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NON-ARCHIMEDEAN GELFAND THEORY

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It is shown that under certain conditions a specified subalgebra X_0 of a non-Archimedean Banach algebra is isometrically isomorphic to the space of all continuous functions from a compact zero-dimensional Hausdorff space to the ground field; this generalizes a recent result in which X_0 is assumed to coincide with the entire algebra.

It is well-known that a complex commutative B^*-algebra with identity is isometrically isomorphic to the space of all continuous functions on the compact Hausdorff space of its maximal ideals with the Gelfand topology. Narici [7] has proved a similar representation theorem for non-Archimedean algebras (see Corollary 6.2). In order that the Gelfand topology be defined as for complex Banach algebras, Narici assumed that the quotient field of the algebra modulo each of its maximal ideals was isomorphic to the ground field. In the present paper an extension of Narici's result is given in which no assumption is made concerning these quotient fields. As preliminaries to this result, a subalgebra of the given algebra, having several pleasant properties, will be introduced, and some natural generalizations of the definition of the Gelfand topology will be considered.

In §1, X will denote a commutative algebra with identity e over the field F; $\mathscr{M}$ will denote the set of maximal ideals of X; $x(M)$, where $x \in X$ and $M \in \mathscr{M}$, will denote the coset of M to which x belongs; $x(\mathscr{M})$ will denote the set $\{x(M) \mid M \in \mathscr{M}\}$; and $\mathscr{R}$ will denote the Jacobson radical, $\cap \mathscr{M}$. The field X/M, $M \in \mathscr{M}$, will be considered as an extension field of F (which may or may not be proper) under the identification $\alpha \leftrightarrow \alpha e(M)$ for each $\alpha \in F$. Then $\sigma(x)$, the spectrum of $x \in X$, is equal to $x(\mathscr{M}) \cap F$.

In §2 and §3, it will be assumed, moreover, that F is complete with respect to a nontrivial rank one valuation, $|\ \ |$, and X is a non-Archimedean commutative Banach algebra with identity e over F; i.e.,

(1) X is a non-Archimedean Banach space satisfying the ultrametric inequality $\|x + y\| \leqq \max(\|x\|, \|y\|)$;

(2) $\|xy\| \leqq \|x\| \|y\|$;

(3) $\|e\| = 1$.

1. The Gelfand subalgebra.

DEFINITION. The set

$$X_0 = \{x \in X \mid x(M) \in F \quad \text{for all} \quad M \in \mathscr{M}\}$$

will be called the *Gelfand subalgebra of* X.

Objects related to X_0 will be subscripted with a zero; e.g., $\mathscr{M}_0$ will denote the set of all maximal ideals of X_0.

Certainly X_0 is a subalgebra containing the identity.

The set of maximal ideals M in X such that $X/M = F$ will be called $\mathscr{M}'$; the set of maximal ideals M_0 in X_0 such that $X_0/M_0 = F$ will be called $\mathscr{M}_0'$.

THEOREM 1. (a) *If $x \in X_0$ is invertible in X, then $x^{-1} \in X_0$.*

(b) *The function $\Psi: M \to M_0 = M \cap X_0$ carries maximal ideals of X into maximal ideals of X_0 having the property $X_0/M_0 = F$; thus, denoting $\Psi(\mathscr{M})$ by $\mathscr{M}_0''$, we have $\mathscr{M}_0'' \subset \mathscr{M}_0' \subset \mathscr{M}_0$.*

(c) *If $M \in \mathscr{M}$, $M_0 = M \cap X_0$ and $x \in X_0$, then $x(M) = x(M_0)$.*

(d) *If $x \in X_0$*

$$\sigma(x) = \sigma_{X_0}(x) = x(\mathscr{M}) = x(\mathscr{M}_0) \cap F = x(\mathscr{M}_0') = x(\mathscr{M}_0'') .$$

(e) *$\mathscr{R}_0 \subset \mathscr{R} \subset X_0$. Hence $\mathscr{R} \oplus F \subset X_0$, and X_0 is semisimple if X is.*

(f) *If $\sigma(x) = \phi$, $x \notin X_0$.*

(g) *If $X \supset K \supset F$, where K is a field, then $K \cap X_0 = F$. Consequently, if X is a field $X_0 = F$.*

(h) *$X_0 = X$ if and only if $X/M = F$ for all $M \in \mathscr{M}$ (i.e., if and only if $\mathscr{M} = \mathscr{M}'$).*

(i) *$X_0 = \bigcap_{M \in \mathscr{M}} M \oplus F$.*

Proof. (a) $x^{-1}(M) = x(M)^{-1}$.

