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ON MAJORANTS FOR SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS IN REGIONS OF THE COMPLEX PLANE

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In this paper, we investigate the rate of growth of functions which are analytic in an arbitrary simplyconnected region of the complex plane and which are solutions of first order algebraic differential equations (i.e., equations of the form $\Omega(z, y, dy/dz) = 0$, where Ω is a polynomial in z, y and dy/dz).

In the course of constructing an example for second order equations, Vijayaraghavan in [5] showed that for *any* realvalued increasing function $\Phi(x)$ on the interval $(0, +\infty)$, it is possible to find a complex function $h(z)$, which is analytic in a simply-connected region R of the plane containing $(0, +\infty)$, and satisfies a first order algebraic differential equation, and which has the property that $|h(x)| > \Phi(x)$ at a sequence of real x tending to $+\infty$[1]. For a given $\Phi(x)$, the function $h(z)$ constructed was of the form $P(az)$ where $P(u)$ is the Weierstrass P-function with primitive periods w and iw' (w, w' real), and where the constant a was of the form $a = w + ib$, where b depends on Φ and b/w' is irrational. Since $P(az)$ has poles at all points $(mw/a) + (niw'/a)$ where m and n are integers, clearly the region R associated with the solution $h(z) = P(az)$ depends on a and hence on $\Phi(x)$. A natural question is thus raised, namely, can such examples be constructed where, for all $\Phi(x)$, the simply-connected region R remains the same. That is, does there exist a simply-connected region R containing $(0, +\infty)$ with the property that for any real-valued increasing function $\Phi(x)$ on $(0, +\infty)$, there is a solution $h(z)$, analytic on R, of a first order algebraic differential equation, such that $|h(x)| > \Phi(x)$ at a sequence of real x tending to $+\infty$? In this paper we answer this question in the *negative* by proving the following more general result (§2 below): If R is any simply-connected region, then there exists a real-valued continuous function $\Psi(z)$ on R with the property that for any function $h(z)$, analytic on R, which satisfies a first order algebraic differential equation, there is a compact set K contained in R such that $|h(z)| < \Psi(z)$ on R-K. In the case where R is not the whole plane, we show that $\Psi(z)$ may be taken to be

[1] None of the solutions $h(z)$ constructed by Vijayaraghavan are real-valued on any interval $(x_0, +\infty)$. Of course this is in accord with the well-known result of Lindelöf [**2**; p. 213] that a *real-valued* solution on an interval $(x_0, +\infty)$ is majorized, on some interval $(x_1, +\infty)$, by $\exp(\exp x)$.

$\exp(\exp((1-|f(z)|)^{-1}))$, where f is a univalent analytic mapping of R onto the unit disk (which exists by the Riemann Mapping Theorem). In the case where R is the whole plane, a wellknown result of Valiron [3; p. 41] states that any entire function satisfying a first order algebraic differential equation is of finite order, so $\Psi(z)$ can be taken to be $\exp(\exp|z|)$ in this case.

To prove our main result (assuming R is not the whole plane), the equation is transformed to the unit disk using f^{-1}. The resulting equation no longer has polynomial coefficients, but careful estimates on the growth of the coefficients can be made using certain standard estimates on the growth of the univalent analytic function f^{-1} on the unit disk. Then by using results of Valiron [4; p. 299] concerning certain analytic functions in the unit disk, we obtain (Lemma C below) the same growth condition on solutions in the unit disk as was obtained in [4; p. 294] in the case where the coefficients were polynomials. Our main result then follows by retransforming back to R.

2. We now state our main result:

THEOREM. *Let R be a simply-connected region of the complex plane which is not the whole plane. Let f be a univalent analytic mapping of R onto the unit disk. (Such a mapping exists by the Riemann Mapping Theorem.) Then, if $h(z)$ is any analytic function on R which is a solution of a first order algebraic differential equation, then there exist a constant $A \geqq 0$ and a compact set X contained in R, such that the inequality $|h(z)| \leqq \exp((1-|f(z)|)^{-A})$ holds for all $z \in R - X$. (Thus clearly, for each such $h(z)$, there is a compact set K contained in R such that the inequality*

$$|h(z)| < \exp(\exp((1-|f(z)|)^{-1})) \text{ holds on } R-K.)$$

REMARK. If R is the whole plane, then by a result of Valiron [3; p. 41], any analytic function on R (i.e., any entire function) which is a solution of a first order algebraic differential equation is of finite order.

