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PRODUCT INTEGRAL REPRESENTATION OF TIME DEPENDENT NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

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The object of this paper is to use the method of product integration to treat the time dependent evolution equation $u'(t) = A(t)(u(t))$, $t \geqq 0$, where u is a function from $[0, \infty)$ to a Banach space S and A is a function from $[0, \infty)$ to the set of mappings (possibly nonlinear) on S. The basic requirements made on A are that for each $t \geqq 0$ $A(t)$ is the infinitesimal generator of a semi-group of nonlinear nonexpansive transformations on S and a continuity condition on $A(t)$ as a function of t.

The product integration method has been used by T. Kato in [5] to treat evolution equations in which $A(t)$ is the infinitesimal generator of a semi-group of linear contraction operators. In [6] Kato treats the nonlinear evolution equation in which $A(t)$ is m-monotone and the Banach space S is uniformly convex. For other investigations of nonlinear evolution equations one should see P. Sobolevski [9], F. Browder [1], J. Neuberger [8], and J. Dorroh [3].

1. Definitions and theorems.

In this section definitions and theorems will be stated. For examples satisfying the definitions and theorems below, one should see § 4. Let S denote a real Banach space.

DEFINITION 1.1. The function T from $[0, \infty)$ to the set of mappings (possibly nonlinear) on S will be said to be a $\mathscr{C}$-semi-groups of mappings on S provided that the following are true:

(1) $T(x + y) = T(x)T(y)$ for $x, y \geqq 0$.

(2) $T(x)$ is nonexpansive for $x \geqq 0$.

(3) If $p \in S$ and $g_p(x)$ is defined as $T(x)p$ for $x \geqq 0$ then g_p is continuous and $g_p(0) = p$.

(4) The infinitesimal generator A of T is defined on a dense subset D_A of S (i.e., if $p \in D_A g_p'^{+}(0)$ exists and $Ap = g_p'^{+}(0)$) and if $p \in D_A g_p'^{+}(x) = Ag_p(x)$ for $x \geqq 0$, $g_p(x) = p + \int_0^x Ag_p(u)du$ for $x \geqq 0$, $g_p'^{+}$ is continuous from the right on $[0, \infty)$, and $\| g_p'^{+} \|$ is nonincreasing on $[0, \infty)$.

DEFINITION 1.2. The mapping A from a subset of S to S will be said to be a $\mathscr{C}$-mapping on S provided that the following are true:

(1) The domain D_A of A is dense in S.

(2) A is monotone on S, i.e., if $\varepsilon > 0$ and

$$p, q \in D_A \, \| (I - \varepsilon A)p - (I - \varepsilon A)q \| \geqq \| p - q \| .$$

(3) A is m-monotone on S, i.e. A is monotone on S and if $\varepsilon > 0$ then Range $(I - \varepsilon A) = S$.

(4) A is the infinitesimal generator of a $\mathscr{C}$-semi-group of mappings on S.

DEFINITION 1.3. Let each of m and n be a nonnegative integer and for each integer i in $[m, n]$ let K_i be a mapping from S to S. If $m > n$ define $\prod_{i=m}^{n} K_i = I$. If $m \leqq n$ define $\prod_{i=m}^{m} K_i = K_m$ and if $m + 1 \leqq j \leqq n$ define $\prod_{i=m}^{j} K_i = K_j \prod_{i=m}^{j-1} K_i$. Define $\prod_{n}^{i=m} K_i = \prod_{i=m}^{n} K_{n+m-i}$. If each of a and b is a nonnegative number then a chain $\{s_i\}_{i=0}^{2m}$ from a to b is a nondecreasing or nonincreasing number-sequence such that $s_0 = a$ and $s_{2m} = b$. The norm of $\{s_i\}_{i=0}^{2m}$ is $\max\{| s_{2i} - s_{2i-2} | \mid i \in [1, m]\}$.

DEFINITION 1.4. Let F be a function from $[0, \infty) \times [0, \infty)$ to the set of mappings on S. Suppose that $p \in S$, $a, b \geqq 0$, and u is a point in S such that if $\varepsilon > 0$ there exists a chain $\{s_i\}_{i=0}^{2m}$ from a to b such that if $\{t_i\}_{i=0}^{2n}$ is a refinement of $\{s_i\}_{i=0}^{2m}$ then

$$\left\| u - \prod_{i=1}^{n} F(t_{2i-1}, | t_{2i} - t_{2i-2} |)p \right\| < \varepsilon .$$

Then u is said to be the product integral of F from a to b with respect to p and is denoted by $\prod_a^b F(I, dI)p$.

REMARK 1.1. Let A be a $\mathscr{C}$-mapping on S and define the function F from $[0, \infty) \times [0, \infty)$ to the set of mappings on S by $F(u, v) = (I - vA)^{-1}$ for $u, v \geqq 0$ (Note that $(I - vA)^{-1}$ exists and has domain S by virtue of the m-monotonicity of A). The following result in [10] will be used in the theorems below:

If A is a $\mathscr{C}$-mapping on S, T is the $\mathscr{C}$-semi-group generated by A, and F is defined as above, then for $p \in S$ and $x \geqq 0$ $T(x)p = \prod_0^x F(I, dI)p$.

