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A STUDY OF ABSOLUTE EXTENSOR SPACES

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In the enclosed paper, we will prove, among others, the following results: (a) A sufficient condition that a space L be an absolute extensor for the class of stratifiable spaces in that L be hyperconnected (this is a refinement of the concept of equiconnected space). This condition is also necessary if L is a metrizable space. (b) Every hypogeodesic space is hyperconnected. (c) Every equiconnected space is ∞-hyperconnected. (d) Every ∞-hyperconnected space is an absolute extensor for the class of CW-complexes of Whitehead.

Some of the merits of our results are the following: (1) The well-known extension theorem of Dugundji (Theorem 4.1 of [2]) as well as part of Theorem 4.3 of [1] are immediate consequences of our Theorem 4.1. (2) It is easily verified that the space Y of Theorem 3.4 of [3] (of course, we need to interchange the roles of the first and second variables of F in order to agree with Dugundji's definition of λ-stable), is locally hypogeodesic and, therefore, Theorem 3.4 of [3] is an easy consequence of our Theorems 3.3 and 4.4. (3) Our results generalize Theorems 3 and 4 of Himmelberg [4] and Theorem 3.4 of Dugundji [3] by removing the stringent hypothesis that the range space be metrizable.

Our many attempts to solve the question "Is every equiconnected metrizable space an AE (metrizable)[1]", which is raised in [3], have, so far, ended in failure. However, our Theorem 4.3 offers a partial solution which leads us to conjecture an affirmative answer to this question, especially in view of the "replacement-by-polytopes" technique of Dugundji [2] (If only we could do it!).

2. Definitions. Throughout, let P_{n-1} denote the unit simplex in Euclidean n-space R^n (i.e., $P_{n-1} = \{t \in R^n \mid \sum_{i=1}^n t_i = 1 \text{ and each } t_i \geqq 0\}$, I the closed unit interval, and A^n the n-fold cartesian product of any set A. Furthermore, let $\delta_i: A^n \to A^{n-1}$ be the map defined by

$$\delta_i(a_1, \cdots, a_n) = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$$

for $i = 1, 2, \cdots, n$.

It seems appropriate to start our definitions by recollecting the concept of an "equiconnected space", which was introduced by Fox [4] and is called a UC-space by Serre [9, p. 490], not only because

[1] We use the following abbreviations: AE $\equiv$ absolute extensor, ANE $\equiv$ absolute neighborhood extensor, AR $\equiv$ absolute retract, ANR $\equiv$ absolute neighborhood retract.

it is closely related to the new concepts we will introduce but also because it will make them more plausible. It should also be observed that the similarity of our Definitions 2.2 and 2.3 with Definitions 1.1 and 5.1 of Michael [8] is, by far, not accidental.

DEFINITION 2.1. A space L is *equiconnected* if there exists a continuous map $F: L \times L \times I \to L$ such that $F(a, b, 0) = a$, $F(a, b, 1) = b$ and $F(a, a, t) = a$ for all $(a, b) \in L \times L$ and $t \in I$. (The function F will be called an equiconnecting function.) The space L is said to be *locally equiconnected* if F is defined only on $U \times I$ with U some neighborhood of the diagonal of $L \times L$.

DEFINITION 2.2. A space L will be called respectively hyperconnected, m-hyperconnected, if there exists functions $h_i: L^i \times P_{i-1} \to L$, for $i = 1, 2, \cdots$, which satisfy conditions (a), (b), (c) and conditions (a), (b), (d) respectively:

(a) $t \in P_{n-1}$ and $t_i = 0$ implies $h_n(x, t) = h_{n-1}(\delta_i x, \delta_i t)$ for each $x \in L^n$ and $n = 2, 3, \cdots$,

(b) for each $x \in L^n$, the map $t \to h_n(x, t)$, from P_{n-1} to L, is continuous,

(c) for each $x \in L$ and neighborhood U of x, there exists a neighborhood V of x such that

$$\bigcup_{i=1}^{\infty} h_i(V^i \times P_{i-1}) \subset U$$

and $V \subset U$,

(d) for each $x \in L$ and neighborhood U of x, there exists a neighborhood V of x such that

$$\bigcup_{i=1}^{m} h_i(V^i \times P_{i-1}) \subset U$$

and $V \subset U$ (we should observe that, in this case, the functions h_k, for $k \geqq m + 1$, may be assumed to be constant functions). The space L will be called *∞-hyperconnected* provided that L is m-hyperconnected for $m = 1, 2, \cdots$. The space L will be called *locally hyperconnected* provided that, for each $x \in X$, there exists a neighborhood of x which is hyperconnected. We similarly define *locally m-hyperconnected* and *locally ∞-hyperconnected.*

