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RINGS OF FUNCTIONS WITH CERTAIN LIPSCHITZ PROPERTIES

C. H. SCANLON

Let (X, d) denote a metric space, $L_c(X)$ the ring of real valued functions on X which are Lipschitz on each compact subset of $X, L_1(X)$ the ring of real valued functions on X which are locally Lipschitz relative to the completion of X, and $L_c^*(X), L_1^*(X)$ the bounded elements of $L_c(X), L_1(X)$. The relations between equality of these rings and the topological properties of X are studied. It is shown that a subspace (S, d) of (X, d) is L_c-embedded (or L_c^*-embedded) in (X, d) if and only if S is closed. Further, every subspace of (X, d) is L_1- and L_1^*-embedded in (X, d).

Su [3] investigated algebraic properties of the rings $L_c(X)$ and $L_c^*(X)$ similar to those of $C(X)$ and $C^*(X)$ by Gillman and Jerrison [2].

2. Equality of rings. Let f denote a real valued function defined on X. f is Lipschitz on $S \subset X$ if and only if there is a real number m, called a Lipschitz constant for f on S, such that if x, $y \in S$, then $|f(x) - f(y)| \leqq md(x, y)$. f is locally Lipschitz on X if and only if for each $x \in X$, there is a neighborhood N of x such that f is Lipschitz on N. If comp X denotes the completion of X, then f is locally Lipschitz with respect to comp X if and only if for each $x \in \text{comp } X$ there is a neighborhood N of x such that f is Lipschitz on $N \cap X$.

THEOREM 2.1. *$f \in L_c(X)$ if and only if f is locally Lipschitz on X.*

Sufficiency. Let f be locally Lipschitz on X and S a compact subset of X. Then there exists a finite collection $N_1, N_2, \cdots, N_m$ of open sets covering S, on each of which f is Lipschitz and thus bounded. Assuming f is not Lipschitz on S implies that there exists a sequence $\{x_n\}$ from S converging to $x \in S$ and a sequence $\{y_n\}$ from S such that $|f(x_n) - f(y_n)|/d(x_n, y_n) > n$ for each positive integer n. Since f is bounded on S, it follows that $\{y_n\}$ converges to x. Since $x \in N_j$ for some $j = 1, 2, \cdots, m, f$ is not Lipschitz on N_j which contradicts the definition of N_j.

Necessity. Let $f \in L_c(X)$ and $x \in X$. Assuming f is not locally Lipschitz at x implies there exists sequences $\{x_n\}$ and $\{y_n\}$ such that

$d(x, x_n) < 1/n$, $d(x, y_n) < 1/n$, and $|f(x_n) - f(y_n)|/d(x_n, y_n) > n$. Then $\{p: p \in \{x_n\}, p \in \{y_n\}$, or $p = x\}$ is a compact subset of X on which f is not Lipschitz.

COROLLARY 2.2. *$f \in L_c^*(x)$ if and only if f is locally Lipschitz on X and bounded.*

Proof. Follows immediately from the definition of $L_c^*(X)$.

COROLLARY 2.3. *$L_1(X) \subset L_c(X)$ and $L_1^*(X) \subset L_c^*(X)$.*

Proof. If f is locally Lipschitz relative to com X, then f is locally Lipschitz.

LEMMA 2.4. *If K is a uniformly bounded set of Lipschitz functions defined on $S \subset X$ and there is a real number m which is a Lipschitz constant for each element of K, then $f(x) = \sup\{g(x): g \in K\}$ for each $x \in S$ is Lipschitz on S and m is a Lipschitz constant for f on S.*

Proof. f exists since K is a uniformly bounded set. Assume $x \in S$, $y \in S$, and

$$f(y) - f(x) - md(x, y) = e > 0 . \tag{1}$$

Let $g \in K$ such that

$$f(y) - g(y) < e , \tag{2}$$

then

$$g(y) - g(x) \leqq md(x, y) . \tag{3}$$

Combining (2) and (3) yields $f(y) - g(x) - md(x, y) < e$, which when combined with (1) gives $f(x) < g(x)$. This contradicts the definition of f.

LEMMA 2.5. *Suppose each of c and $r > 0$, $p \in X$, and for*

$$\text{each } x \in X,\ f(x) = \begin{cases} (c/r)\{r - d(x, p)\} & \text{for } \ d(x, p) \leqq r, \\ 0 & \text{otherwise} \end{cases}$$

then f is Lipschitz on X and (c/r) is a Lipschitz constant for f on X.

Proof. Let $g(x) = (c/r)\{r - d(x, p)\}$ for each $x \in X$. Then for x, $y \in X$,

$$g(x) - g(y) = g(x) - g(p) + g(p) - g(y) ,$$
$$g(x) - g(y) = -(c/r)d(x, p) + (c/r)d(y, p) ,$$

and $g(x) - g(y) \leqq (c/r)d(x, y)$ by the triangle property. Since $\sup\{g, 0\}$ is Lipschitz with a Lipschitz constant $\sup\{(c/r), 0\}$ by Lemma 2.4, the conclusion follows.