In this case let $T(x)$ be denoted by $\exp(xA)$ for $x \geqq 0$.

Let A be a function from $[0, \infty)$ to the set of mappings on S such that the following are true:

(I) For each $t \geqq 0$ $A(t)$ is a $\mathscr{C}$-mapping on S

(II) There is a dense subset D of S such that if $t \geqq 0$ the domain of $A(t)$ is D

(III) A is continuous in the following sense: If $a, b \geqq 0$, M is a bounded subset of D, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta$ then $\| A(u)z - A(v)z \| < \varepsilon$ for each $z \in M$.

THEOREM 1. *Let* A *satisfy conditions* (I), (II) *and* (III). *If* $p \in S$ *and* $a, b \geqq 0$ *the following are true*:

(1) *If* $T(u, v) = \exp(vA(u))$ *for* $u, v \geqq 0$, *then* $\prod_a^b T(I, dI)p$ *exists.*

(2) *If* $L(u, v) = (I - vA(u))^{-1}$ *for* $u, v \geqq 0$, *then* $\prod_a^b L(I, dI)p$ *exists and* $\prod_a^b L(I, dI)p = \prod_a^b T(L, dI)p$.

THEOREM 2. *Let* A *satisfy conditions* (I), (II) *and* (III) *and define* $U(b, a)p = \prod_a^b T(I, dI)p$ *for* $p \in S$ *and* $a, b \geqq 0$. *The following are true*:

(1) $U(b, a)$ *is nonexpansive for* $a, b \geqq 0$.

(2) $U(b, c)U(c, a) = U(b, a)$ *for* $a, b \geqq 0$ *and* $c \in [a, b]$ *and* $U(a, a) = I$ *for* $a \geqq 0$.

(3) *If* $p \in S$ *and* $a \geqq 0$ *then* $U(a, t)p$ *is continuous in* t

(4) *If* $p \in S, 0 \leqq a \leqq t$, *and* $U(t, a)p \in D$, *then* $\partial^+ U(t, a)p/\partial t = A(t)U(t, a)p$ *and if* $p \in S, 0 < s \leqq b$, *and* $U(s, b)p \in D$, *then*

$$\partial^- U(s, b)p/\partial s = -A(s)U(s, b)p \; .$$

2. Product integral representations. In this section, Theorems 1 and 2 will be proved. Before proving part (1) of Theorem 1 three lemmas will be proved each under the hypothesis of Theorem 1.

LEMMA 1.1. *If* $p \in D, a, b \geqq 0$, *and* $\{s_i\}_{i=0}^{2m}$ *is a chain from* a *to* b *then*

$$\left\|\prod_{i=1}^{m} T(s_{2i-1}, |s_{2i} - s_{2i-2}|)p - p\right\| \leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \, \|A(s_{2i-1})p\| \; .$$

Proof.

$$\begin{aligned}
&\left\|\prod_{i=1}^{m} T(s_{2i-1}, |s_{2i} - s_{2i-2}|)p - p\right\| \\
&\leqq \sum_{i=1}^{m}\left\|\prod_{j=i}^{m} T(s_{2j-1}, |s_{2j} - s_{2j-2}|)p - \prod_{j=i+1}^{m} T(_{2j-1}, |s_{2j} - s_{2j-2}|)p\right\| \\
&\leqq \sum_{i=1}^{m} \| T(s_{2i-1}, |s_{2i} - s_{2i-2}|)p - p \| \\
&= \sum_{i=1}^{m}\left\|\int_0^{|s_{2i}-s_{2i-2}|} A(s_{2i-1})T(s_{2i-1}, t)pdt\right\| \\
&\leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \cdot \| A(s_{2i-1})p \| \; .
\end{aligned}$$

LEMMA 1.2. *If* $p \in D, a, b \geqq 0, \{s_i\}_{i=0}^{2m}$ *is a chain from* a *to* b, *and* $\{s_i'\}_{i=1}^{m}$ *is a sequence in* $[a, b]$, *then*

$$\left\|\prod_{i=1}^{m} L(s_i', |s_{2i} - s_{2i-2}|)p - p\right\| \leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \, \|A(s_i')p\| \; .$$

Proof.

$$\left\|\prod_{i=1}^{m} L(s'_i, |s_{2i}-s_{2i-2}|)p - p\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|\prod_{j=i}^{m} L(s'_j, |s_{2j}-s_{2j-2}|)p - \prod_{j=i+1}^{m} L(s'_j, |s_{2j}-s_{2j-2}|)p\right\|$$
$$\leqq \sum_{i=1}^{m} \| L(s'_i, |s_{2i}-s_{2i-2}|)p - p \|$$
$$= \sum_{i=1}^{m} \| L(s'_i, |s_{2i}-s_{2i-2}|)p$$
$$- L(s'_i, |s_{2i}-s_{2i-2}|)(I - |s_{2i}-s_{2i-2}| A(s'_i))p \|$$
$$\leqq \sum_{i=1}^{m} |s_{2i}-s_{2i-2}| \cdot \| A(s'_i)p \| .$$