DEFINITION 2.3. A space L is said to be *hypogeodesic* if there exists a function $F: L \times L \times I \to L$ satisfying the following conditions:

(a) for $x, y \in L$ and $t \in I$, $F(x, y, 0) = x$, $F(x, y, 1) = y$ and $F(x, x, t) = x$,

(b) for each $y \in L$, the map $(x, t) \to F(x, y, t)$, from $L \times I$ to L, is continuous,

(c) for each $x, a, y \in L$ and neighborhood U of y, there exist neighborhoods V of a and W of y (V and W depend on x) with $W \subset U$, such that

$$F(a, x, t) \in W \quad \text{implies} \quad F(b, x, t) \in U$$

for any $b \in V$,

(d) for each $x \in L$ there exist neighborhood bases $\{V_\alpha\}$ and $\{U_\alpha\}$ of x, with $V_\alpha \subset U_\alpha$ for each α, such that

$$y \in V_\alpha, z \in U_\alpha \quad \text{implies} \quad F(z, y, t) \in U_\alpha$$

for each $t \in I$.

(The function F will be called an hypogeodesic function for L.)
The space L is said to be locally hypogeodesic if F is defined only on $U \times I$ with U some neighborhood of the diagonal on $L \times L$.

Clearly, every locally convex linear topological space is hypogeodesic, and every linear topological space is equiconnected. Furthermore, every equiconnecting function is easily seen to satisfy conditions (a), (b), and (c) of Definition 2.3.

3. Hypo, equi and hyper.

THEOREM 3.1. *If L is hypogeodesic then L is hyperconnected.*

Proof. (Similar to the proof of Proposition 5.3 of Michael [8].) Throughout this proof we will make use of the following notation:

(1) If $x \in L^{n+1}$, then $\hat{x} \in L^n$ is defined by $\hat{x}_i = x_i$ for $i = 1, \cdots, n$.

(2) If $t \in P_n$ and $t_{n+1} \neq 1$ then $\hat{t} \in P_{n-1}$ is defined by

$$\hat{t}_i = \frac{t_i}{1 - t_{n+1}} \quad \text{for} \quad i = 1, \cdots, n .$$

Clearly $\delta_i\hat{x} = \widehat{(\delta_i x)}$ and $\delta_i\hat{t} = \widehat{(\delta_i t)}$, whenever $t \in P_{n-1}$, $x \in L^n$ and $1 < i < n$.

Let $h_1: L \times \{1\} \to L$ be defined by $h_1(x, 1) = x$, and let F be an hypogeodesic function for L. By induction, assume we have defined maps $h_i: L^i \times P_{i-1} \to L$, for $i = 1, \cdots, n$, which satisfy parts (a) and (b) of Definition 2.2. Now let

$$(3) \qquad h_{n+1}(x, t) = \begin{cases} x_{n+1} & \text{if } t_{n+1} = 1 \\ F(h_n(\hat{x}, \hat{t}), x_{n+1}, t_{n+1}) & \text{if } t_{n+1} \neq 1 . \end{cases}$$

Let us check that the functions h_n satisfy conditions (a), (b) and (c) of Definition 2.2.

2.2(a). Clearly $h_2(x, t) = h_1(\delta_i x, \delta_i t)$ whenever $t_i = 0$. Hence let us assume that $h_n(x, t) = h_{n-1}(\delta_i x, \delta_i t)$ whenever $t_i = 0$ and let us shown that $h_{n+1}(x, t) = h_n(\delta_i x, \delta_i t)$ whenever $t_i = 0\,(1 \leqq i \leqq n+1)$: If $i = n+1$ then $h_{n+1}(x, t) = F(h_n(\hat{x}, \delta_{n+1} t), x_{n+1}, 0) = h_n(\hat{x}, \delta_{n+1} \mid t \mid) = h_n(\delta_{n+1} x, \delta_{n+1} t)$ by (3) and Definition 2.3(a). If $i < n+1$ and $t_{n+1} = 1$, then $h_{n+1}(x, t) = x_{n+1} = h_n(\delta_i x, \delta_i t)$, by (3). If $i < n+1$ and $t_{n+1} \neq 1$, then

$$\begin{aligned} h_{n+1}(x, t) &= F(h_n(\hat{x}, \hat{t}), x_{n+1}, t_{n+1}) \\ &= F(h_{n-1}(\delta_i \hat{x}, \delta_i \hat{t}), x_{n+1}, t_{n+1}) \\ &= F(h_{n-1}(\widehat{(\delta_i x)}, \widehat{(\delta_i t)}), x_{n+1}, t_{n+1}) = h_n(\delta_i x, \delta_i t) . \end{aligned}$$

By induction, we get that the functions h_n satisfy Definition 2.2(a).