3. *Proof of the theorem.* Let $h(z)$ be analytic on the simply-connected region R and be a solution of a first order algebraic differential equation $\Omega(z, y, y') = 0$. We may write $\Omega(z, y, y')$ in the form,

$$\Omega(z, y, y') = \sum H_{kj}(z) y^k (y')^j \tag{1}$$

where the functions $H_{kj}(z)$ are polynomials in z. Thus,

$$\sum H_{kj}(z)(h(z))^k (h'(z))^j \equiv 0 \quad \text{on} \quad R\,. \tag{2}$$

Let g be the inverse of f. Thus g is a univalent analytic mapping of the unit disk onto R. Since $h(z)$ is analytic on R, clearly the function

$$\varphi(\zeta) = h(g(\zeta)) \quad \text{is analytic on} \quad |\zeta| < 1 . \tag{3}$$

Since $\varphi'(\zeta) = h'(g(\zeta))g'(\zeta)$, we have by (2) that $\varphi(\zeta)$ satisfies the relation,

$$\sum F_{kj}(\zeta)(\varphi(\zeta))^k(\varphi'(\zeta))^j \equiv 0 \quad \text{on} \quad |\zeta| < 1, \text{ where} \tag{4}$$

$$F_{kj}(\zeta) = H_{kj}(g(\zeta))/(g'(\zeta))^j \quad \text{for each} \quad (k, j) . \tag{5}$$

For each (k, j) such that $H_{kj} \not\equiv 0$, let $d(k, j)$ be the degree of the polynomial H_{kj}. Define

$$q = 1 + \max\{j + 2d(k, j) \colon H_{kj} \not\equiv 0\} \tag{6}$$

Thus clearly,

$$q > 0 . \tag{7}$$

We now prove,

LEMMA A. *There exists a constant $K^* > 0$ such that on any circle $|\zeta| = r$, where $r \in [0, 1)$, we have $|F_{kj}(\zeta)| \leqq K^*(1 - r)^{-q}$ for all (k, j).*

Proof. Since $g(\zeta)$ is univalent on $|\zeta| < 1$, the function $G(\zeta) = (g(\zeta) - g(0))/g'(0)$ is also univalent on $|\zeta| < 1$ and $G(0) = 0$ while $G'(0) = 1$. Thus by [1; Th. 17. 4. 7, p. 353], $|G(\zeta)| \leqq r(1 - r)^{-2}$ on any circle $|\zeta| = r < 1$. Since r < 1 and $(1 - r)^2 \leqq 1$, we clearly obtain

$$|g(\zeta)| \leqq L(1 - r)^{-2} \quad \text{on} \quad |\zeta| = r < 1 \tag{8}$$

where $L = |g'(0)| + |g(0)| > 0$ (since $g'(0) \neq 0$). Again, since $G(\zeta)$ is univalent on $|\zeta| < 1$ and $G(0) = 0$ while $G'(0) = 1$ we have by [1; Th. 17. 4. 6, p. 351] that $(1 - r)(1 + r)^{-3} \leqq |G'(\zeta)| \leqq (1 + r)(1 - r)^{-3}$ on any circle $|\zeta| = r < 1$. Setting $K_1 = 2|g'(0)|$ and $K_2 = |g'(0)|/8$, we have

$$K_1 > 0 \quad \text{and} \quad K_2 > 0 \text{ (since } g'(0) \neq 0\text{), and} \tag{9}$$

$$K_2(1 - r) \leqq |g'(\zeta)| \leqq K_1(1 - r)^{-3} \quad \text{on} \quad |\zeta| = r < 1 . \tag{10}$$

Now consider any (k, j). If $H_{kj} \equiv 0$ clearly $F_{kj} \equiv 0$ by (5), so we may assume $H_{kj} \not\equiv 0$. Since H_{kj} is of degree $d = d(k, j)$, we may write, $H_{kj}(z) = B_0 + B_1 z + \cdots + B_d z^d$ where the B_j are constants. Thus by (8), we have on any circle $|\zeta| = r < 1$,

$$(11) \qquad |H_{kj}(g(\zeta))| \leqq |B_0| + |B_1| L(1-r)^{-2} + \cdots + |B_d| L^d(1-r)^{-2d} .$$

