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LATTICES WITH NO INTERVAL HOMOMORPHISMS

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This paper arose from the following analogous questions: (1) Does a distributive topological lattice on a continuum admit sufficiently many continuous lattice homomorphisms onto the unit interval to separate points, and (2) does a topological semilattice on a continuum admit sufficiently many continuous semilattice homomorphisms onto the unit interval to separate points? Earlier investigations of topological lattices and semilattices have provided partial positive solutions. However, examples of an infinite-dimensional distributive lattice and a one-dimensional semilattice which admit only trivial homomorphisms into the interval are presented in this paper.

A topological lattice consists of a Hausdorff space L together with a pair of continuous lattice operations $\wedge, \vee: L \times L \rightarrow L$. A topological semilattice consists of a Hausdorff space S together with a continuous semilattice operation $\wedge: S \times S \rightarrow S$. In the theory of topological lattices and semilattices, the following problem, raised by Dyer and Shields in [8], has received considerable attention: Does a distributive topological lattice (a topological semilattice) on a continuum admit sufficiently many continuous lattice (semilattice) homomorphisms onto the unit interval $[0, 1]$ to separate points?

Anderson [2] has given an affirmative answer for finite-dimensional lattices; Davies [7] and Strauss [12] have made further contributions to the problem for the lattice case. The semilattice question has been answered affirmatively for finite-dimensional semilattices on Peano continua [11]. The purpose of this paper is to provide examples that show the answer is not yes in general. We give examples of an infinite-dimensional distributive lattice and a one-dimensional semilattice which admit only trivial homomorphisms into the interval.

Since the idempotents of an abelian topological semigroup form a semilattice, these examples have ramifications with regard to representations of such semigroups. In particular, Brown and Friedberg [6] have a range space for representations (or semicharacters) of a special class of compact abelian semigroups. These representations separate points if and only if the homomorphisms of the idempotents into the interval separate points.

1. Preliminaries.

Let S be a (lower) semilattice. If $A \subset S$, we define

$$L(A) = \{y \in S: y \leqq x \text{ for some } x \in A\}$$

and

$$M(A) = \{z \in S: x \leqq z \text{ for some } x \in A\} .$$

The set A is an *ideal* if $L(A) = A$. The set A is *convex* if $x, z \in A$ and $x \leqq y \leqq z$ imply $y \in A$. The following theorem is a slight modification of a theorem of Borrego [3]. It will simplify somewhat showing multiplication is continuous at a later stage.

THEOREM 1.1. *Let S be a compact, Hausdorff space which is algebraically a semilattice. If the graph of the partial order is closed and the operation $\wedge$ is continuous at all points of the graph, then S is a topological semilattice.*

Proof. Let $\{x_\alpha\}$ and $\{y_\beta\}$ be nets converging to x and y respectively. Let z be a cluster point of the net $\{x_\alpha \wedge y_\beta\}$. Since $(x_\alpha \wedge y_\beta, x_\alpha)$ clusters to (z, x), we conclude that $z \leqq x$. Similarly we conclude $z \leqq y$; hence $z \leqq x \wedge y$. By continuity of $\wedge$ on the graph, we conclude that $x_\alpha \wedge (x \wedge y)$ converges to $x \wedge y$, $y_\beta \wedge (x \wedge y)$ converges to $x \wedge y$, and hence $(x_\alpha \wedge y_\beta) \wedge x \wedge y$ converges to $x \wedge y$. Thus $x \wedge y \leqq z$ since $x_\alpha \wedge y_\beta$ clusters to z. Hence $z = x \wedge y$. Since $x \wedge y$ is the only cluster point, multiplication is continuous.

The next theorem is an unpublished result of D. R. Brown although apparently other researchers in the area of topological lattices and semilattices were aware of it independently.