THEOREM 2.6. *Each of the following is equivalent to each of the others*:

(1) $L_1(X) = L_c(x)$,

(2) $L_1^*(X) = L_c^*(X)$, *and*

(3) X *is complete.*

Proof. (1) $\Rightarrow$ (2) obviously. The remaining order is (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Assume (2) and that X is not complete. Then there exists an $x \in (\text{comp } X) - X$ and a sequence $\{x_n\}$ of distinct points in X such that $\{x_n\}$ converges to x. For each odd integer n, let

$$r_n = \frac{1}{3}\inf\{y\colon y = d(x_n, x_m) \quad \text{for} \quad m \neq n \quad \text{or} \quad y = (1/n)\} ,$$

$$C(x_n, r_n) = \{t \in X\colon d(t, x_n) \leqq r_n\} ,$$

and

$$f_n(t) = \begin{cases} (1/r_n)\{r_n - d(x_n, t)\} & \text{for} \quad t \in C(x_n, r_n) \\ 0 \quad \text{otherwise} \end{cases}$$

for each $t \in X$. Let $f(t) = \sup\{f_n(t)\}$ for each $t \in X$. If S is a compact subset of X, then S can intersect at most a finite number of the elements of $\{C(x_n, r_n)\}$ and since only a finite number of elements of $\{f_n\}$ are nonzero on S, by Lemma 2.4 f is Lipschitz on S and $f \in L_c^*(X)$. For each neighborhood N in comp X of x, there is a point $t \in N$ and a point $y \in N$ such that $f(t) = 1$ and $f(y) = 0$. Thus $f \notin L_1(X)$ and by contradiction, (2) $\Rightarrow$ (3).

If (3) is true, $f \in L_1(X)$ if and only if f is locally Lipschitz. Thus by Theorem 2.1, $L_1(X) = L_c(X)$ and (3) $\Rightarrow$ (1).

THEOREM 2.7. $L_c(X) = L_c^*(X)$ *if and only if* X *is compact.*

Proof. If X is compact, then each element of $L_c(X)$ is bounded.

Assume $L_c(X) = L_c^*(X)$ and X is not compact. Then there exists a sequence $\{x_n\}$ of distinct points in X which has no convergent subsequence. Let

$$r_n = \frac{1}{3}\inf\left\{y\colon y = d(x_n, x_m) \quad \text{for} \quad n \neq m \quad \text{or} \quad y = \frac{1}{n}\right\} ,$$

and

$$f(x) = \begin{cases} (n/r_n)\{r_n - d(x_n, x)\} & \text{for} \quad d(x_n, x) \leqq r_n \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in X$. By an argument similar to the one for Theorem 2.6, $f \in L_c(X)$. Since $f(x_n) = n$ for each n, $f \in L_c(X) - L_c^*(X)$ which contradicts the assumption.

THEOREM 2.8. $L_1(X) = L_1^*(X)$ *if and only if* comp X *is compact.*

Proof. Each element of $L_1(X)$, $L_1^*(X)$ can be uniquely extended to an element of $L_1(\text{comp } X) = L_c(\text{comp } X)$, $L_1^*(\text{comp } X) = L_c^*(\text{comp } X)$. Since $L_c(\text{comp } X) = L_c^*(\text{comp } X)$ if and only if comp X is compact by Theorem 2.7, the conclusion follows.

3. If A denotes one of L_1, L_1^*, L_c, L_c^* and $S \subset X$, then the statement that S is A-embedded in X means that if $f \in A(S)$, there is a $g \in A(X)$ such that $g \mid S = f$ where $g \mid S = \{(x, y) \in g : x \in S\}$.

THEOREM 3.1. *If S is a subset of X, then each of the following is equivalent to each of the others*:

(1) *S is L_c-embedded in X,*

(2) *S is L_c^*-embedded in X, and*

(3) *S is closed.*

Proof. Czipszer and Geher [1] proved that if S is a closed subset of X and f is a real valued locally Lipschitz function with domain S, then there is a real valued locally Lipschitz function g with domain X such that $g \mid S = f$. Furthermore, they proved that if f is bounded, then there exists a bounded such g. Consequently, by Theorem 2.1, (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2).

Assume (2) and S is not closed. Then there exists a sequence $\{x_n\}$ of distinct points in S and a point $x \in X - S$ such that $\{x_n\}$ converges to x. Construct f as in Theorem 2.6. Then $f \in L_c^*(S)$ which has no extension to X in $L_c(X)$. Thus (2) $\Rightarrow$ (3). Note that this also shows (1) $\Rightarrow$ (3).

COROLLARY 3.2. *Every subset of X is L_1-embedded and L_1^*-embedded in X.*

Proof. If $S \subset X$, then every element of $L_1(S)$ has a unique extension to the closure of S in comp X and by Theorems 2.6 and 3.1

an extension in $L_1(\text{comp}\, X)$ which when restricted to X is an element of $L_1(X)$.

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THE UNIVERSITY OF OKLAHOMA