LEMMA 1.3. *If M is a bounded subset of D, $a, b \geqq 0$, $\gamma > 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$, $|u - v| < \delta$, $0 \leqq x < \gamma$, and $z \in M$, then $\| T(u, x)z - T(v, x)z \| \leqq x \cdot \varepsilon$.*

Proof. Let $M' = \{\prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \mid z \in M, v \in [a, b], 0 \leqq x < \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from 0 to $x\}$. Let $z_0 \in M$, let $z \in M$, let $v \in [a, b]$, let $0 \leqq x < \gamma$, and let $\{s_i\}_{i=0}^{2m}$ be a chain from 0 to x. Then,

$$\left\|\prod_{i=1}^{m} L(v, s_{2i}-s_{2i-2})z - \prod_{i=1}^{m} L(v, s_{2i}-s_{2i-2})z_0\right\| \leqq \| z - z_0 \| .$$

Further, by Lemma 1.2,

$$\left\|\prod_{i=1}^{m} L(v, s_{2i}-s_{2i-i})z_0 - z_0\right\| \leqq x \cdot \max_{u \in [0,x]} \| A(u)z_0 \| .$$

Then, $\| \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \| \leqq \| z - z_0 \| + \| z_0 \| + x \cdot \max_{u \in [0,\gamma]} \| A(u)z_0 \|$ and so M' is bounded. There exists $\delta > 0$ such that if $u, v \in [a, b]$, $|u - v| < \delta$, and $z \in M'$, then $\| A(u)z - A(v)z \| < \varepsilon$. Then if $0 \leqq x < \gamma$, $z \in M$, $\{s_i\}_{i=0}^{2m}$ is a chain from 0 to x, $u, v \in [a, b]$, and $|u - v| < \delta$,

$$\left\|\prod_{i=1}^{m} L(u, s_{2i}-s_{2i-2})z - \prod_{i=1}^{m} L(v, s_{2i}-s_{2i-2})z\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|\prod_{j=i}^{m} L(u, s_{2j}-s_{2j-2}) \prod_{k=1}^{i-1} L(v, s_{2k}-s_{2k-2})z\right.$$
$$\left. - \prod_{j=i+1}^{m} L(u, s_{2j}-s_{2j-2}) \prod_{k=1}^{i} L(v, s_{2k}-s_{2k-2})z\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|L(u, s_{2i}-s_{2i-2}) \prod_{k=1}^{i-1} L(v, s_{2k}-s_{2k-2})z\right.$$
$$\left. - \prod_{k=1}^{i} L(v, s_{2k}-s_{2k-2})z\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|\prod_{k=1}^{i-1} L(v, s_{2k}-s_{2k-2})z\right.$$
$$\left. - (I - (s_{2i}-s_{2i-2})A(u)) \prod_{k=1}^{i} L(v, s_{2k}-s_{2k-2})z\right\|$$

$$= \sum_{i=1}^{m} (s_{2i} - s_{2i-2})\Big\| A(v) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z$$
$$- A(u) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z \Big\|$$
$$< \sum_{i=1}^{m} (s_{2i} - s_{2i-2}) \cdot \varepsilon$$
$$= x \cdot \varepsilon .$$

Then, since $T(u, x)z = \prod_0^x L(u, dI)z$ and $T(v, x)z = \prod_0^x L(v, dI)z$ (see Remark 1.1), $\| T(u, x)z - T(v, x)z \| \leqq x \cdot \varepsilon$.

Proof of Part (1) *of Theorem* 1. Let $p \in D$, let $a, b \geqq 0$, and let $\varepsilon > 0$. Let $M = \{\prod_{i=1}^m T(r_{2i-1}, | r_{2i} - r_{2i-2} |)p \mid x \in [a, b]$ and $\{r_i\}_{i=0}^{2m}$ is a chain from a to $x\}$. Then M is a bounded subset of D by Lemma 1.1. There exists $\delta > 0$ such that if $u, v \in [a, b]$, $| u - v | < \delta$, $0 \leqq x \leqq 1$ and $z \in M$, then $\| T(u, x)z - T(v, x)z \| \leqq \varepsilon \cdot x$. Let $\{s_i\}_{i=0}^{2m}$ be a chain from a to b with norm $< \min\{\delta, 1\}$ and let $\{t_i\}_{i=0}^{2n}$ be a refinement of $\{s_i\}_{i=0}^{2m}$, i.e., there is an increasing sequence u such that $u_0 = 0$, $u_m = n$, and if $1 \leqq i \leqq m$ $s_{2i} = t_{2u_i}$. For $1 \leqq i \leqq m$ let $K_i = T(s_{2i-1}, | s_{2i} - s_{2i-2} |)$ and let $J_i = \prod_{j=u_{i-1}+1}^{u_i} T(t_{2j-1}, | t_{2j} - t_{2j-2} |)$. Then,