(b) Apply the fundamental theorem of homomorphisms to the mapping

$$\begin{aligned} X_0 &\longrightarrow X/M \\ x &\longrightarrow x(M) . \end{aligned}$$

(c) If $x \in X_0$ and $\alpha \in F$,

$$\begin{aligned} x(M) = \alpha &\Longleftrightarrow x - \alpha \in M \\ &\Longleftrightarrow \text{(since } x, \alpha \in X_0\text{)}\ x - \alpha \in M_0 \\ &\Longleftrightarrow x(M_0) = \alpha . \end{aligned}$$

(d) From (c) it follows that for $x \in X_0$,

$$\sigma(x) = x(\mathscr{M}) = x(\mathscr{M}_0'') \subset x(\mathscr{M}_0') \subset x(\mathscr{M}_0) \cap F = \sigma_{X_0}(x) .$$

However, (a) implies $\sigma(x) = \sigma_{X_0}(x)$.

(e) These statements are all easy.

(f) If $x \in X_0$, $\sigma(x) = x(\mathscr{M})$, which cannot be empty.

(g) This follows from (f), since each $x \in K - F$ has empty spectrum.

(h) and (i) are clear.

Parts (g) and (h) show the extreme cases $X_0 = F$ and $X_0 = X$ are possible.

LEMMA 2.1. *Let M be a maximal ideal of X, and let x be an algebraic element of X having minimal polynomial $m(t) = \prod p_i(t)^{k_i}$, where the polynomials $p_i(t)$ are irreducible over F. Then $x(M)(\in X/M)$ is algebraic over F, and the irreducible polynomial over F which $x(M)$ satisfies is one of the p_i. Conversely, for each of the irreducible factors p_i of m, there is a maximal ideal M_i such that $x(M_i)(\in X/M_i)$ satisfies p_i.*

Proof. Since $m(x) = 0$, $m(x(M)) = \prod p_i(x(M))^{k_i} = 0$, for any maximal ideal M, and, indeed, one of the p_i is the irreducible polynomial which $x(M)$ satisfies.

Each of the elements $p_i(x)$ is a divisor of zero, so that each of these elements is a nonunit of X and is contained in some maximal ideal M_i. Therefore $p_i(x(M_i)) = 0$.

From this one readily obtains

THEOREM 2. *An algebraic element $x \in X$ is in X_0 if and only if its minimal polynomial factors into a product of linear polynomials.*

COROLLARY 2.1. *If F is algebraically closed, X_0 contains all algebraic elements of X.*

COROLLARY 2.2. (a) *X_0 and $(X_0)_0$ have the same algebraic elements.*

(b) *If X is algebraic (i.e., all elements of X are algebraic), $(X_0)_0 = X_0$.*

(c) *If X is finite dimensional, $(X_0)_0 = X_0$.*

Corollary 2.2 suggests the question is $(X_0)_0 = X_0$ for every algebra X? An answer to this question has not been found.

EXAMPLE. Let X be a commutative Banach algebra with identity over the real numbers, R. Then X_0 is complete by the completeness of R (and the fact $\{x_n\}$ being Cauchy implies $\{(x_n(M)\}$ is Cauchy for each $M \in \mathscr{M}$). Thus, for each $M_0 \in \mathscr{M}_0$, X_0/M_0 is a commutative Banach algebra which is a field. Hence by the Mazur-Gelfand theorem,

X_0/M_0 is isomorphic to either the real numbers or the complex numbers, C. But according to the above, $X_0/M_0 = C$ is impossible. Therefore $X_0/M_0 = R$ for each $M_0 \in \mathscr{M}_0$ or, equivalently, $(X_0)_0 = X_0$.