2.2(b). Proof by induction. Clearly the map $t \to h_1(x, t) = x$, from P_0 to L, is continuous. Suppose that the map $t \to h_n(w, t)$, from P_{n-1} to X, is continuous, for each $w \in L^n$. Pick a fixed point $x \in L^{n+1}$ and let us show that the map $t \to h_{n+1}(x, t)$, from P_n to L, is also continuous: Let $s \in P_n$. Then

Case 1. $s_{n+1} \neq 1$. For some neighborhood U of s in P_n, $t \in U$ implies $t_{n+1} \neq 1$. Therefore, for $t \in U$, $h_{n+1}(x, t) = F(h_n(\hat{x}, \hat{t}), x_{n+1}, t_{n+1})$ and hence the continuity of the map $t \to h_{n+1}(x, t)$ at $s \in P_n$, follows from Definition 2.3(b) and the inductive hypothesis. (Indeed, pick sequence $t(1), t(2), \cdots$ in $U \subset P_n$ such that $\lim_i t(i) = s$. Then one easily sees that $\lim_i \widehat{t(i)} = \hat{s}$ and thus $\lim_i h_n(\hat{x}, \widehat{t(i)}) = h_n(\hat{x}, \hat{s})$ by the inductive hypothesis. Consequently, $\lim_i h_{n+1}(x, (t(i)) = \lim_i F(h_n(\hat{x}, \widehat{t(i)}), x_{n+1}, (t(i)_{n+1}) = F(h_n(\hat{x}, \hat{s}), x_{n+1}, s_{n+1}) = h_{n+1}(x, s)$ which shows that the map $t \to h_{n+1}(x, t)$ is continuous at s.)

Case 2. $s_{n+1} = 1$. Then $h_{n+1}(x, s) = x_{n+1}$, by (3). We will show that $h_{n+1}(x, t)$ is close to x_{n+1} if t_{n+1} is close to 1, as follows: Let $A = \{h_n(\hat{x}, t) \mid t \in P_{n-1}\}$ and, for each $\tau \in I$, let $f_\tau\colon A \to X$ be defined by

$$f_\tau(a) = F(a, x_{n+1}, \tau) .$$

Then

$$\begin{aligned} &h_{n+1}(x, t) = f_{t_{n+1}}(a_t) \qquad \text{for some } a_t \in A \text{ if } t_{n+1} \neq t , \\ &h_{n+1}(x, t) = x_{n+1} \qquad \text{if } t_{n+1} = 1 . \end{aligned}$$

Therefore, in order to prove that the map $t \to h_{n+1}(x, t)$ is continuous at s, it suffices to show that $f_\tau \to f_1$ uniformly as $\tau \to 1$ (note that $f_1(a) = x_{n+1}$ for each $a \in A$, by Definition 2.3(a)): From Definition 2.3(b) we get that $f_\tau \to f_1$ pointwise. Moreover A is compact, because h_n is continuous and $\{x\} \times P_n$ is compact, and the family $\{f_\tau \mid \tau \in I\}$ is evenly continuous (see page 235 of [7]), by Definition 2.3(c). (Let $a \in A \subset L$ and $y \in L$ and let U be a neighborhood of y. By Definition 2.3(c), pick neighborhoods V of a and W of y (V and W depend on x_{n+1}) with

$W \subset U$ such that whenever $F(a, x_{n+1}, t) \in W$ then $F(b, x_{n+1}, t) \in U$ for all $b \in V$. Then we obviously get that $f_\tau(V) \subset U$ whenever $f_\tau(a) \in W$, which shows that the family $\{f_\tau \mid \tau \in I\}$ is evenly continuous.) By Theorem 23 (page 237) of [7], the family $\{f_\tau \mid \tau \in I\}$ is equicontinuous (note that, for each $a \in A$, $\{f_\tau(a) \mid \tau \in I\} \subset F(A \times \{x_{n+1}\} \times I)$ a compact subset of L, because of Definition 2.3(b)). Therefore, the topologies of pointwise convergence and uniform convergence coincide on $\{f_\tau \mid \tau \in I\}$, by Theorem 15 (page 232) of [7], and hence $f_\tau \to f_1$ uniformly as $\tau \to 1$.

It is now easily seen that the map $t \to h_{n+1}(x, t)$ is continuous at each $s \in P_n$.