$$\Big\| \prod_{i=1}^{m} T(t_{2i-1}, | t_{2i} - t_{2i-2} |)p - \prod_{i=1}^{m} T(s_{2i-1}, | s_{2i} - s_{2i-2} |)p \Big\|$$
$$= \Big\| \prod_{i=1}^{n} J_i p - \prod_{i=1}^{m} K_i p \Big\|$$
$$\leqq \sum_{i=1}^{m} \Big\| \prod_{j=i}^{m} J_j \prod_{k=1}^{i-1} K_k p - \prod_{j=i+1}^{m} J_j \prod_{k=1}^{i} K_k p \Big\|$$
$$\leqq \sum_{i=1}^{m} \Big\| J_i \prod_{k=1}^{i-1} K_k p - K_i \prod_{k=1}^{i-1} K_k p \Big\|$$
$$= \sum_{i=1}^{m} \Big\| \prod_{j=u_{i-1}+1}^{u_i} T(t_{2j-1}, | t_{2j} - t_{2j-2} |) \prod_{k=1}^{i-1} K_k p$$
$$- \prod_{j=u_{i-1}+1}^{u_i} T(s_{2i-1}, | t_{2j} - t_{2j-2} |) \prod_{k=1}^{i-1} K_k p \Big\|$$
$$\leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} \Big\| \prod_{r=j}^{u_i} T(s_{2i-1}, | t_{2r} - t_{2r-2} |) \prod_{h=u_{i-1}+1}^{j-1} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p$$
$$- \prod_{r=j+1}^{u_i} T(s_{2i-1}, | t_{2r} - t_{2r-2} |) \prod_{h=u_{i-1}+1}^{j} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p \Big\|$$
$$\leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} \Big\| T(s_{2i-1}, | t_{2j} - t_{2j-2} |) \prod_{h=u_{i-1}+1}^{j-1} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p$$
$$- T(t_{2j-1}, | t_{2j} - t_{2j-2} |) \prod_{h=u_{i-1}+1}^{j-1} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p \Big\|$$
$$\leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} | t_{2j} - t_{2j-2} | \cdot \varepsilon = | b - a | \cdot \varepsilon .$$

Hence, $\prod_a^b T(I, dI)p$ exists. Further, using the fact that D is dense in S and $T(u, x)$ is nonexpansive for $u, x \geqq 0$ one sees that if $p \in S$, $a, b \geqq 0$, then $\prod_a^b T(I, dI)p$ exists and thus part (1) of Theorem 1 is proved.

Before proving part (2) of Theorem 1 three lemmas will be proved each under the hypothesis of Theorem 1.

LEMMA 1.4. *If $p, q \in S$, $a, c \geqq 0$, and $b \in [a, c]$, then the following are true*:

(i) $\| \prod_a^b T(I, dI)p - \prod_a^b T(I, dI)q \| \leqq \| p - q \|$.

(ii) $\prod_b^c T(I, dI) \prod_a^b T(I, dI)p = \prod_a^c T(I, dI)p$.

(iii) *If $p \in D$ then* $\| \prod_a^b T(I, dI)p - p \| \leqq |b - a| \cdot \max_{u \in [a,b]} \| A(u)p \|$.

Proof. Parts (i) and (ii) follow from the nonexpansive property of $T(u, x)$, $u, x \geqq 0$. Part (iii) follows from Lemma 1.1.

LEMMA 1.5. *If M is a bounded subset of D, $a, b \geqq 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$, $|v - u| < \delta$, $w \in [u, v]$, and $z \in M$, then*

$$\left\| \prod_u^v T(I, dI)z - T(w, |v - u|)z \right\| \leqq |v - u| \cdot \varepsilon .$$

Proof. Let $M' = \{\prod_{i=1}^m T(s_{2i-1}, |s_{2i} - s_{2i-2}|)z \mid z \in M, x, y \in [a, b], \{s_i\}_{i=0}^{2m}$ is a chain from y to $x\}$. Then M' is a bounded subset of D by Lemma 1.1. By Lemma 1.3 there exists $\delta > 0$ such that if $u, v \in [a, b]$, $|u - v| < \delta$, $z \in M'$ and $0 \leqq x \leqq 1$, then $\| T(u, x)z - T(v, x)z \| \leqq x \cdot \varepsilon$. Let $u, v \in [a, b]$, $|v - u| < \min\{\delta, 1\}$, $w \in [u, v]$, $z \in M$, and let $\{s_i\}_{i=0}^{2m}$ be a chain from u to v. Then,

$$\begin{aligned}
&\left\| \prod_{i=1}^m T(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - T(w, |v - u|)z \right\| \\
&= \left\| \prod_{i=1}^m T(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^m T(w, |s_{2i} - s_{2i-2}|)z \right\| \\
&\leqq \sum_{i=1}^m \left\| T(s_{2i-1}, |s_{2i} - s_{2i-2}|) \prod_{j=1}^{i-1} T(s_{2j-1}, |s_{2j} - s_{2j-2}|)z \right. \\
&\quad \left. - T(w, |s_{2i} - s_{2i-2}|) \prod_{j=1}^{i-1} T(s_{2j-1}, |s_{2j} - s_{j-2}|)z \right\| \\
&\leqq \sum_{i=1}^m |s_{2i} - s_{2i-2}| \cdot \varepsilon \\
&= |v - u| \cdot \varepsilon .
\end{aligned}$$

Thus, $\| \prod_u^v T(I, dI)z - T(w, |v - u|)z \| \leqq |v - u| \cdot \varepsilon$.