2. Spaces of maximal ideals.

DEFINITION. The weakest topology on $\mathscr{M}'$ for which each of the functions

$$\begin{aligned} \hat{x}\colon \mathscr{M}' &\longrightarrow F \\ M &\longrightarrow x(M) \end{aligned} \qquad (x \in X)$$

is continuous will be called the *Gelfand topology*. This definition may be applied to the algebra X_0 to obtain a Gelfand topology on $\mathscr{M}_0'$.

If $X_0 = X$, then $\mathscr{M}_0' = \mathscr{M}' = \mathscr{M}$ and the above definitions reduce to the usual definition of the Gelfand topology.

These topological spaces will be regarded as uniform spaces with the uniformity which has a base consisting of sets of the form

$$U = \{(M, M') \mid \ |x_i(M) - x_i(M')| < \varepsilon,\ i = 1, \cdots, n\} .$$

As usual, the set $U[M]$ will be written as $V(M; x_1, \cdots, x_n; \varepsilon)$.

Naturally all statements concerning $\mathscr{M}'$ have analogues concerning $\mathscr{M}_0'$.

THEOREM 3. *$\mathscr{M}'$ is complete in the Gelfand topology.*

Proof. Let $\{M_\alpha\}$ be a Cauchy net in $\mathscr{M}'$. We show that $M = \{x \in X \mid x(M_\alpha) \to 0\}$ is a maximal ideal in $\mathscr{M}'$ and that $M_\alpha \to M$.

A modification of the usual proof shows that the *spectral radius*, $r_\sigma(x) = \sup_{\alpha \in \sigma(x)} |\alpha|$ satisfies

$$r_\sigma(x) \leqq \|x\| .$$

Thus if $x \in M$ and $y \in X$, $|y(M_\alpha)x(M_\alpha)| \leqq \|y\|\,|x(M_\alpha)| \to 0$, from which it readily follows that M is an ideal. The mapping

$$\begin{aligned} F &\longrightarrow X/M \\ \alpha &\longrightarrow \alpha e + M \end{aligned}$$

is an isomorphism, because $\alpha e \in M$ implies $\alpha = \lim_\alpha \alpha e(M_\alpha) = 0$. But, since $\{M_\alpha\}$ is a Cauchy net, the M_α are eventually close of order $U = \{(M, M') \mid \ |x(M) - x(M')| < \varepsilon\}$. Therefore, for each x there is an $\alpha \in F$ such that

$$x(M_a) \longrightarrow \alpha\ ,$$
$$(x - \alpha e)(M_a) \longrightarrow 0\ ,$$
$$x + M = \alpha e + M;$$

which implies the isomorphism $\alpha \to \alpha e + M$ is onto. Hence $M \in \mathscr{M}'$.

For a point $x \in X$, let $\alpha = \lim x(M_a)$. Then $\lim (x - \alpha e)(M_a) = 0$, $x - \alpha e \in M$ and $x(M) = \alpha e(M) = \alpha = \lim x(M_a)$. Thus $M_a \in V(M; x; \varepsilon)$ for a sufficiently large. Hence $M_a \to M$, since neighborhoods of this type constitute a subbase.

An immediate consequence of Theorem 3 is that $\mathscr{M}'$ is compact if and only if it is totally bounded. In connection with the compactness of $\mathscr{M}'$, note that a modification of the proof for complex Banach algebras shows that if F is locally compact, $\mathscr{M}'$ is compact.

THEOREM 4. *The Gelfand topology on $\mathscr{M}'$ is 0-dimensional, totally disconnected and Hausdorff.*

Proof. It follows from the ultrametric inequality which the valuation satisfies that the collection of sets of the form $V(M; x_1, \cdots, x_n; \varepsilon)$ forms a base consisting of sets which are simultaneously open and closed. Hence $\mathscr{M}'$ is 0-dimensional.

