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ANALYTIC INTERPOLATION OF CERTAIN MULTIPLIER SPACES

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Let W_p denote the space of all functions on the circle which are the uniform limit of a sequence of trigonometric polynomials which is bounded as a sequence of multipliers for $l_p, 1 \leqq p \leqq 2$. Let U_s be the interpolation space $[W_2, W_1]_s$ (see 1.1). Our main result, Theorem 2.4, states that for a compact subset E of the circle, $U_s | E = C(E)$ if and only if $W_1 | E = C(E)$. A major step in the proof is a maximum principle for interpolation, Theorem 1.7. We also give a direct proof that $U_s \neq W_p$ (see Theorem 2.7) for corresponding s and p.

1. Some properties of analytic interpolation.

1.1. Let B^0 and B^1 be two Banach spaces continuously embedded in a topological vector space V such that $B^0 \cap B^1$ is dense in both B^0 and B^1. For $0 < s < 1$, let $\mathfrak{F}$, $[B^0, B^1]_s$ and $B^0 + B^1$ denote the spaces as defined in [1, § 1]. For two Banach spaces X and Y we let $O(X, Y)$ denote the Banach space of bounded linear operators from X into Y where the norm is the usual operator norm. Let $O(X)$ denote $O(X, X)$.

1.2. Assume the notation and conditions of paragraph 1.1 and for convenience let B_s denote the space $[B^0, B^1]_s, 0 < s < 1$. Let V' denote the Banach space

$$O(B^0 \cap B^1, B^0 + B^1) .$$

Let A_j be a closed subspace of $O(B^j), j = 0, 1$. By restricting the elements in A_j to $B^0 \cap B^1$ in the obvious way we may regard A_j as continuously embedded in the topological vector space V', and it is with respect to this embedding that we understand $[A_0, A_1]_s$; in particular, $[A_0, A_1]_s$ is a subspace of V'. We will assume that $A_0 \cap A_1$ is dense in A_j with respect to the norm of A_j ,$j = 0, 1$, when these spaces are embedded in V' as described. Since $B^0 \cap B^1$ is dense in B^0 and B^1, we know from [1, § 9.3] that $B^0 \cap B^1$ is dense in B_s; thus, since $B_s \subset B^0 + B^1$, the restriction of elements of $O(B_s)$ to $B^0 \cap B^1$ gives a continuous embedding of $O(B_s)$ in V' in the obvious manner. Note that each element of $A_0 \cap A_1$ is bounded with respect to the norm $\| \|_{B_s}$ restricted to $B^0 \cap B^1$ and is, therefore, contained in the enbedded $O(B_s)$. Let A_s denote the closure of $A_0 \cap A_1$ in $O(B_s)$ where $O(B_s)$ is regarded as embedded in V' in the manner just described. Finally, we let M_s and N_s denote the norms of the spaces A_s and $[A_0, A_1]_s$, respectively.

LEMMA 1.3. *Assuming* 1.2, $[A_0, A_1]_s \subset A_s$ *and* $M_s \leqq N_s$, $0 < s < 1$.

This lemma is an immediate consequence of [1, § 11.1].

1.4. Assume the notation and conditions of 1.1. Let J be a closed subspace of $B^0 + B^1$. We will assume that

(1.4.1) $I^j = J \cap B_j$, is closed in B^j, $j = 0, 1$. Clearly the map α defined by

$$\alpha(x + I^j) = x + J \qquad j = 0, 1$$

is a continuous one to one linear map from B^j/I^j into V/J. Let

$$D_s = [\alpha(B^0/I^0), \alpha(B^1/I^1)]_s .$$

LEMMA 1.5. *Assuming* 1.4, *if* $x \in B_s$, $0 < s < 1$, *then* $x + J \in D_s$ *and*

$$\| x + J \|_{D_s} \leqq \| x + (J \cap B_s) \|_{B_s/(J \cap B_s)} . \tag{1.5.1}$$

Proof. Let $x \in B_s$, $h \in J \cap B_s$ and $\varepsilon > 0$. Choose $f \in \mathfrak{F} = \mathfrak{F}(B^0, B^1)$ such that $f(s) = x + h$ and

$$\| f \|_{\mathfrak{F}} \leqq \varepsilon + \| x + h \|_{B_s} . \tag{1.5.2}$$

Let $g(\xi) = f(\xi) + J$ for $1 \leqq |\xi| \leqq \varepsilon$. Then it is clear that $g \in \mathfrak{F}_1$ where

$$\mathfrak{F}_1 = \mathfrak{F}(\alpha(B^0/I^0), \alpha(B^1/I^1))$$

and that

$$g(s) = x + J . \tag{1.5.3}$$

Hence, $x + J \in D_s$. Furthermore, since it is clear that

$$\| g \|_{\mathfrak{F}_1} \leqq \| f \|_{\mathfrak{F}} , \tag{1.5.4}$$

(1.5.1) follows from (1.5.2), (1.5.3), (1.5.4) and the fact that h and ε were chosen arbitrarily.