LEMMA 1.6. *If M is a bounded subset of D, $a, b \geqq 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$, $w \in [u, v]$, $|v - u| < \delta$, $z \in M$,* *and* $\{s_i\}_{i=0}^{2m}$ *is a chain from* u *to* v, *then*

$$\left\|\prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^{m} L(w, |s_{2i} - s_{2i-2}|)z\right\| \leqq |v - u|\cdot\varepsilon .$$

Proof. An argument similar to the one in Lemma 1.3 proves Lemma 1.6.

Proof of Part (2) *of Theorem* 1. Let $p \in D$, $a, b \geqq 0$, and $\varepsilon > 0$. Let $M = \{\prod_a^x T(I, dI)p \mid x \in [a, b]\}$. Then M is a bounded subset of D by Lemma 1.4. By Lemmas 1.5 and 1.6 there exists $\delta > 0$ such that if $u, v \in [a, b]$, $w \in [u, v]$, $|u - v| < \delta$, $z \in M$, and $\{s_i\}_{i=0}^{2m}$ is a chain from u to v, then

$$\left\|\prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^{m} L(w, |s_{2i} - s_{2i-2}|)z\right\| \leqq |v - u|\cdot\varepsilon/3|b - a|$$

and $\| \prod_u^v T(I, dI)z - T(w, |v - u|)z \| \leqq |v - u|\cdot\varepsilon/3|b - a|$. Let $\{r_i\}_{i=0}^{2q}$ be a chain from a to b with norm $< \delta$. Let $\{s_i\}_{i=0}^{2m}$ be a refinement of $\{r_i\}_{i=0}^{2q}$ such that there exists an increasing sequence u such that $u_0 = 0, u_q = m$, if $1 \leqq i \leqq q$ $r_{2i} = s_{2u_i}$, and if $1 \leqq i \leqq q$ and $\{t_k\}_{k=0}^{2n}$ is a refinement of $\{s_j\}_{j=2u_{i-1}}^{2u_i}$, then

$$\Big\|\prod_{k=1}^{n} L(r_{2i-1}, |t_{2k} - t_{2k-2}|) \prod_a^{r_{2i-2}} T(I, dI)p$$
$$- T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_a^{r_{2i-2}} T(I, dI)p\Big\| \leqq |r_{2i} - r_{2i-2}|\cdot\varepsilon/3|b - a| .$$

(Note that if

$$1 \leqq i \leqq q \; T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_a^{r_{2i-2}} T(I, dI)p$$
$$= \prod_{r_{2i-2}}^{r_{2i}} L(r_{2i-1}, dI) \prod_a^{r_{2i-2}} T(I, dI)p = \prod_{r_{2i}}^{r_{2i-2}} L(r_{2i-1}, dI) \prod_a^{r_{2i-2}} T(I, dI)p$$

—see Remark 1.1). Let $\{t_i\}_{i=0}^{2n}$ be a refinement of $\{s_i\}_{i=0}^{2m}$ and let v be an increasing sequence such that $v_0 = 0, v_m = n$, and if $1 \leqq i \leqq m$ $s_{2i} = t_{2v_i}$. Then,

$$\left\|\prod_{i=1}^{n} L(t_{2i-1}, |t_{2i} - t_{2i-2}|)p - \prod_a^b T(I, dI)p\right\|$$
$$= \Big\|\prod_{i=1}^{q} \prod_{j=u_{i-1}+1}^{u_i} \prod_{k=v_{j-1}+1}^{v_j} L(t_{2k-1}, |t_{2k} - t_{2k-2}|)p$$
$$- \prod_{i=1}^{q} \prod_{r_{2i-2}}^{r_{2i}} T(I, dI)p\Big\|$$
$$\leqq \sum_{i=1}^{q} \Big\| \prod_{j=u_{i-1}+1}^{u_i} \prod_{k=v_{j-1}+1}^{v_j} L(t_{2k-1}, |t_{2k} - t_{2k-2}|) \prod_a^{r_{2i-2}} T(I, dI)p$$
$$- \prod_{r_{2i-2}}^{r_{2i}} T(I, dI) \prod_a^{r_{2i-2}} T(I, dI)p\Big\|$$

$$\leqq \sum_{i=1}^{q} | r_{2i} - r_{2i-2} | \cdot \varepsilon/3 | b - a |$$
$$+ \sum_{i=1}^{q} \left\| \prod_{j=u_{i-1}+1}^{u_i} \prod_{k=v_{j-1}+1}^{v_j} L(r_{2i-1}, | t_{2k} - t_{2k-2} |) \prod_{a}^{r_{2i-2}} T(I, dI)p \right.$$
$$\left. - T(r_{2i-1}, | r_{2i} - r_{2i-2} |) \prod_{a}^{r_{2i-2}} T(I, dI)p \right\|$$
$$+ \sum_{i=1}^{q} | r_{2i} - r_{2i-2} | \cdot \varepsilon/3 | b - a |$$
$$\leqq \varepsilon .$$

Thus, $\prod_a^b L(I, dI)p$ exists and is $\prod_a^b T(I, dI)p$ for $p \in D$. Further, using the fact that D is dense in S and $L(u, x)$ is nonexpansive for $u, x \geqq 0$ one sees that $\prod_a^b L(I, dI)p = \prod_a^b T(I, dI)p$ for all $p \in S$.