Once it is shown that $\mathscr{M}'$ is Hausdorff, it will follow that $\mathscr{M}'$ is totally disconnected. For if A is a component of a 0-dimensional Hausdorff space, and it is assumed there are two distinct points x, $y \in A$, then an open and closed neighborhood of x not containing y contradicts the connectedness of A. But $\mathscr{M}'$ is Hausdorff because the family $\{\hat{x}\}$ separates points.

We state below three results that are simple extensions of results due to Beckenstein [2]:

The topology on $\mathscr{M}'$ induced by the hull-kernel topology on $\mathscr{M}$ is weaker than the Gelfand topology on $\mathscr{M}'$.

The family $\{\hat{x}\}, x \in X$, is regular with respect to the Gelfand topology (i.e., it separates point and closed sets of the Gelfand topology) if and only if the hull-kernel topology and the Gelfand topology coincide.

If $\mathscr{M}'$ is compact in the Gelfand topology, $\{\hat{x}\}, x \in X$, is regular if and only if $\mathscr{M}'$ is Hausdorff in the hull-kernel topology.

One might topologize $\mathscr{M}$ with the functions $\{\hat{x}\}$, $x \in X_0$; however, $\mathscr{M}$ topologized in this fashion need not enjoy the same properties that $\mathscr{M}'$ does. For example, $\mathscr{M}$ may receive the trivial topology (this happens if and only if $X_0 = \mathscr{R} \oplus F$).

3. Representation theorems. For a closed subspace A of a normed linear space Y, the symbol $\|\ \ \|$ on Y/A will denote the infimum norm $\|y + A\| = \inf_{a \in A} \|y + a\|$. Then the identification of $\alpha \in F$ with $\alpha e(M) \in (X/M, \|\ \ \|)$, $M \in \mathscr{M}$, is an isometry as well as an isomorphism.

We define the rings $V = \{x \in X \mid \|x\| \leqq 1\}$ and

$$W = \{x \in X \mid \|x(M)\| \leqq 1 \quad \text{for all} \quad M \in \mathscr{M}\} .$$

Certainly $V \subset W$.

For T a compact topological space we denote by $F(T)$ the Banach algebra of all continuous F-valued functions on T with pointwise operations and $\|f\| = \max_{t \in T} |f(t)|$.

Note that X_0 is a closed subalgebra since F is complete.

In this section $\mathscr{M}_0'$ (which coincides with $\mathscr{M}$ if $X_0 = X$) will be assumed to have the Gelfand topology.

LEMMA 5.1. *If $(X_0)_0 = X_0$, $V_0 = W_0$ and $\mathscr{M}_0'$ is compact, then $r_\sigma(x) = \|x\|$ for each $x \in X$.*

Proof. Since $(X_0)_0 = X_0$, $W_0 = \{x \in X_0 \mid r_\sigma(x) \leqq 1\}$. Suppose $x \in X_0$ is such that

$$r_\sigma(x) < \|x\| .$$

Since $|x(\mathscr{M}_0')|$ is a continuous image of a compact set, there exists $\alpha \in x(\mathscr{M}_0')$ such that $|\alpha| = \sup |x(\mathscr{M}_0')| = r_\sigma(x)$. Then

$$1 = r_\sigma(x/\alpha) < \|x/\alpha\| ,$$

which implies $x/\alpha \in W_0$ and $x/\alpha \notin V_0$, contradicting the hypothesis.

THEOREM 5. *If $(X_0)_0 = X_0$, $V_0 = W_0$ and $\mathscr{M}_0'$ is compact, then X_0 is isometrically isomorphic to $F(\mathscr{M}_0')$.*

Proof. Consider

$$\begin{aligned} h\colon X_0 &\longrightarrow F(\mathscr{M}_0') \\ x &\longrightarrow \hat{x} . \end{aligned}$$

This is clearly a homomorphism. But

$$\|\hat{x}\| = \max_{M \in \mathscr{M}_0'} |\hat{x}(M)| = r_\sigma(x) = \|x\| ,$$

so that X_0 is isometrically isomorphic to $h(X_0)$. But $h(X_0)$ separates points, contains constants and, by virtue of the map h, is complete. The theorem now follows from a result of Kaplansky [5] (see also Cantor [3] and Chernoff et al. [4]): *If B is a subalgebra of $F(T)$, where T is a compact, totally disconnected topological space, and B separates points and contains constants, then B is dense in $F(T)$.*