The following lemma can be proved by the usual method of successive approximations.

LEMMA 1.6. *Suppose that* D_1 *is a Banach space that is continuously embedded in a Banach space* D_0 *such that* D_1 *is dense in* D_0 *with respect to the norm of* D_0. *Suppose that there exist constants* c, c_1, $c < 1$, *with the property that for each* $x \in D_1$ *there is a corresponding element* z *in* D_1 *such that*

$$|z|_1 \leqq c_1|x|_0 \quad \textit{and} \quad |x - z|_0 \leqq c|x|_0 .$$

Then $D_1 = D_0$.

We will now establish a "maximum principle" for analytic interpolation.

THEOREM 1.7. *If, in addition to the assumptions of paragraph* 1.1, $B^0 = [B^0, B^1]_s$ *for some* s $(0 < s < 1)$, *then* $B^0 = B^1$.

Proof. From the fact that B^0 and B^1 are continuously embedded in V and the closed graph theorem we conclude that the norms $|\ |_0$ and $|\ |_s$ on B^0 and $[B^0, B^1]_s$, respectively, are equivalent. In particular, there is a constant c such that

(1.7.1) $$|x|_0 \leqq c|x|_s \quad \text{for all } x \text{ in } B^0 .$$

From [1, 9.4. (ii)] we conclude that

(1.7.2) $$|x|_s \leqq |x|_0^{1-s}|x|_1^s \quad \text{for all } x \text{ in } B^0 \cap B^1 .$$

We conclude from (1.7.1) and (1.7.2) that

$$|x|_0 \leqq c^{1/s}|x|_1 \quad \text{for all } x \text{ in } B^0 \cap B^1 .$$

Thus, B_1 is continuously embedded in B^0. We shall now prove that (1.7.3) there is a constant c_1 with the property that for each x in B^1 there is a corresponding y in B^1 such that

$$|y|_1 \leqq c_1|x|_0 \quad \text{and} \quad |y - x|_0 \leqq (1/2)|x|_0 .$$

Let $x \in B^1$. In particular, $x \in [B^0, B^1]_s$ and, therefore, there exists an $f \in \mathfrak{F}(B^0, B^1)$ such that $f(s) = x$ and $|f|_{\mathfrak{F}(B^0,B^1)} \leqq 2|x|_s$. Since the norms $|\ |_0$ and $|\ |_s$ are equivalent we can choose a real number λ so that $2|u|_s e^{\lambda s} \leqq (1/2)|u|_0$ for every u in B^0. Let $g(\xi) = f(\xi)e^{-\lambda(\xi - s)}$ where $0 \leqq \operatorname{Re} \xi \leqq 1$. Then

(1.7.4) $$x = g(s) = \int_{-\infty}^{\infty} g(it)\mu_0(s, t)dt + \int_{-\infty}^{\infty} g(1 + it)\mu_1(s, t)dt$$

where μ_0 and μ_1 are the Poisson kernels for the strip $0 \leqq \operatorname{Re} \xi \leqq 1$ (see [1, 9.4]). Let y and z denote the first and second integrals, respectively, appearing in (1.7.4). Since $\int_{-\infty}^{\infty} |\mu_i(s, t)|dt \leqq 1$ $(i = 0, 1)$, $|g(it)|_0 \leqq 2|x|_s e^{\lambda s} \leqq (1/2)|x|_0$ (all real t), and

$$|g(1 + it)|_1 \leqq 2|x|_s e^{-\lambda(1-s)} \leqq (1/2)e^{-\lambda}|x|_0$$

(all real t), it follows that $|x - z|_0 \leqq (1/2)|x|_0$ and $|z|_1 \leqq (1/2)e^{-\lambda}|x|_0$. This proves (1.7.3). Since B^1 is continuously embedded as a dense subspace in B^0 and (1.7.3) holds, the conclusion of Theorem 1.7 follows from Lemma 1.6.

