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A NOTE ON GROUPS WITH FINITE DUAL SPACES

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If a locally compact group has only a finite number of inequivalent irreducible unitary representations, then one is tempted to conjecture that it is a finite group. The conjecture is known to be true in certain special cases. We present here a proof in case the group satisfies the second axiom of countability.

PROPOSITION 1.1. *If G is an abelian locally compact group having only a finite number of inequivalent irreducible unitary representations, then G is a finite group.*

This follows immediately from the Pontrjagin duality theorem.

PROPOSITION 1.2. *If G is a compact group having only a finite number of inequivalent irreducible unitary representations, then G is a finite group.*

We may deduce a proof of this from the Peter-Weyl theorem, for example, as follows: $L^2(G)$ is the direct sum $\sum_i I_i$ of finite dimensional subspaces $[I_i]$, where, for each i, I_i is a minimal two-sided ideal in $L^2(G)$. Further, there is a one-to-one correspondence between the set $[I_i]$ of these ideals and the set of all equivalence classes of irreducible unitary representations of G. If the latter set is finite, as assumed, then $L^2(G)$ is finite dimensional, and G is necessarily a finite group.

The proof we give here for the second countable case depends on Dixmier's theory of square-integrable representations, which, in turn, depends on some rather technical results concerning Hilbert algebras. It would be desirable, of course, to discover an elementary proof to what appears to be such an elementary theorem. I have devised a fairly elementary proof—"elementary" in the sense that, beyond the notion of Haar measure, the only deep result needed is Kadison's theorem on the algebraic irreducibility of a topologically irreducible *-representation of a C^*-algebra. This proof, however, still suffers from being quite long, so I do not include it.

Regarding the situation when G is an arbitrary locally compact group, there is no direct integral theory available in general, and we therefore lose an important tool for moving from hypotheses about the dual space to conclusions, for example, about the regular representation. I can not make headway in resolving this conjecture even in the case when G is an uncountable discrete group. Of course, if the conjecture is true, then the hypothesis of a finite dual must imply that the dual space is actually discrete. Even this subsidiary implication is apparently nontrivial in general.

2. Discrete decomposition of the regular representation.

Let G be a unimodular group, and let R denote the left regular representation of G. The following theorem can be deduced from § 14 of [1].

THEOREM 2.1. *Let M be a closed subspace of $L^2(G)$ which is irreducible under R. Then the mapping $(f, \phi) \rightarrow f * \phi$ is defined on $L^2(G) \times M$ into M, and there exists a constant k_M such that $\| f * \phi \|_2 \leqq k_M \| f \|_2 \| \phi \|_2$ for every f in $L^2(G)$ and every ϕ in M.*

LEMMA 2.2. *Let M be a closed irreducible subspace of $L^2(G)$, and let k_M be a constant which satisfies the inequality in Theorem 2.1 above. Suppose N is a closed subspace of $L^2(G)$ for which $R|_N$ is equivalent to $R|_M$. If f is an element of $L^2(G)$ and ϕ is an element of N, then $\| f * \phi \|_2 \leqq k_M \| f \|_2 \| \phi \|_2$, so that k_N may be taken to equal k_M.*

Proof. Let θ be an equivalence between $R|_M$ and $R|_N$. Let ϕ be an element of N, f be an element of $L^2(G)$, and $[f_n]$ be a sequence of elements of $L^1(G) \cap L^2(G)$ which converges to f in $L^2(G)$. Then:

$$\begin{aligned}&\| f * \phi \|_2 = (\text{by Theorem 2.1}) \lim \| f_n * \phi \|_2 = \lim \| f_n * \theta(\theta^{-1}(\phi)) \|_2 \\ &= \lim \| \theta(f_n * \theta^{-1}(\phi)) \|_2 = \| f_n * \theta^{-1}(\phi) \|_2 \leqq \lim k_M \| f_n \|_2 \| \theta^{-1}(\phi) \|_2 \\ &= k_M \| f \|_2 \| \phi \|_2 .\end{aligned}$$

THEOREM 2.3. *Let G be a unimodular group. Assume that R is a direct sum of irreducible subrepresentations and that only a finite number of inequivalent irreducible representations occurs in this decomposition. Then G is a finite group.*

Proof. We prove that $L^2(G)$ is a convolution algebra, whence, by [5], G is compact, whence, by Proposition 1.2, G is finite.

Thus, decompose $L^2(G)$ as the direct sum $\sum_i M_i$, where, for each i, M_i is a closed irreducible subspace. For each i, let k_i denote a constant $k_{(M_i)}$ as guaranteed by Theorem 2.1. By Lemma 2.2 we may assume that, if $R|_{(M_i)}$ is equivalent to $R|_{(M_j)}$, then $k_i = k_j$. By the hypothesis that only a finite number of inequivalent irreducible representations occurs in R, we may conclude that the set $[k_i]$ of these constants is finite, whence uniformly bounded by a positive number k.

Now let f and g be two elements of $L^2(G)$. Denote by g_i the projection of g onto the subspace M_i. Then:

$\| f*g \|_2^2 = \| f*\sum_i g_i \|_2^2 =$ (by the orthogonality of the $[M_i]$ and by Theorem 2.1)

$$\sum_i \| f*g_i \|_2^2 \leqq \sum_i (k_i)^2 \| f \|_2^2 \| g_i \|_2^2 \leqq k^2 \| f \|_2^2 \sum_i \| g_i \|_2^2 = k^2 \| f \|_2^2 \| g \|_2^2 .$$

Hence $f*g$ is again an element of $L^2(G)$.

3. Finiteness properties in the dual space.

By the *dual space* $\hat{G}$, we shall mean the set of all equivalence classes of irreducible unitary representations of a locally compact group G.

THEOREM 3.1. *Let G be an infinite, second countable, unimodular group. Then the spectrum of the regular representation R of G is infinite, i.e., R weakly contains an infinite number of elements of G.* (For the definitions of "spectrum" and "weak containment" see [2].)

Proof. Assume that R weakly contains only a finite number of inequivalent irreducible representations of G. By the second countability of G, R is equivalent to a direct integral of irreducible representations, [4], and we may assume, by [2], that the only irreducible unitary representations which occur in this direct integral decomposition are the elements of $\hat{G}$ which R weakly contains. Having assumed that R weakly contains only a finite number of elements of $\hat{G}$, this direct integral is equivalent to a direct sum of irreducible unitary representations only finitely many of which are inequivalent. We now have the hypotheses of Theorem 2.3. This implies that G is finite, which is a contradiction.

COROLLARY 3.2. *Let G be a second countable group. Then G is finite if and only if $\hat{G}$ is finite.*

Proof. Of course "G finite $\rightarrow \hat{G}$ finite" is classical. Conversely, if $\hat{G}$ is finite, then the spectrum of the regular representation is finite, and the proof will be complete, by the theorem above, if we can show that G is unimodular.

If δ denotes the modular function of G, then $\delta(G)$ is an abelian group (a subgroup of the group of positive reals) whose dual space, being in one-to-one correspondence with a subset of the dual space of G, is finite. Hence, by Proposition 1.1, $\delta(G)$ is finite, whence δ is identically 1.

COROLLARY 3.3. *Let G be a sigma-compact group. Then G is finite if and only if $\hat{G}$ is finite.*

A proof follows from Corollary 3.2 together with the theorem of [3] which states that, if G is a sigma-compact group, then there exists a compact normal subgroup H for which G/H is second countable.

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