Thus clearly there is a constant $\alpha(k, j) > 0$ such that,

$$(12) \qquad |H_{kj}(g(\zeta))| \leqq \alpha(k, j)(1-r)^{-2d} \quad \text{on} \quad |\zeta| = r < 1 .$$

By (10), we have $|g'(\zeta)|^j \geqq K_2^j(1-r)^j$ on $|\zeta| = r < 1$, so by (12) and (5),

$$(13) \qquad |F_{kj}(\zeta)| \leqq (\alpha(k, j)/K_2^j)(1-r)^{-(j+2d)} \quad \text{on} \quad |\zeta| = r < 1 .$$

Setting

$$K^* = 1 + \max \{\alpha(k, j)/K_2^j \colon H_{kj} \not\equiv 0\}$$

and noting that $q > j + 2d(k, j)$ by (6), we have by (13) that for any (k, j)

$$(14) \qquad |F_{kj}(\zeta)| \leqq K^*(1-r)^{-q} \quad \text{on any circle} \quad |\zeta| = r < 1 ,$$

proving Lemma A.

Now set

$$(15) \qquad p = \max \{k + j \colon H_{kj} \not\equiv 0\} , \quad \text{and}$$

$$(16) \qquad m = \max \{j \colon H_{p-j,j} \not\equiv 0\}.$$

We now prove,

LEMMA B. *There exist constants $K^\# > 0, \sigma \geqq 0$ and $r_0 \in [0, 1)$ such that on any circle $|\zeta| = r$, where $r_0 < r < 1$, we have $|F_{p-m,m}(\zeta)| \geqq K^\#(1-r)^\sigma$.*

Proof. If the degree d of $H_{p-m,m}$ is zero, then $H_{p-m,m}$ is a non-zero constant function, say $H_{p-m,m}(z) \equiv L_1$. Then by (5) and (10), we have $|F_{p-m,m}(\zeta)| \geqq (|L_1|/K_1^m)(1-r)^{3m}$ on $|\zeta| = r < 1$ which proves the lemma in this case.

Thus we may assume $d > 0$. Since $H_{p-m,m}(z)$ is a polynomial of degree $d \geqq 1$, we may write,

$$(17) \qquad H_{p-m,m}(z) = B(z - a_1) \cdots (z - a_d) \quad \text{where} \quad B \neq 0 ,$$

and hence

$$(18) \qquad H_{p-m,m}(g(\zeta)) = B(g(\zeta) - a_1) \cdots (g(\zeta) - a_d) \quad \text{for} \quad |\zeta| < 1 .$$

We partition the set of roots $\{a_1, \cdots, a_d\}$ of $H_{p-m,m}(z)$ into three classes and arrange them as follows: $a_1, \cdots, a_b$ lie in the complement of the closure of R; $a_{b+1}, \cdots, a_c$ lie in R and $a_{c+1}, \cdots, a_d$ lie on the boundary of R, for some b and c with $0 \leqq b \leqq c \leqq d$.

We first consider any root $a_k, 1 \leqq k \leqq b$. Since a_k lies in the complement of the closure of R, the distance ε_k from a_k to R is strictly positive. But if $|\zeta| < 1$ then $g(\zeta)$ lies in R so we have,

$$(19) \qquad |g(\zeta) - a_k| \geqq \varepsilon_k > 0 \quad \text{for} \quad |\zeta| < 1 \quad \text{and} \quad k \in \{1, \cdots, b\} .$$