2. The spaces W_p and U_s. Let l_p, $1 \leqq p < \infty$, denote the Banach space of complex valued functions x on the integers such that

$$||x||_{l_p} = (\sum |x(n)|^p)^{1/p} < \infty$$

where the sum is over all integers n. Each function α on the integers which vanishes outside some finite set determines a linear transformation T_α on l_p defined by

$$T_\alpha x(n) = \sum_{-\infty < k < \infty} x(n - k)\alpha(k) .$$

Let W_p' denote the closure of the operators T_α in $O(l_p)$. Since l_1 is a dense subspace of each space l_p, $1 \leqq p < \infty$, the restriction of elements in $O(l_p)$, $1 \leqq p \leqq 2$, to the subspace l_1 gives a one-to-one continuous linear embedding of $O(l_p)$, $1 \leqq p \leqq 2$, into the space

$$R = O(l_1, l_2) .$$

Throughout this section we will identify $O(l_p)$ with its image under this embedding without further comment. Let U_s' denote the space $[W_2', W_1']_s$ where V in 1.1 is, in this case, R.

Our immediate purpose is to define a "Fourier transform" on W_p' and to prove Lemmas 2.2 and 2.3.

If x is a complex valued function on the integers Z, let $\tau_n x(k) = x(k - n)$. Let δ_n denote the function on Z such that $\delta_n(n) = 1$ and $\delta_n(k) = 0, k \neq n$. If x and y are two complex valued function on Z let

$$x*y(m) = \sum_{n \in Z} x(m - n)y(n)$$

define the function $x*y$ provided the sum converges absolutely for each $m \in Z$. For each H in W_p' let $H^\sim$ denote the function $H(\delta_0)$ in l_p. The following lemma states the needed properties of the map $H \to H^\sim$. Note that $\tau_n x = \delta_n * x$ for each $n \in Z$ and for each complex valued function x on z.

LEMMA 2.1.
(2.1.1) *$H \to H^\sim$ is a one-to-one linear transformation from W_p' into l_p.*
(2.1.2) *$Hx = H^\sim * x$, $H \in W_p'$, $x \in l_p$.*
(2.1.3) *$(HK)^\sim = H^\sim * K^\sim$, $H, K \in W_p'$.*

Proof. The map $H \to H^\sim$ is clearly linear. Evidently, each H in

W_p' commutes with all operators τ_m, $m \in Z$, since the operators of the form T_α commute with the operators τ_m, $m \in Z$. Thus for $H \in W_p'$ and $m \in Z$, we see that

$$H(\delta_m) = H(\tau_m \delta_0) = \tau_m H(\delta_0) = \tau_m H^\sim = H^\sim * \delta_m . \tag{2.1.4}$$

From this we see that since the linear span of the elements δ_m is dense in l_p, the map $H \to H^\sim$ is one-to-one. Obviously, $H^\sim$ is in l_p. To establish (2.1.2) we first note that since $H^\sim$ is in $l_q(q^{-1} + p^{-1} = 1)$ the map $x \to H^\sim * x$ is a continuous linear map from l_p into c_0, the space of complex valued functions on Z which tend to 0 at $\pm\infty$. The map $x \to Hx$ is also a continuous linear map from l_p into c_0. These observations together with (2.1.4) and the density property of the δ_m's noted above complete the proof of (2.1.2). To prove (2.1.3) we note that for H and K in W_p', $K^\sim \in l_p$, so by (2.1.2) we have

$$H^\sim * K^\sim = H(K^\sim) = H(K\delta_0) = (HK)\delta_0 = (HK)^\sim .$$

This completes the proof of the lemma.

Let $L_p(1 \leqq p < \infty)$ denote the Banach space of measurable functions $g(\theta)$ on the circle (reals mod 2π) whose norm $\|g\|_{L_p}$,

$$\|g\|_{L_p} = \left((1/2\pi)\int_0^{2\pi} |g(\theta)|^{1/p} d\theta \,|\right)^{1/p} ,$$

is finite. Let L_∞ denote the space of essentially bounded measurable functions g with $\|g\|_{L_\infty}$ denoting the essential supremum of g.

Since each function $H^\sim$, $H \in W_p'$, is in l_p, which is contained in l_2, there is a unique function $H^\wedge$ in L_2 such that $\sum H^\sim(n)e^{in\theta}$ is the Fourier series of $H^\wedge$.