Define $U(b, a)p = \prod_a^b T(I, dI)p$ for $p \in S$ and $a, b \geqq 0$.

Proof of Theorem 2. Parts (1), (2), and (3) of Theorem 2 follow from Lemma 1.4. Suppose that $p \in S, 0 \leqq a \leqq t$, and $U(t, a)p \in D$. Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < h < \delta_1$

$$|| A(t)T(t, h)U(t, a)p - A(t)U(t, a)p || < \varepsilon/2$$

(see Definition 1.1, part (4)). By Lemma 1.5 there exists $\delta_2 > 0$ such that if $0 < h < \delta_2 || U(t + h, t)U(t, a)p - T(t, h)U(t, a)p || < h \cdot \varepsilon/2$. Then, if $0 < h < \min \{\delta_1, \delta_2\}$,

$$\begin{aligned} &|| (1/h)(U(t + h, a)p - U(t, a)p) - A(t)U(t, a)p || \\ &\quad = || (1/h)(U(t + h,t)U(t, a)p - U(t, a)p) - A(t)U(t, a)p || \\ &\quad\quad < \varepsilon/2 + || (1/h)(T(t, h)U(t, a)p - U(t, a)p) - A(t)U(t, a)p || \\ &\quad = \varepsilon/2 + \left\| 1/h \int_0^h [A(t)T(t, u)U(t, a)p - A(t)U(t, a)p] du \right\| < \varepsilon . \end{aligned}$$

Hence, $\partial^+ U(t, a)p/\partial t = A(t)U(t, a)p$. Suppose that $p \in S, 0 < s \leqq b$, and $U(s, b)p \in D$. Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < h < \delta_1$ then $0 \leqq s - h$ and $|| A(s)T(s, h)U(s, b)p - A(s)U(s, b)p || < \varepsilon/2$. By Lemma 1.5 there exists $\delta_2 > 0$ such that if $0 < h < \delta_2$

$$|| U(s - h, s)U(s, b)p - T(s, h)U(s, b)p || < h \cdot \varepsilon/2 .$$

Then, if $0 < h < \min \{\delta_1, \delta_2\}$

$$\begin{aligned} &|| (1/-h)(U(s - h, b)p - U(s, b)p) - (-A(s)U(s, b)p) || \\ &\quad = || (1/h)(U(s - h, s)U(s, b)p - U(s, b)p) - A(s)U(s, b)p || \\ &\quad\quad < \varepsilon/2 + || (1/h)(T(s, h)U(s, b)p - U(s, b)p) - A(s)U(s, b)p || \\ &\quad = \varepsilon/2 + \left\| 1/h \int_0^h [A(s)T(s, u)U(s, b)p - A(s)U(s, b)p] du \right\| < \varepsilon . \end{aligned}$$

Hence, $\partial^- U(s, b)p/\partial s = -A(s)U(s, b)p$.

3. Product integral representation in the uniform case. For Theorem 3 A is required to satisfy, in addition to conditions (I), (II), (III) of § 1, the following:

(IV) For each $t \geqq 0$ $A(t)$ has domain all of S.

(V) If $0 \leqq a \leqq b$, M is a bounded subset of S, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \in [a, b]$, $z, w \in M$, and $\| z - w \| < \delta$, then

$$\| A(u)z - A(u)w \| < \varepsilon .$$

THEOREM 3. *Let A satisfy conditions* (I)—(V) *and define*

$$M(u, v) = (I + vA(u))$$

for $u, v \geqq 0$. If $p \in S$ and $a, b \geqq 0$, then $\prod_a^b M(I, dI)p = U(b, a)p$.

Before proving Theorem 3, three lemmas will be proved each under the hypothesis of Theorem 3.

LEMMA 3.1. *Let $p \in S$ and let $a, b \geqq 0$. There is a neighborhood $N_{p,\delta}$ about p, a positive number γ, and a positive number K such that if $q \in N_{p,\delta}$, $x, y \in [a, b]$, $| y - x | < \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from x to y, then*

$$\left\| \prod_{i=1}^{m} M(s_{2i-1}, | s_{2i} - s_{2i-2} |)q - q \right\| \leqq | y - x | \cdot K .$$