THEOREM 1.2. *Let S be a compact topological semilattice. Then the space S' of all closed ideals, ordered by inclusion, is a compact distributive topological lattice. The mapping sending s into $L(s)$ is a topological isomorphism from S into S'. If S is connected (metrizable), then S' is connected (metrizable).*

Proof. The space S' of closed ideals is known to be a compact topological semigroup with respect to the operation

$$A \cdot B = \{a \wedge b: a \in A, b \in B\}$$

[9, A-7.2]. Since A and B are ideals, $A \cdot B \subset A \cap B \subset (A \cap B)^2 \subset A \cdot B$. Hence $A \cdot B = A \cap B$. The union of two closed ideals is another such, and a straightforward argument shows that this operation is continuous. Hence S' is a compact distributive topological lattice.

The mapping G sending s into $L(s)$ is an isomorphism since

$$L(s \wedge t) = L(s) \cap L(t) .$$

By continuity of multiplication on the space of all closed subsets, if a net s_α converges to s, then $S \wedge s_\alpha$ converges to $S \wedge s$. Since $L(s) = S \wedge s$, the mapping G is continuous and hence a homeomorphism.

If S is connected, then S' is connected since as a compact lattice it is generated by $G(S)$. If S is metric, it is well-known that the space of closed subsets is metrizable; hence the subset S' is metrizable.

We now define a series which will be employed in the definition of the first example. For each positive integer n larger than 1, we set

$$\alpha_n = 1/m2^{m-1} \text{ where } 2^{m-1} < n \leqq 2^m .$$

This series may be thought of as dividing the m-th term of the harmonic series into 2^{m-1} equal parts. Hence this series is divergent.

PROPOSITION 1.3. *For any $\varepsilon > 0$, there exists a positive integer P such that if $k \geqq P$, then $\sum_{n=2}^{k} \alpha_n + \varepsilon > \sum_{n=2}^{2k} \alpha_n$.*

Proof. We first note that $\sum_{n \in A} \alpha_n = 1/m$ if $A = \{n: 2^{m-1} < n \leqq 2^m\}$. Choose q and P such that $2/\varepsilon < q$ and $2^{q-1} < P$. If $k \geqq P$, there exists an unique m such that $2^{m-1} < k \leqq 2^m$. Then

$$\sum_{n=2}^{2k} \alpha_n \leqq \sum_{n=2}^{2^{m+1}} \alpha_n = \sum_{n=2}^{2^{m-1}} \alpha_n + \left(\frac{1}{m} + \frac{1}{m+1}\right) \leqq \sum_{n=2}^{k} \alpha_n + \frac{2}{m} .$$

Since $m \geqq q$, we have $2/m \leqq 2/q < \varepsilon$; this completes the proof.

2. Examples with no interval homomorphisms.

We first define some basic building blocks from which we construct our examples. Let H^* denote $[0, \infty]$, the extended nonnegative reals; H^* is a topological lattice with respect to its natural order. For each positive integer i, let $s(i)$ be the least integer with the property that $i \leqq \sum_{n=2}^{s(i)} \alpha_n$; such an integer exists since $\sum \alpha_n$ is divergent. We set $S_i = \prod_{n=1}^{s(i)} \{0, 1\}$; each S_i is a finite lattice with respect to the coordinatewise order with $0 < 1$.

For $x \in S_i$, $\theta(x)$ will denote the number of zero entries of x. We define $\sigma_i: S_i \to H^*$ by (i) $\sigma_i(x) = \infty$ if $\theta(x) = 0$, (ii) $\sigma_i(x) = i$ if $\theta(x) = 1$, (iii) $\sigma_i(x) = 0$ if $\theta(x) = s(i)$, and (iv) $\sigma_i(x) = i - \sum_{n=2}^{\theta(x)} \alpha_n$ for all other cases.

LEMMA 2.1. *Each σ_i is an order preserving function from S_i into H^*. If $\tau > \varepsilon > 0$ are fixed positive numbers, there exists a positive integer Q such that if $i \geqq Q$, $x, y \in S_i$, $\sigma_i(x) > \tau$, $\sigma_i(y) > \tau$, then $\sigma_i(x \wedge y) > \tau - \varepsilon$.*

Proof. That each σ_i is order preserving is a straightforward consequence of its definition.

Assume that $\tau > \varepsilon > 0$. Choose the P guaranteed by Proposition 1.3 which corresponds to ε. Choose Q larger than $\tau + \sum_{n=2}^{2P} \alpha_n$.