We now consider any root $a_j, b + 1 \leqq j \leqq c$. Thus $a_j \in R$ so there exists t_j in the unit disk $|\zeta| < 1$ such that $g(t_j) = a_j$. Let $\delta = \max \{|t_j| : b + 1 \leqq j \leqq c\}$ and let $r_0 = (1 + \delta)/2$. Thus $\delta < r_0 < 1$. Let N be the image under g of the annulus $r_0 \leqq |\zeta| < 1$. Then no a_j (for $b + 1 \leqq j \leqq c$) can lie in the closure of N, for otherwise there would be a sequence $\{\zeta_n\}$ with $r_0 \leqq |\zeta_n| < 1$ such that $\{g(\zeta_n)\} \to a_j$. By the continuity of g^{-1} we would obtain $\{\zeta_n\} \to t_j$. Since $|\zeta_n| \geqq r_0$ for all n, we would then have $|t_j| \geqq r_0$ which contradicts $|t_j| \leqq \delta$ (since $\delta < r_0$). Thus the distance ε_j from a_j to N is strictly positive for $j = b + 1, \cdots, c$. Thus for $j \in \{b + 1, \cdots, c\}$ we have

$$(20) \qquad |g(\zeta) - a_j| \geqq \varepsilon_j > 0 \quad \text{for} \quad r_0 \leqq |\zeta| < 1 .$$

Finally, we consider any root $a_s, c + 1 \leqq s \leqq d$, and we consider the function $G_s(\zeta) = (g(\zeta) - a_s)^{-1}$ defined on $|\zeta| < 1$. Since a_s lies on the boundary of R, the equation $g(\zeta) = a_s$ has no roots in $|\zeta| < 1$ by definition of the map $g(\zeta)$. Thus $G_s(\zeta)$ is analytic on $|\zeta| < 1$ and is univalent on $|\zeta| < 1$ since $g(\zeta)$ is. Thus the function

$$\psi_s(\zeta) = (G_s(\zeta) - G_s(0))/G_s'(0)$$

is univalent and analytic on $|\zeta| < 1$ and $\psi_s(0) = 0$ while $\psi_s'(0) = 1$. Thus by [1; Th. 17. 4. 6, p. 351], we have $|\psi_s'(\zeta)| \leqq (1 + r)(1 - r)^{-3}$ on any circle $|\zeta| = r < 1$. Hence $|G_s'(\zeta)| \leqq 2 |G_s'(0)| (1 - r)^{-3}$ on $|\zeta| = r < 1$. But $G_s'(\zeta) = -g'(\zeta)(g(\zeta) - a_s)^{-2}$, so we obtain,

$$(21) \qquad |g(\zeta) - a_s|^2 \geqq (1 - r)^3 |g'(\zeta)|/2 |G_s'(0)| \quad \text{on} \quad |\zeta| = r < 1 .$$

But by (10), $|g'(\zeta)| \geqq K_2(1 - r)$ on $|\zeta| = r < 1$, where $K_2 > 0$. Thus clearly,

$$(22) \qquad |g(\zeta) - a_s|^2 \geqq (K_2/2 |G_s'(0)|)(1 - r)^4 \quad \text{on} \quad |\zeta| = r < 1 .$$

Setting $M_s = (K_2/2 |G_s'(0)|)^{1/2}$, we have $M_s > 0$ and

$$(23) \qquad |g(\zeta) - a_s| \geqq M_s(1 - r)^2 \quad \text{on} \quad |\zeta| = r < 1 \quad \text{for} \quad s \in \{c + 1, \cdots, d\} .$$

Thus in view of (18), (19), (20) and (23), we obtain

$$(24) \qquad |H_{p-m,m}(g(\zeta))| \geqq K_0(1 - r)^{2(d-c)} \quad \text{on} \quad |\zeta| = r ,$$

when

$$r_0 \leqq r < 1, \text{ where } K_0 = |B| \varepsilon_1 \cdots \varepsilon_c M_{c+1} \cdots M_d > 0 .$$

Now by (10), $|g'(\zeta)|^m \leqq K_1^m(1-r)^{-3m}$ on $|\zeta| = r < 1$, so in view of (5) and (24), we obtain

$$(25) \quad |F_{p-m,m}(\zeta)| \geqq K^{\#}(1-r)^{\sigma} \quad \text{on} \quad |\zeta| = r, \quad \text{when} \quad r_0 \leqq r < 1 \,,$$

where $K^{\#} = K_0/K_1^m > 0$ and $\sigma = 2(d-c) + 3m \geqq 0$. This proves Lemma B.