Proof. There exists a positive number K such that if $u \in [a, b]$ and $q \in N_{p,1}$ then $\| A(u)q \| \leqq K$. Let $\delta = 1/2$ and let $\gamma = 1/2K$. Let $q \in N_{p,\delta}$, $x, y \in [a, b]$, $| y - x | < \gamma$, $\{s_i\}_{i=0}^{2m}$ a chain from x to y, $1 \leqq j \leqq m - 1$, and suppose that $\| \prod_{i=1}^{j} M(s_{2i-1}, | s_{2i} - s_{2i-2} |)q - q \| \leqq | s_{2j} - s_0 | \cdot K$. Then, $\prod_{i=1}^{j} M(s_{2i-1}, | s_{2i} - s_{2i-2} |)q \in N_{p,1}$ and so

$$\begin{aligned} &\left\| \prod_{i=1}^{j+1} M(s_{2i-1}, | s_{2i} - s_{2i-2} |)q - q \right\| \\ &\leqq \left\| \prod_{i=1}^{j} M(s_{2i-1}, | s_{2i} - s_{2i-2} |)q - q \right\| \\ &\quad + | s_{2j+2} - s_{2j} | \cdot \left\| A(s_{2j+1}) \prod_{i=1}^{j} M(s_{2i-1}, | s_{2i} - s_{2i-2} |)q \right\| \\ &\leqq | s_{2j+2} - s_0 | \cdot K . \end{aligned}$$

LEMMA 3.2. *If $p \in S$ and $a \geqq 0$ then $U(t, a)p$ is continuous in t.*

Proof. Let $p \in S$ and $a, b \geqq 0$. In a manner similar to Lemma 3.1 one proves the following: There is a neighborhood $N_{q,\delta}$ about $q = U(b, a)p$, $\gamma > 0$, and $K > 0$ such that if $z \in N_{q,\delta}$, $x, y \in [a, b]$, $| y - x | < \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from x to y then

$$\left\|\prod_{m}^{i=1}(I - |s_{2i} - s_{2i-2}|A(s_{2i-1}))z - z\right\| \leqq |y - x|\cdot K .$$

Let $\varepsilon > 0$, let $x \in [a, b]$ such that $|x - b| < \gamma$, let $\{s_i\}_{i=0}^{2m}$ be a chain from a to b and $k \leqq m$ an integer such that $s_{2k} = x$ and

$$\left\|U(b, a)p - \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p\right\| < \min\{\varepsilon, \delta\}$$

and

$$\left\|U(x, a)p - \prod_{i=1}^{k} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p\right\| < \varepsilon .$$

Then,

$$\begin{aligned}
&\| U(x, a)p - U(b, a)p \| \\
&\quad < 2\varepsilon + \left\|\prod_{m}^{i=k+1}(I - |s_{2i} - s_{2i-2}|A(s_{2i-1}))\prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p \right.\\
&\quad \left. - \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p\right\| \\
&\quad < 2\varepsilon + |b - x|\cdot K .
\end{aligned}$$

Then, $\lim_{x\to b} U(x, a)p = U(b, a)p$ for $x \in [a, b]$. Further, by Lemma 1.4 $\lim_{x\to b} U(x, a)p = U(b, a)p$ for $x \notin [a, b]$.

LEMMA 3.3. *Let $p \in S$ and $a \geqq 0$. There exists a neighborhood $N_{p,\delta}$ about p and $\gamma > 0$ such that the following are true*:

(1) *If $\varepsilon > 0$ there exists $\alpha > 0$ such that if $q \in N_{p,\delta}$, $a \leqq x \leqq a + \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to x with* norm $< \alpha$, *then*

$$\left\|\prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - U(x, a)q\right\| < \varepsilon .$$

and

(2) *If $\varepsilon > 0$ there exists $\alpha > 0$ such that if $q \in N_{p,\delta}$*, $\max\{0, a - \gamma\} \leqq$ *$x \leqq a$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to x with* norm $< \alpha$, *then*

$$\left\|\prod_{i=1}^{m} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - U(x, a)q\right\| < \varepsilon .$$

Proof. By Lemma 3.1 there exists $\delta > 0$ and $\gamma > 0$ such that if $q \in N_{p,\delta}$, $a \leqq x \leqq a + \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to x then

$$\prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q \in N_{p,2\delta} .$$

Let $\varepsilon > 0$. By Lemma 1.5 there exists $\alpha_1 > 0$ such that if

$$u, v \in [a, a + \gamma], 0 \leqq v - u < \alpha_1, u \leqq w \leqq v ,$$

and $q \in N_{p,2\delta}$, then $\| U(v, u)q - T(w, v-u)q \| \leqq (v-u)\cdot\varepsilon/2\gamma$. There exists $\alpha_2 > 0$ such that if $q \in N_{p,2\delta}$, $u \in [a, a+\gamma]$, and $0 \leqq x < \alpha_2$, then $\| A(u)T(u, x)q - A(u)q \| < \varepsilon/2\gamma$ (Note that

$$\| T(u, x)q - q \| = \left\| \int_0^x A(u)T(u, t)q dt \right\| \leqq x \cdot \| A(u)q \| \leqq x \cdot \times (\max \| A(t)z \|, t \in [a, a+\gamma], z \in N_{p,2\delta})) .$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, let $q \in N_{p,\delta}$, let $a \leqq x \leqq a + \gamma$, and let $\{s_i\}_{i=0}^{2m}$ be a chain from a to x with norm $< \alpha$. Then,