We suppose that $i \geqq Q$, $x, y \in S_i$, $\sigma_i(x) > \tau$ and $\sigma_i(y) > \tau$; we denote $x \wedge y$ by z. Either $\theta(z) \leqq 2\theta(x)$ or $\theta(z) \leqq 2\theta(y)$ obtains; we arbitrarily assume $\theta(z) \leqq 2\theta(x)$ (the reason one of the inequalities prevails is that $x \wedge y$ can have at most twice as many zero entries as one of x or y). We note from the definition of σ_i that in all cases $\sigma_i(z) \geqq i - \sum_{n=2}^{\theta(z)} \alpha_n$ if the summation is interpreted to be 0 for $\theta(z)$ equal to 0 or 1.

If $\theta(x) \leqq P$, then

$$\sigma_i(z) \geqq i - \sum_{n=2}^{\theta(z)} \alpha_n \geqq Q - \sum_{n=2}^{2\theta(x)} \alpha_n \geqq Q - \sum_{n=2}^{2P} \alpha_n \geqq \tau \, ;$$

the last inequality follows from the choice of Q. Hence $\sigma_i(z) > \tau - \varepsilon$ if $\theta(x) \leqq P$.

If $P < \theta(x)$, then

$$\sigma_i(z) \geqq i - \sum_{n=2}^{\theta(z)} \alpha_n \geqq i - \sum_{n=2}^{2\theta(x)} \alpha_n \geqq i - \left(\sum_{n=2}^{\theta(x)} \alpha_n + \varepsilon\right) = \sigma_i(x) - \varepsilon > \tau - \varepsilon \, .$$

Hence $\sigma_i(z) > \tau - \varepsilon$ for both cases. We now define the first example. We denote $H^* \times \prod_{i=1}^{\infty} S_i$ by K. With coordinatewise order K is a topological lattice homeomorphic to the Cartesian product of an interval and the Cantor set.

EXAMPLE 1. We define $L = \{(t, (x_i)_{i=1}^{\infty}) \in K \colon t \leqq \sigma_i(x_i) \text{ for all } i\}$. With respect to the order inherited from K, L is a compact, one-dimensional topological lattice. If A is a subsemilattice of L with respect to the cap operation and if $1 \in A^\circ$ (where 1 denotes the largest element of L), then $A \cap (0 \times \prod S_i) \neq \varnothing$.

Proof. (1) L is compact.

Suppose $(t, (x_i)) \notin L$. Then $t > \sigma_n(x_n)$ for some n. There exists an open neighborhood U of $(t, (x_i))$ such that if $(s, (y_i)) \in U$, then $s > \sigma_n(x_n)$ and $y_n = x_n$; then $s > \sigma_n(x_n) = \sigma_n(y_n)$ implies $(s, (y_i)) \notin L$. Hence $K \backslash L$ is open and L is compact.

(2) L is algebraically a lattice.

With respect to the cup operation, L is a subsemilattice of K. This follows from the fact each σ_i is order preserving.

To complete this part, we show that if $(s, (x_i)), (t, (y_i)) \in L$, then $(u, (z_i))$ is a greatest lower bound in L where

$$u = s \wedge t \wedge \inf. \{\sigma_i(z_i) \colon 1 \leqq i\}$$

and $z_i = x_i \wedge y_i$. By its definition $(u, (z_i))$ is a lower bound and a member of L. Let $(r, (w_i))$ be another lower bound for $(s, (x_i))$ and $(t, (y_i))$ in L. Then $r \leqq s \wedge t$ and $w_i \leqq x_i \wedge y_i = z_i$ for each i; hence $\sigma_i(w_i) \leqq \sigma_i(z_i)$ for each i. Since $(r, (w_i)) \in L$, then $r \leqq \inf \{\sigma_i(w_i) \colon 1 \leqq i\} \leqq \inf \{\sigma_i(z_i) \colon 1 \leqq i\}$.

Hence $r \leqq u$; and thus $(u, (z_i))$ is a *glb* in L.

(3) L is a topological lattice.

The cup operation is continuous since L is a subsemilattice of K with respect to this operation. This implies that the partial order on L has closed graph.