Now $\varphi(\zeta)$ satisfies the relation (4) in $|\zeta| < 1$. We introduce the notation $M(r, \varphi) = \max_{|\zeta|=r} |\varphi(\zeta)|$ for $r \in [0, 1)$. We now prove,

LEMMA C. *There exist real numbers $A \geqq 0$ and $r^* \in [0, 1)$ such that for any $r \in [r^*, 1)$, we have $M(r, \varphi) \leqq \exp((1-r)^{-A})$.*

Proof. If $M(r, \varphi)$ is bounded on $[0, 1)$, then clearly the result holds. Hence we may assume $M(r, \varphi)$ is unbounded so $M(r, \varphi) \to +\infty$ as $r \to 1$.

We proceed by contradiction and assume the lemma is false. Then in the terminology of [4; p. 294], $\varphi(\zeta)$ is of order ∞ in $|\zeta| < 1$. Let $\sum_{j=0}^{\infty} c_j \zeta^j$ be the power series expansion of $\varphi(\zeta)$. For each $r \in [0, 1)$, Let $N(r) = \max_{j \geqq 0} |c_j| r^j$ and $n(r) = \max\{j : |c_j| r^j = N(r)\}$. Then since $\varphi(\zeta)$ is of order ∞, by [4; p. 299], for any $b \in (0, 1)$, there is in $(0, 1)$ a sequence of values of r (called *remarkable*) tending to one, such that

$$(26) \qquad \log M(r, \varphi) > \gamma'(n(r))^b \quad \text{and} \quad n(r) > \gamma''(1-r)^{-\delta}$$

where $\delta = (1-b)^{-1}$ and γ', γ'' are strictly positive constants independent of r, and such that at every point of $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$ we have

$$(27) \qquad \varphi'(\zeta) = (1 + \in(\zeta))(n(r)/\zeta)\varphi(\zeta)$$

where $\varepsilon(\zeta)$ tends uniformly to zero as $r = |\zeta|$ tends to one.

Let q be as in (6), so $q > 0$ by (7). Let σ be as in Lemma B, so $\sigma \geqq 0$. Thus $q + \sigma > 0$, so $b = (q+\sigma)/(q+\sigma+1)$ belongs to $(0, 1)$. It is for this value of b that we apply the Valiron theory (26), (27). Then the corresponding $\delta = q + \sigma + 1$, so

$$(28) \qquad \delta > q + \sigma > 0 \,.$$

Let p and m be as in (15) and (16). Then clearly by (5),

$$(29) \quad p = \max\{k + j : F_{kj} \not\equiv 0\} \quad \text{and} \quad m = \max\{j : F_{p-j,j} \not\equiv 0\} \,.$$

Now let $r \in (0, 1)$ and let ζ be a point on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$. Then $\varphi(\zeta) \neq 0$ and so by dividing (4) through by

$(\varphi(\zeta))^p$ (and using (29)), we can write (4) as

$$(30)\qquad \sum_{j=0}^{m} F_{p-j,j}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^j \equiv -\sum_{k+j<p} F_{kj}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^j(\varphi(\zeta))^{k+j-p} .$$

We will denote the left side of (30) by $\Lambda(\zeta)$, and the right side by $\Gamma(\zeta)$.

We now assert that there exist real numbers $L^* > 0$ and $r^\# \in (1/2, 1)$ such that if $r \in (r^\#, 1)$ is a remarkable value, then

$$(31)\qquad |\Gamma(\zeta)| \leqq L^*(1-r)^{-q}(M(r,\varphi))^{-1/2}$$

for any point on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$.

To prove (31), we note first that for remarkable $r \to 1$, we have $n(r) \to +\infty$ by (26). Thus since $M(r, \varphi) \to +\infty$ and $\varepsilon(\zeta) \to 0$ for remarkable $r = |\zeta| \to 1$, we can choose $r^* \in (1/2, 1)$ such that for remarkable $r \in (r^*, 1)$, we have $M(r, \varphi) > 1$, $n(r) > 1$ and $|\varepsilon(\zeta)| < 1/2$ whenever $|\zeta| = r$ and $|\varphi(\zeta)| = M(r, \varphi)$. Now let $r \in (r^*, 1)$ be a remarkable value and let ζ be a point on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$. We refer to the right side of (30). If $k + j < p$, then $p - (k + j) \geqq 1$, so