LEMMA 2.2. *For $1 \leqq p \leqq 2$ the map $H \to H^\wedge$ is a norm decreasing algebraic isomorphism from W_p' into L_∞.*

Proof. The fact that $H \to H^\wedge$ is a one-to-one linear map from W_p' into L_2 is clear from (2.1.1) and the fact that each function in L_2 is uniquely determined by its Fourier coefficients. For each $f \in L_1$, let $\lambda(f)$ denote the function on Z defined by:

$$\lambda(f)(n) = (1/2\pi)\int_0^{2\pi} f(\theta)e^{-in\theta} d\theta .$$

It is clear from the Schwarz inequality that the map $(f, g) \to \lambda(f \cdot g)(n)$ is a continuous bilinear functional on $L_2 \oplus L_2$ for each integer n. On the other hand, the map

$$(f, g) \to (\lambda(f) * \lambda(g))(n)$$

is also a continuous bilinear functional on $L_2 \oplus L_2$. Since these functionals (for each n) clearly agree when f and g are trigonometric polynomials, they must agree on $L_2 \oplus L_2$. Since λ is a one-to-one map, the multiplicative property of $H \to H^\wedge$ now follows from (2.1.3). To prove that the map is norm decreasing we first note the following inequalities:

$$\|H^n\|_{W'_p} \geqq \|H^n \delta_0\|_{l_p} = \|(H^n)^\sim\|_{l_p} \geqq \|(H^n)^\sim\|_{l_2} = \|(H^n)^\wedge\|_{L_2} = \|(H^\wedge)^n\|_{L_2} .$$

It is well known that $(\|H^n\|_{W'_p})^{1/n}$ converges to the spectral radius of H, which is dominated by $\|H\|_{W'_p}$, and that $(\|(H^\wedge)^n\|_{L_2})^{1/n}$ converges to $\|H^\wedge\|_{L_\infty}$ as $n \to \infty$. This proves the lemma.

Let W_p and U_s denote the functions on the circle of the form $H^\wedge$ where $H \in W'_p$, U'_s, respectively. The following lemma is an immediate consequence of Lemma 2.2.

LEMMA 2.3. *W_p consists precisely of the functions on the circle which are the uniform limits of sequences $H_n^\wedge$ of trigonometric polynomials such that H_n is a Cuachy sequence in W'_p.*

For any subset E of the circle group $U_s \mid E$ denotes the functions on E obtained by restricting the functions of U_s to E and $C(E)$ denotes the continuous complex valued functions on E.

THEOREM 2.4. *Suppose that E is a compact subset of the circle group and $0 < s < 1$. Then $U_s \mid E = C(E)$ if and only if $W_1 \mid E = C(E)$.*

Proof. First assume that $W_1 \mid E = C(E)$. By Lemma 1.3, $U'_s \subset W'_p$; consequently, $U_s \subset W_p$. We conclude from Lemma 2.3 that $W_p \subset C(T)$. Thus, $U_s \mid E \subset C(E)$. Since $W'_2 \supset W'_1$, it is clear from the definition of interpolation that $U'_s \supset W'_1$. Thus, $U_s \mid E \supset C(E)$.

Consider the converse and assume that $U_s \mid E = C(E)$. In 1.4 we let $B^0 = W'_2$, $B^1 = W'_1$, $V = R$ and

$$J = \{a \in W'_2 \colon \hat{a}(\theta) = 0,\ \theta \in E\} .$$

The assumptions on J in 1.4 are clearly satisfied since by Lemma 2.2, the maps $a \to \hat{a}$ are continuous on W'_1 and W'_2. By Theorem 1.5, if $x \in U'_s$, then $x + J$ is in the space

$$[\alpha(W'_2/J),\ \alpha(W'_1/(J \cap W'_1))]_s . \tag{2.4.1}$$

However, by hypothesis, the cosets in V of the form $x + J$,, $x \in U'_s$, are the same as the cosets $y + J$, $y \in W'_2$. Therefore, the space in (2.4.1) is $\alpha(W'_2/J)$. Since $W'_2 \supset W'_1$,

$$\alpha(W'_2/J) \supset \alpha(W'_1/(J \cap W'_1)) ;$$

therefore, we conclude from 1.7 that

$$\alpha(W_2'/J) = \alpha(W_1'/(J \cap W_1')) ;$$

or, what is the same thing, that $W_1 | E = C(E)$. This completes the proof.