LEMMA 6.1. *If $V = W$ and $\mathscr{M}_0'$ is compact, then $r_\sigma(x) = \|x\|$ for each $x \in X_0$.*

Proof. As in Lemma 5.1, $r_\sigma(x) < \|x\|$ would lead to $x/\alpha \in W \cap X_0 = \{x \in X_0 \mid r_\sigma(x) \leqq 1\}$ and $x/\alpha \notin V \cap X_0$.

THEOREM 6. *If $V = W$ and $\mathscr{M}_0'$ is compact, then X_0 is isometrically isomorphic to $F(\mathscr{M}_0')$.*

Proof. Choose h as above.

COROLLARY 6.1. *If $V = W$ and $\mathscr{M}_0'$ is compact, then $(X_0)_0 = X_0$.*

Proof. By Theorems 4 and 6 we may assume $X = F(T)$, where T is a 0-dimensional compact Hausdorff space. The corollary is a consequence of the following (Narici [7], Ths. 2 and 3): *The maximal ideals of $F(T)$ are the sets of the form $M_t = \{f \in F(T) \mid f(t) = 0\}$,* and *$F(T)/M_t$ is isomorphic to F.*

COROLLARY 6.2. (*Narici*) *Suppose that $X/M = F$ for every maximal ideal M, that $V = W$, and that $\mathscr{M}$ is compact in the Gelfand topology. Then X is isometrically isomorphic to $F(\mathscr{M})$ under the mapping $x \to \hat{x}$.*

Note that $V = W$ implies $V_0 = W_0$. For certainly $V_0 = V \cap X_0$; if $x \in W_0$, then

$$\sup_{M \in \mathscr{M}} \|x(M)\| = \sup_{M \in \mathscr{M}_0'} \|x(M)\| \leqq \sup_{M \in \mathscr{M}_0} \|x(M)\| \leqq 1 ,$$

which implies $x \in W$. Hence $W_0 \subset W \cap X_0 = V \cap X_0 = V_0$.

Let $\hat{X}$ be a commutative Banach algebra without identity (over a complete nontrivially valued field), and suppose that X is the adjunction of identity to $\hat{X}$: X is the orthogonal sum of F and $\hat{X}$ (see Monna and Springer [6]), and multiplication in X is defined by $(\alpha + x)(\beta + y) = \alpha\beta + \alpha y + \beta x + xy$. Let $\hat{V} = \{x \in \hat{X} \mid \|x\| \leqq 1\}$. Then $V = \mathfrak{o} + \hat{V}$, where $\mathfrak{o}$ is the ring of integers of F. $\hat{\mathscr{M}}$ will denote the set of all regular maximal ideals of $\hat{X}$, and $x(\hat{M})$ will denote $x + \hat{M}$, where $\hat{M} \in \hat{\mathscr{M}}$. For each $\hat{M} \in \hat{\mathscr{M}}$, $\hat{X}/\hat{M}$ contains the field $Fe_{\hat{M}}$, where $e_{\hat{M}}$ is the identity of $\hat{X}/\hat{M}$. $Fe_{\hat{M}}$ is isomorphic to F under $\alpha \leftrightarrow \alpha e_{\hat{M}}$, but $\| \alpha e_{\hat{M}} \| = |\alpha| \, \| e_{\hat{M}} \|$, so that this is not an isometry unless $\| e_{\hat{M}} \| = 1$. Let $\hat{\mathscr{M}}'$ denote the set of regular maximal ideals of $\hat{X}$ whose quotient fields are (algebra) isomorphic to F. The mapping $M \to M \cap \hat{X}$ is a 1-to-1 correspondence between $\mathscr{M} - \{\hat{X}\}$ and $\hat{\mathscr{M}}$ which preserves quotient fields (and hence is also a 1-to-1 correspondence between $\mathscr{M}' - \{\hat{X}\}$ and $\hat{\mathscr{M}}'$).