$$\begin{aligned}
&\left\| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - U(x, a)q \right\| \\
&= \left\| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - \prod_{i=1}^{m} U(s_{2i}, s_{2i-2})q \right\| \\
&\leqq \sum_{i=1}^{m} \Big\| U(s_{2i}, s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \\
&\quad - M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \Big\| \\
&< \varepsilon/2 + \sum_{i=1}^{m} \Big\| T(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \\
&\quad - M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \Big\| \\
&= \varepsilon/2 + \sum_{i=1}^{m} \Big\| \int_0^{s_{2i}-s_{2i-2}} [A(s_{2i-1})T(s_{2i-1}, t) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \\
&\quad - A(s_{2i-1}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q] dt \Big\| \\
&< \varepsilon/2 + \sum_{i=1}^{m} (s_{2i} - s_{2i-2})\cdot\varepsilon/2\gamma < \varepsilon .
\end{aligned}$$

A similar argument proves part (2) of the lemma.

Proof of Theorem 3. Let $p \in S$ and $0 \leqq a < b$. Suppose that if $a \leqq x < b \prod_a^x M(I, dI)p$ exists and is $U(x, a)p$. Let $a \leqq x < b$, let $\{s_i\}_{i=0}^{2m}$ be a chain from a to b, and let $j < m$ such that $s_{2j} = x$. One uses the inequality

$$\begin{aligned}
&\left\| U(b, a)p - \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\
&\leqq \left\| U(b, a)p - \prod_{a}^{x} M(I, dI)p \right\| \\
&\quad + \left\| \prod_{a}^{x} M(I, dI)p - \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\
&\quad + \Big\| \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \\
&\quad - \prod_{i=j+1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \Big\|
\end{aligned}$$

and Lemmas 3.1 and 3.2 to show $\prod_a^b M(I, dI)p$ exists and is $U(b, a)p$. Suppose now that for $a \leqq x \leqq b$ $\prod_a^x M(I, dI)p = U(x, a)p$. Let $b < x$, let $\{s_i\}_{i=0}^{2m}$ be a chain from a to x, and let $j < m$ such that $s_{2j} = b$. One uses the inequality

$$\begin{aligned} &\left\| U(x, a)p - \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\ &\leqq \left\| U(x, b)U(b, a)p - U(x, b) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\ &\quad + \left\| U(x, b) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right. \\ &\quad \left. - \prod_{i=j+1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \end{aligned}$$

and Lemma 3.3 to show that there exists $\gamma > 0$ such that if $b \leqq x < b + \gamma$ then $\prod_a^x M(I, dI)p$ exists and is $U(x, a)p$. Thus, if $p \in S$ and $0 \leqq a \leqq b$ then $\prod_a^b M(I, dI)p$ exists and is $U(b, a)p$. With a similar argument one shows that for $p \in S$ and $0 \leqq a \leqq b$ $\prod_b^a M(I, dI)p$ exists and is $U(a, b)p$.

4. Examples.

In conclusion two examples will be given.

EXAMPLE 1. Let S be the Hilbert space and let A be densely defined and m-monotone on S (Definition 1.2). In M. Crandall and A. Pazy [2] and in T. Kato [6], it is shown that B is the infinitesimal generator of a $\mathscr{C}$-semi-group on S (Definition 1.1). Let X be a function from $[0, \infty)$ to S such that X is continuous. Define $A(t)p = Bp + X(t)$ for $p \in$ Domain (B) and $t \geqq 0$. Then A satisfies conditions (I)—(III).

EXAMPLE 2. Let S be a Banach space and let B be a mapping from S to S such that B is m-monotone S and uniformly continuous on bounded subsets of S. In [11] it is shown that B is the infinitesimal generator of a $\mathscr{C}$-semi-group of mappings on S. Let C be a continuous mapping from $[0, \infty)$ to $[0, \infty)$, let D be a continuous mapping from $[0, \infty)$ to $(0, \infty)$, and let each of E and F be a continuous mapping from $[0, \infty)$ to S. Define $A(t)p = C(t)\cdot B(D(t)\cdot p + E(t)) + F(t)$ for $t \geqq 0$ and $p \in S$. Suppose $t \geqq 0$, $\varepsilon > 0$, and $p, q \in S$. Then,

$$\begin{aligned} &\| (I - \varepsilon A(t))p - (I - \varepsilon A(t))q \| \\ &\quad = (1/D(t)) \| (I - \varepsilon C(t)D(t)B)(D(t)p + E(t)) \\ &\qquad - (I - \varepsilon C(t)D(t)B)(D(t)q + E(t)) \| \\ &\quad \geqq (1/D(t)) \| (D(t)p + E(t)) - (D(t)q + E(t)) \| \\ &\quad = \| p - q \| \end{aligned}$$

and so $A(t)$ is monotone for $t \geqq 0$. Suppose $t \geqq 0$, $\varepsilon > 0$, and $p \in S$. Let q' be in S such that $(I - \varepsilon C(t)D(t)B)q' = D(t)p + E(t) + \varepsilon D(t)F(t)$.

Let $q = (1/D(t))(q' - E(t))$. Then $(I - \varepsilon A(t))q = p$ and so $A(t)$ is m-monotone. Then A satisfies conditions (I)—(V).

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