$$(32)\qquad |\varphi(\zeta)|^{k+j-p} \leqq (M(r,\varphi))^{-1} .$$

Since $|\varepsilon(\zeta)| < 1/2$ and $|\zeta| = r > 1/2$, we have by (27) that

$$(33)\qquad |\varphi'(\zeta)/\varphi(\zeta)| \leqq 4n(r) .$$

By Lemma A, there is a constant $K^* > 0$ such that $|F_{kj}(\zeta)| \leqq K^*(1-r)^{-q}$ for all (k, j). Thus by (32), (33) and the definition of $\Gamma(\zeta)$, it follows that

$$(34)\qquad |\Gamma(\zeta)| \leqq K_3(1-r)^{-q}(n(r))^{\theta}(M(r,\varphi))^{-1}$$

where $K_3 > 0$ and $\theta \geqq 0$ are constants (independent of r). But by (26), $\gamma'(n(r))^b < \log M(r, \varphi)$, so by (34),

$$(35)\qquad |\Gamma(\zeta)| \leqq L^*(1-r)^{-q}(\log M(r,\varphi))^{\theta/b}(M(r,\varphi))^{-1}$$

where $L^* > 0$ is a constant (independent of r). But since $M(r, \varphi) \to +\infty$ as $r \to 1$, clearly there exists $r' \in (0, 1)$ such that if $r \in (r', 1)$, then $(\log M(r, \varphi))^{\theta/b} < (M(r, \varphi))^{1/2}$. Setting $r^\# = \max\{r^*, r'\}$, we see that (31) follows from (35).

We now consider $\Lambda(\zeta) = \sum_{j=0}^{m} F_{p-j,j}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^j$.

Case I. $m = 0$. Then $\Lambda(\zeta) = F_{p-m,m}(\zeta)$. By Lemma B there exist constants $K^\# > 0$ and $r_0 \in (r^\#, 1)$, such that on any circle $|\zeta| = r$ where $r \in (r_0, 1)$, we have

$$|\Lambda(\zeta)| = |F_{p-m,m}(\zeta)| \geqq K^{\#}(1-r)^{\sigma} . \tag{36}$$

Since $\Lambda(\zeta) = \Gamma(\zeta)$ (by (30)), we have by (31) and (36) that for all remarkable $r \in (r_0, 1)$, $K^{\#}(1-r)^{\sigma} \leqq L^{*}(1-r)^{-q}(M(r, \varphi))^{-1/2}$ and hence

$$(1-r)^{2(q+\sigma)} M(r, \varphi) \leqq (L^{*}/K^{\#})^2 . \tag{37}$$

But by (26), there exist constants $\gamma' > 0, \gamma'' > 0$, independent of r, such that $\log M(r, \varphi) > \gamma'(\gamma'')^b(1-r)^{-b\delta}$. Hence with (37), we see that for all remarkable $r \in (r_0, 1)$,

$$(1-r)^{2(q+\sigma)} \exp(\gamma(1-r)^{-b\delta}) \leqq (L^{*}/K^{\#})^2 \tag{38}$$

where $\gamma = \gamma'(\gamma'')^b > 0$. Since $b\delta = q + \sigma > 0$ (by (28)), it is clear that (38) is impossible since the left side of (38) tends to $+\infty$ as $r \to 1$. This contradiction proves Lemma C in the case $m = 0$.

Case II. $m > 0$. Since $\varepsilon(\zeta)$ (in (27)) tends to zero as $r = |\zeta| \to 1$, we can choose $r_1 \in (0, 1)$ such that for any remarkable $r \in (r_1, 1)$, we have $|\varepsilon(\zeta)| < 1/2$ for those points ζ on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$. Hence for those ζ, we have by (27),

$$|\zeta\varphi'(\zeta)/n(r)\varphi(\zeta)| \geqq 1 - |\varepsilon(\zeta)| > 1/2 .$$

Since $|\zeta| < 1$, we have $|\varphi'(\zeta)/\varphi(\zeta)| > n(r)/2$ and so by (26), we have

$$|\varphi'(\zeta)/\varphi(\zeta)| > (\gamma''/2)(1-r)^{-\delta} \quad \text{where} \quad \gamma'' > 0 . \tag{39}$$

We now assert that there exists a constant $K_0 > 0$, such that for all sufficiently large remarkable r, we have

$$|\Lambda(\zeta)| \geqq K_0(1-r)^{\sigma - m\delta} \tag{40}$$

at all points of $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$.