DEFINITION. $\hat{X}_0 = \{x \in \hat{X} \mid x(\hat{M}) \in Fe_{\hat{M}} \text{ for all } \hat{M} \in \hat{\mathscr{M}}\}$.

$$\hat{W} = \{x \in \hat{X} \mid \; \| x(\hat{M}) \| \leqq \| e_{\hat{M}} \| \quad \text{for all} \quad \hat{M} \in \hat{\mathscr{M}}\} .$$

Clearly $F + \hat{X}_0 \subset X$ is the adjunction of identity to $\hat{X}_0$.

LEMMA. (a) $X_0 = F \oplus \hat{X}_0$.

(b) *$M \to M \cap \hat{X}_0$ is a 1-to-1 correspondence between $\mathscr{M}_0' - \{\hat{X}_0\}$ and $\hat{\mathscr{M}}_0'$.*

(c) *If $\hat{X}/\hat{M} = F$ for each $\hat{M} \in \hat{\mathscr{M}}$, then $W = \mathfrak{o} + \hat{W}$.*

Proof. (a) Since $X/\hat{X} = F$, $\alpha + x \in X_0$ if and only if $(\alpha + x)(M) \in F$ for all $M \in \mathscr{M} - \{\hat{X}\}$. Now suppose $M \in \mathscr{M} - \{\hat{X}\}$, $\hat{M} = M \cap \hat{X}$ and $u \in \hat{X}$ is an identity modulo $\hat{M}$ (i.e., $u(\hat{M}) = e_{\hat{M}}$). Then $M = \{\alpha + x \in X \mid (\alpha + x)u \in \hat{M}\}$, which implies

$$((1 - u)u)(\hat{M}) = e_{\hat{M}} - e_{\hat{M}}^2 = 0 \quad \text{and} \quad 1 - u \in M .$$

Therefore, if $\beta \in F$,

$$\begin{aligned} x(M) = \beta &\iff 0 \equiv x - \beta \equiv x - \beta u \quad (M) \\ &\iff (\text{since } x, \beta u \in \hat{X}) x \equiv \beta u \quad (\hat{M}) \\ &\iff x(\hat{M}) = \beta e_{\hat{M}} . \end{aligned}$$

Thus by the 1-to-1 correspondence between $\mathscr{M} - \{\hat{X}\}$ and $\hat{\mathscr{M}}$, $\alpha + x \in X_0$ if and only if $x \in \hat{X}_0$.

(b) follows from (a).

(c) The assumption $\hat{X} = \hat{X}_0$ implies $X = X_0$, so that $\| (\alpha + x)(M) \|$, $M \in \mathscr{M}$, may be written $|(\alpha + x)(M)|$. Therefore

$$\begin{aligned} \alpha + x \in W &\iff |(\alpha + x)(M)| = |\alpha + x(M)| \leqq 1 \quad \text{for all} \quad M \in \mathscr{M} \\ &\iff \alpha \in \mathfrak{o} \quad \text{and} \quad x(M) \in \mathfrak{o} \quad \text{for all} \quad M \in \mathscr{M} - \{\hat{X}\} \\ &\iff \alpha \in \mathfrak{o} \quad \text{and} \quad x(\hat{M}) \in \mathfrak{o} e_{\hat{M}} \quad \text{for all} \quad \hat{M} \in \hat{\mathscr{M}} \\ &\iff \alpha + x \in \mathfrak{o} + \hat{W} . \end{aligned}$$

COROLLARY 6.3. *If $\hat{X}_0 = \hat{X}$, $\hat{V} = \hat{W}$ and $\mathscr{M}$ is compact, then $\hat{X}$ is isometrically isomorphic to the subalgebra of functions in $F(\mathscr{M})$ which vanish at $\hat{X}$.*

Proof. If $\hat{X} = \hat{X}_0$, then $W = \mathfrak{o} + \hat{W} = \mathfrak{o} + \hat{V} = V$ and $X = X_0$, so Corollary 6.2 may be used.