2.2(c). Let $x \in L$ and open $U \subset L$ such that $x \in U$. Then, by Definition 2.3(d), there exists neighborhoods V and W of x such that $V \subset W \subset U$ and $F(z, y, t) \in W$ whenever $y \in V$ and $z \in W$. By an inductive argument, one easily sees that

$$\bigcup_{n=1}^{\infty} h_n(V^n \times P_{n-1}) \subset W \subset U ,$$

which completes the proof.

THEOREM 3.2. *If L is equiconnected then L is ∞-hyperconnected.*

Proof. Exactly the same as the proof of Theorem 3.1, except that we must verify that the functions h_n, $n = 1, 2, \cdots$, satisfy condition (d) of Definition 2.2: Let us first observe that, for each neighborhood U of x one easily finds a neighborhood V of x with $V \subset U$,

$$F(V \times V \times I) \subset U ,$$

because F is continuous, $F(x, x, t) = x$ for each $t \in I$ and I is compact. Clearly L is 1-hyperconnected. Therefore let us assume that L is n-hyperconnected and show that L is $(n + 1)$-hyperconnected. Pick $x \in L$ and neighborhood U of x. Then pick neighborhoods V and W of x such that

$$F(V \times V \times I) \subset U,\ h_n(W^n \times P_{n-1}) \subset V,\ W \subset V \subset U .$$

Using the definition of h_{n+1}, it is trivial to check that

$$h_{n+1}(W^{n+1} \times P_n) \subset U .$$

This concludes our inductive argument and completes the proof.

Because of the proofs of Theorems 3.1 and 3.2, the following result is clearly valid and easily verified:

THEOREM 3.3. *If L is locally hypogeodesic (locally equiconnected) then L is locally hyperconnected (locally ∞-hyperconnected).*

4. Extension theorems.

THEOREM 4.1. *Every hyperconnected space L is an* AE *(stratifiable).*

Proof. The proof of this result is very similar to the proof of Theorem 4.3 in [1]. We will thus indicate the general procedure without details. We briefly comment on some *recurring notation throughout ensuing* proof: Letting $U \to \{U_n\}_{n=1}^{\infty}$ be a stratification of X (see footnote 1), we let $n(U, x) = \min\{n \mid x \in U_n\}$ and $U_x = U_{n(U,x)} - (X - \{x\})^{-}_{n(U,x)}$, for each open $U \subset X$ and $x \in U$. It is easily seen (see Lemma 4.1 of [1]) that each U_x is an open neighborhood of x, $U_x \cap V_y \neq \emptyset$ implies that $x \in V$ or $y \in U$ (indeed, $n(U, x) \leqq n(V, y)$ implies that $y \in U$).

Now let X be a stratifiable space, A a closed subset of X, $f: A \to L$ a continuous function. Let $W = X - A$, $W' = \{x \in W \mid x \in U_y$ for some $y \in A$ and open U containing $y\}$ and $m(x) = \max\{n(U, y) \mid y \in A$ and $x \in U_y\}$, for each $x \in W'$. It is easily seen that $m(x) < n(W, x) < \infty$.

Using the paracompactness of W, let $\mathscr{V}$ be an open locally finite (with respect to W) refinement of $\{W_x \mid x \in W\}$. For each $V \in \mathscr{V}$ pick $x_V \in W$ with $V \subset W_{x_V}$. If $x_V \in W'$ pick $a_V \in A$ and open S_V containing a_V such that $x_V \in (S_V)_{a_V}$ and $n(S_V, a_V) = m(x_V)$; if $x_V \in W$, let a_V be the fixed point $a_0 \in A$.

Let $\{p_V \mid V \in \mathscr{V}\}$ be a partition of unity subordinated to $\mathscr{V}$, and define $g: X \to L$ by

$$g(x) = f(x) \qquad \text{for } x \in A$$

$$g(x) = h_n((f(a_{V_1}), \cdots, f(a_{V_n})), (p_{V_1}(x), \cdots, p_{V_n}(x)) \qquad \text{for } x \in W,$$

where $V_1, \cdots, V_n$ are the only elements $V \in \mathscr{V}$ such that $p_V(x) \neq 0$, for $x \in X - A$. Clearly g is a well defined function from X to L. It is not quite obvious that g is continuous anywhere, as was the case in Theorem 4.3 of [1].

We will first show that g is continuous at each point $b \in A$. Let 0 be any open subset of L containing $f(b)$ and let H be an open subset of 0 such that $f(b) \in H$ and $\bigcup_{n=1}^{\infty} h_n(H^n \times P_{n-1}) \subset 0$. Since f is continuous there exists an open neighborhood N of b such that $f(A \cap N) \subset H \subset 0$. It is easily seen that $g((N_b)_b) \subset 0$, which shows that g is continuous at each $b \in A$.