To prove (40), we consider the quotients,

$$Q_j(\zeta) = (F_{p-j,j}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^j)/(F_{p-m,m}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^m) ,$$

for $j = 0, 1, \cdots, m-1$. By Lemma A, there is a constant $K^{*} > 0$ such that $|F_{p-j,j}(\zeta)| \leqq K^{*}(1-r)^{-q}$ on any circle $|\zeta| = r < 1$, for all j, while by Lemma B, there is a constant $K^{\#} > 0$ such that for all sufficiently large $r \in (0, 1)$, we have

$$|F_{p-m,m}(\zeta)| \geqq K^{\#}(1-r)^{\sigma} \quad \text{on} \quad |\zeta| = r . \tag{41}$$

Thus, with (39), we see that for $j = 0, 1, \cdots, m-1$, there is a constant $L_j > 0$ such that

$$|Q_j(\zeta)| \leqq L_j(1-r)^{(m-j)\delta - (q+\sigma)} \tag{42}$$

for all sufficiently large remarkable r and all points ζ on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$. Since $(m-j)\delta \geqq \delta > q + \sigma$ (by (28)), we see that the right side of (42) tends to zero as $r \to 1$, so for all sufficiently large remarkable r, we have

$$(43) \qquad |Q_j(\zeta)| \leqq 1/(m+1) \quad \text{for} \quad j = 0, 1, \cdots, m-1,$$

at all points on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$. Now by definition of $\varLambda$, we have

$$\varLambda(\zeta) = F_{p-m,m}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^m\left(1 + \sum_{j=0}^{m-1} Q_j(\zeta)\right),$$

and so by (43), we have

$$(44) \qquad |\varLambda(\zeta)| \geqq (1/(m+1))\, |F_{p-m,m}(\zeta)(\varphi'(\zeta)/\varphi(\zeta))^m|$$

for all sufficiently large remarkable r and all points ζ on $|\zeta| = r$ at which $|\varphi(\zeta)| = M(r, \varphi)$. Now it is clear that (40) follows from (39), (41) and (44).

Since $\varGamma(\zeta) = \varLambda(\zeta)$ (by (30)), we have by (31) and (40) that for all sufficiently large remarkable r, $K_0(1-r)^{\sigma - m\delta} \leqq L^*(1-r)^{-q}(M(r, \varphi))^{-1/2}$, and so

$$(45) \qquad (1-r)^{2(q+\sigma-m\delta)} M(r, \varphi) \leqq (L^*/K_0)^2 .$$

But by (26), $\log M(r, \varphi) > \gamma'(\gamma'')^b(1-r)^{-b\delta}$ (where $\gamma', \gamma'' > 0$), and so for all sufficiently large remarkable r, we have

$$(46) \qquad (1-r)^{2(q+\sigma-m\delta)} \exp(\gamma(1-r)^{-b\delta}) \leqq (L^*/K_0)^2$$

where $\gamma = \gamma'(\gamma'')^b > 0$. Since $b\delta = q + \sigma > 0$ (by (28)), it is clear that (46) is impossible, since the left side of (46) tends to $+\infty$ as $r \to 1$. This contradiction proves Lemma C in Case II, and thus the proof of Lemma C is complete.

We now conclude the proof of the theorem. By Lemma C, there exist constants $A \geqq 0$ and $r^* \in [0, 1)$, such that if $r \in [r^*, 1)$, then

$$(47) \qquad |\varphi(\zeta)| \leqq \exp((1-r)^{-A}) \quad \text{on} \quad |\zeta| = r < 1 .$$

Let X be the image under g of the closed disk $|\zeta| \leqq r^*$. Then X is a compact set contained in R. If $z \in R - X$, then we have $r^* < |f(z)| < 1$ (since $f = g^{-1}$), and so by (47), we have

$$(48) \qquad |\varphi(f(z))| \leqq \exp((1 - |f(z)|)^{-A}) .$$

But $\varphi = h \circ g$ (by (3)) and so $h(z) = \varphi(f(z))$. Thus

$$| h(z) | \leqq \exp ((1 - | f(z) |)^{-A})$$

if $z \in R - X$, which concludes the proof of the theorem.

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