COROLLARY 5.1. *If $(\hat{X}_0)_0 = \hat{X}_0$, $\hat{V}_0 = \hat{W}_0$ and $\mathscr{M}_0'$ is compact, then $\hat{X}_0$ is isometrically isomorphic to the subalgebra of functions in $F(\mathscr{M}_0')$ which vanish at $\hat{X}_0$.*

Recall that if the ground field is locally compact, $\mathscr{M}$ in Corollary 6.3 and $\mathscr{M}_0'$ in Corollary 5.1 are compact.

4. Miscellaneous.

THEOREM 7. *Let Y be a commutative normed algebra with identity e over the valued field F, and let X be a finite dimensional algebra over Y with Y-basis $\{u_1, \cdots, u_n\}$ and structure constants defined by $u_iu_j = \sum_k c_{ij}^k u_k$. If X has an identity, we assume that it coincides with the identity of Y and that $u_1 = e$. Then, viewing X as a normed linear space with the norm $\| \sum a_iu_i \| = \max \| a_i \|$, $a_i \in Y$, the following are equivalent:*

(a) *X is a normed algebra over F.*

(b) *$\| u_iu_j \| \leqq \| u_i \| \| u_j \|$ for all i and j.*

(c) *$c_{ij}^k \in V_Y$ for all i, j and k, where $V_Y = \{y \in Y \mid \| y \| \leqq 1\}$.*

Proof. The equivalence of (b) and (c) follows from the equalities $\| u_i \| \| u_j \| = \| e \| \| e \| = 1$ (which follows from the representations $u_i = eu_i$, $u_j = eu_j$) and $\| u_iu_j \| = \| \sum_k c_{ij}^k u_k \| = \max_k \| c_{ij}^k \|$.

Certainly (a) implies (b), so that to complete the proof it suffices to show (c) implies (a). It is easy to verify that, independent of the nature of the c_{ij}^k, X is an algebra over F and a normed vector space. In fact X is isomorphic as a normed space to an orthogonal direct sum of n copies of Y (which shows that X is complete if Y is). If e is the identity of X, our assumption $u_1 = e$ guarantees that $\| e \| = 1$. Therefore, to show (c) implies (a), it only remains to show $\| xy \| \leqq \| x \| \| y \|$ for all $x, y \in X$.

Let $x = \sum a_iu_i$ and let $y = \sum b_iu_i$. Then

$$\begin{aligned} \| xy \| &= \| \sum_k (\sum_{i,j} a_ib_jc_{ij}^k)u_k \| \\ &\leqq \max_k (\max_{i,j} \| a_ib_jc_{ij}^k \|) \\ &\leqq \max_{i,j,k} \| a_i \| \| b_j \| \| c_{ij}^k \| \\ &\leqq \| x \| \| y \|. \end{aligned}$$

COROLLARY 7.1. *Let Y be as above, and let $X = Y[z]$, where z*

satisfies a monic polynomial, say of degree n, with coefficients in V_Y (i.e., z is integral over V_Y). Then

(a) *X is a normed algebra with $\|\sum_0^{n-1} y_i z^i\| = \max \|y_i\|$;*

(b) *$\|z\| = 1$, i.e., a unit vector has been adjoined to Y;*

(c) *$\|X\| = \|Y\|$; and*

(d) *X is a commutative Banach algebra with identity if Y is.*

Proof. Choosing $u_i = z^i$, $i = 0, \cdots, n-1$, a simple induction argument shows that $\|z^k\| \leqq 1$ for all k, so that

$$\|u_i u_j\| = \|z^{i+j}\| \leqq 1 = \|u_i\| \|u_j\| .$$

This establishes (a); (b) and (c) are obvious; and (d) was established in the proof of the main theorem.

Using Corollary 7.1 it is easy to construct a variety of Banach algebras X in which $F \subsetneqq X_0 \subsetneqq X$. For example, let F be a p-adic number field not containing $i = \sqrt{-1}$ (see Bachman [1]), let $X = F(i)[x] = F[i][x]$, where x has minimal polynomial $(t-1)^2$, and norm $F[i][x]$ in two stages according to Corollary 7.1.

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