functions f and g respectively, [see 1]. Hence by [1, Corollary 2] we see that

$$C_f(T) = \sum_{T+1} e_n , \quad \text{and}$$
$$C_g(T) = \sum_{T+1} \tilde{e}_n .$$

To verify (13) assume that $C_f(T) = C_g(T)$. Then

(16) $$\sum_{T+1} e_n = \sum_{T+1} \tilde{e}_n .$$

Because T is a T-regressive isol, it follows from (3) that $T+1$ will also be a T-regressive isol. Also both $T + 1 \leqq^* e_n$ and $T + 1 \leqq^* \tilde{e}_n$, since e_n and $\tilde{e}_n$ are each recursive functions. In light of Corollary 1, it follows from (16) that there is a number $m \in E$, such that

(17) $$n \geqq m \Longrightarrow e_n = \tilde{e}_n .$$

If $m = 0$, then it is easy to see from (14) and (15) that $f(n) = g(n)$ for each number $n \in E$; and the desired result follows. Let us assume now that $m \geqq 1$. Let

$$U = \sum_{(T+1-m)} e_{m+n}$$
$$= \sum_{(T+1-m)} \tilde{e}_{m+n} .$$

Then, from (16) we have

$$e_0 + \cdots + e_{m-1} + U = \tilde{e}_0 + \cdots + \tilde{e}_{m-1} + U ;$$

and hence also

(18) $$e_0 + \cdots + e_{m-1} = \tilde{e}_0 + \cdots + \tilde{e}_{m-1} .$$

In view of (14) and (15), it follows from (18) that $f(m-1) = g(m-1)$. Finally, combining this fact with (14), (15) and (17) we see that

$$n \geqq m - 1 \Longrightarrow f(n) = g(n) .$$

Therefore the set $\{n \mid f(n) = g(n)\}$ is cofinite, and this completes the proof of the theorem.

THEOREM 2. *There exist cosimple universal regressive isols.*

Proof. Use Theorem 1 and the fact that cosimple T-regressive isols exist.

4. Concluding remarks. (A) The existence of universal regressive isols was first proved by E. Ellentuck in some notes not yet published. Also, in some unpublished notes, J. Barback showed that multiple-free regressive isols exist and that these are also universal.

(B) We have also proved the following result, stated here without proof, of which Theorem 1 is a corollary:

Let T be a T-regressive isol, and let f and g be any recursive combinatorial functions. Then

$$C_f(T) \leqq C_g(T) \Longrightarrow \{x \mid f(x) \leqq g(x)\} \text{ is cofinite .}$$

(C) We wish to state without proofs some additional properties of the collection Λ_{TR} of all T-regressive isols. We will assume that the reader is familiar with the three relations $\leqq$, $\leqq$ and $*$ defined between infinite regressive isols; the first two are defined in [6], and the third in [2].

THEOREM A. *Let* $A \in \Lambda_R - E$ *and* $T \in \Lambda_{TR}$. *Then*
(a) T *is multiple-free,*
(b) $A \leqq T \Longrightarrow A \in \Lambda_{TR}$,
(c) $A \leqq T \Longrightarrow A \in \Lambda_{TR}$.

THEOREM B. *Let* $A, B, T \in \Lambda_{TR}$. *Then*
(a) $A * B \Longrightarrow \min(A, B) \in \Lambda_{TR}$,
(b) $A + B \in \Lambda_R \Longrightarrow \min(A, B) \in \Lambda_{TR}$,
(c) $A, B \leqq T \Longrightarrow \min(A, B) \in \Lambda_{TR}$.

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