Finally we show that g is continuous at each $x \in X \to A$. Let 0 be a neighborhood of

$$g(x) = h_m((f(a_{V_1}), \cdots, f(a_{V_m})), (p_{V_1}(x), \cdots, p_{V_m}(x)))$$

and let N be a neighborhood of x which intersects only finitely many $V \in \mathscr{V}$; say $V_1, \cdots, V_m, \cdots, V_{m+k}$. By Definition 2.2(b), there exists a neighborhood W of $(p_{V_1}(x), \cdots, p_{V_m}(x), 0, \cdots, 0) \in P_{m+k-1}$ which is mapped into 0 by the continuous map

$$t \longrightarrow h_{m+k}((f(a_{V_1}), \cdots, f(a_{V_{m+n}}), t) .$$

Define a map $p: N \rightarrow P_{m+k-1} \subset I^{m+k}$ by $p(y) = (p_{V_1}(y), \cdots, p_{V_{m+k}}(y))$ for each $y \in N$. Since p is clearly continuous, then there exists a neighborhood U of x such that $p(U) \subset W$. It is now easily seen that $g(N \cap U) \subset 0$, with the help of Definition 2.2(a), which shows that g is continuous at each $x \in X - A$, thus completing the proof.

It is easily seen that the preceding proof remains valid if we assume that L is only $(n + 1)$-hyperconnected and $X - A$ is n-dimensional (in the covering sense), for then we can choose the open cover $\mathscr{V}$ to be of order n and thus define the function g in terms of f and h_{n+1} only. We have, therefore, proved the following result.

THEOREM 4.2. *Let X be a stratifiable space, A a closed subset of X, L an $(n + 1)$-hyperconnected space and $f: A \rightarrow L$ a continuous function. If* $\dim (X - A) \leqq n$ *then f has a continuous extension g: $X \rightarrow L$.*

The following result, when combined with Theorem 3.2, provides a partial answer to the following question which is raised in [3]: Is every equiconnected matrizable space an AE (metrizable)?

THEOREM 4.3. *Every ∞-hyperconnected space L is an* AE *(CW-complex of Whitehead).*

Proof. Let K be a CW-complex, A a closed subset of K and f: $A \rightarrow L$ a continuous function. For each n, let K_n be the n-skeleton of K. It is well-known that $\dim K_n$ is finite for each n. Therefore, by theorem 4.2 (note that K is stratifiable because of Theorem 7.2 of [1], for example) and induction, we can find continuous functions

$$g_n: A \cup K_n \rightarrow L \qquad \text{for } n = 1, 2, \cdots$$

such that

$$g_n \mid A = f \text{ and } g_{n+1} \mid (A \cup K_n) = g_n$$

for each n. Now define $g: K \rightarrow L$ by

$$g(x) = g_n(x) \qquad \text{if } x \in A \cup K_n .$$

It is easily seen that g is a well-defined continuous extension of f to all of X, which completes the proof.

It is easily seen that all the preceding results of this section have local analogues. For the sake of completeness, let us state and prove one.

THEOREM 4.4. *Every locally hyperconnected space L is an* ANE *(stratifiable).*

Proof. By Theorem 4.1, L is a local AE (stratifiable) (i.e., for each $a \in L$ there exists a neighborhood N of a such that N is an AE (stratifiable)). Consequently, by Theorem 19.2 of Hanner [5], L is an ANE (stratifiable).

5. Characterization of AR (metrizable) spaces.

THEOREM 5.1. *A metrizable space M is an* AE *(stratifiable) if and only if M is hyperconnected.*

Proof. Since the "if" part is an immediate consequence of Theorem 4.1, we direct our attention to the "only if" part (the basic technique is extracted from Michael [8]): Embed M in a Banach space B and let H be the closed convex hull of M in B. Then there exists a retraction $r: H \to M$. For each n, define $h_n: M^n \times P_n \to M$ by

$$h_n(z, t) = r(\sum_{i=1}^{n} t_i z_i), \qquad \text{for } (z, t) \in M^n \times P_n .$$

Clearly, each h_n satisfies conditions (a) and (b) of Definition 2.2. Since r is continuous, for each $x \in M$ and open subset U of H with $x \in U$, there exists a convex open set $V \subset H$ with $x \in V \subset U$ such that

$$r(V \cap M) \subset r(V) \subset U \cap M ;$$

hence $\bigcup_{i=1}^{\infty} h_i(V \cap M)^i \times P_{i-1}) \subset U \cap M$, and hence the functions h_n satisfy condition (c) of Definition 2.2. Consequently M is hyperconnected.

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