COMMENT 2.5. It is natural to compare U_s and W_p where $[l_2, l_1]_s = l_p$, i.e., $(1 - s)/2 + s = 1/p$. In [3] we showed that Theorem 2.4 is not valid for W_p. To be exact, there is a compact subset E of the circle such that $W_p | E \neq C(E) = W_{4/3} | E, 1 \leqq p < 4/3$. We had originally used this result to show that $W_p \neq U_s$; however, the referee has suggested a direct proof which we will now give.

LEMMA 2.6. *Let h_n be a sequence in U_s, $0 < s < 1$, such that $|| h_n ||_s \leqq M$ (here $|| \ ||_s$ is the norm in U_s) and $h_n \rightarrow h$ almost everywhere. Then h agrees with some continuous function almost everywhere.*

Proof. Since $|| h_n ||_s \leqq M$ there exist functions $f_n(\theta, \xi)$, analytic in ξ for $0 < B(\xi) < 1$ and continuouns in $0 \leqq B(\xi) \leqq 1$, such that for any real number t, $|| f_n(\theta, it) ||_0 \leqq 2M$, $|| f_n(\theta, 1 + it) ||_1 \leqq 2M$ and $f_n(\theta, s) = h_n(\theta)$. Let $g_n(\theta, \xi) = f_n(\theta, \xi)e^{+\lambda(\xi - s)}$. Then

$$\begin{aligned} h_n(\theta) = f_n(\theta, s) = g_n(\theta, s) &= \int_{-\infty}^{+\infty} g_n(\theta, it)\mu_0(s, t)dt \\ &\quad + \int_{-\infty}^{+\infty} g_n(\theta, 1 + it)\mu_1(s, t)dt \\ &= u_n(\theta) + v_n(\theta) \end{aligned}$$

where μ_0 and μ_1 are the Poisson Kernels for the strip (see [1, 9.4]). Evidently $|| u_n ||_0 \leqq 2e^{-\lambda s}M$, $|| v_n ||_1 \leqq 2e^{\lambda(1-s)}M$. Since the v_n are uniformly bounded, by taking a subsequence if necessary, we may assume that v_n converges weakly to a bounded function $v(\theta)$, that is

$$\lim_{n \to \infty} \int v_n(\theta)\varphi(\theta)d\theta = \int v(\theta)\varphi(\theta)d\theta$$

for every integrable φ. Furthermore, as is readily seen, $v(\theta)$ belongs to U_1 and therefore is continuous. Since h_n is uniformly bounded and converges almost everywhere, h_n converges weakly. Since h_n and v_n converge weakly, u_n converges weakly to some function u. From the fact that $| u_n(\theta) | \leqq || u_n ||_0 \leqq 2e^{-\lambda s}M$, it follows that $| u(\theta) | \leqq 2e^{-\lambda s}$ almost everywhere. Since $h = u + v$ almost everywhere and λ can be taken arbitrarily large, h agrees almost everywhere with the uniform limit of continuous functions. This completes the proof of the lemma.

THEOREM 2.7. *U_s is properly contained in W_p for $1 < p < 2$.*

Proof. To prove the theorem it suffices to exhibit a sequence of functions in U_s whose norms in U_s tend to infinity and whose norms in W_p remain bounded. Let $h(e^{it}) = 1$ for $0 \leqq t \leqq \pi$ and $h(e^{it}) = 0$ for $\pi < t < 2\pi$. Then h is a multiplier for l_p (see [2]), which does not agree almost everywhere with any continuous function. Let φ_n be defined by: $\varphi_n(e^{it}) = n$ for $|t| \leqq 1/2n$, $\varphi_n(e^{it}) = 0$ otherwise, $n = 1, 2, \cdots$. Let $h_n = h*\varphi_n$, $n = 1, 2, \cdots$. Since $\int_0^{2\pi} | h_n(e^{it}) | dt = 1$, it follows that the W_p norm of h_n is the same as the W_p norm of h; thus, h_n is bounded in W_p. Since both h and φ_n belong to $L_2(0, 2\pi)$, $h_n \in W_1 \subset U_s$. Obviously, h_n converges to h almost everywhere. Since h does not agree almost everywhere with any continuous function, it follows from Lemma 2.6 that h_n is not bounded in U_s.

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