Let $x = (s, (x_i))$ and $y = (t, (y_i))$ be elements of L. To show continuity of multiplication, we may assume that $y \leqq x$ by Theorem 1.1.

We first consider the case that $0 < t, s < \infty$. For a positive integer N and $\varepsilon > 0$, let $W = \{(u, z_i) \in L: t - 3\varepsilon < u < t + 3\varepsilon, y_i = z_i$ for $i \leqq N\}$ be a basic neighborhood of y where $3\varepsilon < t$. Let Q be the positive integer guaranteed by Lemma 2.1 for $\tau = t - \varepsilon$ and ε; we set $M = \max\{N, Q\}$. We define neighborhoods U and V of x and y resp. by

$U = \{(s', (a_i)) \in L: s - \varepsilon < s' < s + \varepsilon, a_i = x_i$ for $i \leqq M\}$ and

$V = \{(t', (b_i)) \in L: t - \varepsilon < t' < t + \varepsilon, b_i = y_i$ for $i \leqq M\}$.

To complete the proof, we show $U \wedge V \subset W$.

Let $(s', (a_i)) \in U$ and $(t', (b_i)) \in V$ and let $(u, (z_i))$ be their greatest lower bound in L, i.e., $z_i = a_i \wedge b_i$ for all i and

$$u = s' \wedge t' \wedge \inf\{\sigma_i(z_i): 1 \leqq i\} .$$

We have immediately $u \leqq t' < t + 3\varepsilon$ and $z_i = a_i \wedge b_i = x_i \wedge y_i = y_i$ for $i \leqq N$ since $N \leqq M$. Since $(t', (b_i))$ is an element of V and hence of L, we have $t - \varepsilon < t' \leqq \sigma_i(b_i)$ for all i. Similarly since $t \leqq s$, we have $t - \varepsilon \leqq s - \varepsilon < s' \leqq \sigma_i(a_i)$. If $i \leqq M$, then

$$t - 2\varepsilon < t - \varepsilon < \sigma_i(b_i) = \sigma_i(a_i \wedge b_i) = \sigma_i(z_i)$$

since $a_i \wedge b_i = x_i \wedge y_i = y_i = b_i$. If $M < i$, then $Q < i$ and

$$(t - \varepsilon) - \varepsilon < \sigma_i(a_i \wedge b_i)$$

by Lemma 2.1. Hence $t - 3\varepsilon < t - 2\varepsilon \leqq s' \wedge t' \wedge \inf\{\sigma_i(z_i): 1 \leqq i\} = u$. Thus $(u, (z_i)) \in W$.

The case $t = 0$ is straightforward and omitted. The case that one or both of t and s are ∞ can be handled by a slight modification of the above argument.

(4) L is one-dimensional.

This follows from the fact that L is homeomorphic to a closed subset of the Cartesian product of the Cantor set and unit interval.

(5) If A is a subsemilattice and $1 \in A^\circ$, then $A \cap (0 \times \prod S_i) \neq \varnothing$.

Note that $(\infty, (x_i))$ where each x_i has entries all 1 is an element of L, and hence is the 1 for L. There exists at 1 a basis of open sets of the form $U = \{(r, (x_i)) \in L: j < r, x_i$ has entries all 1 for $i \leqq j\}$ where j is a positive integer. We assume j is chosen so that $U \subset A^\circ$. We define T to be all elements of the form $(j + 1, (x_i))$ such that x_i has entries all 1 for $i \neq j + 1$ and x_{j+1} has one zero entry. Then T has $s(j + 1)$ elements. For each element of T,

$$\inf \{\sigma_i(x_i): 1 \leqq i\} = \sigma_{j+1}(x_{j+1}) = j + 1 ;$$

hence $T \subset L$ and thus $T \subset U$. Let $(t, (z_i))$ be the greatest lower bound in L of T. Since A is a subsemilattice, $(t, (z_i)) \in A$. Since $(t, z_i) \in L$, $t \leqq \sigma_{j+1}(z_{j+1}) = 0$ since z_{j+1} has entries all zero. Hence $t = 0$.

EXAMPLE 2. Let I denote all elements of L with first entry zero; I is an ideal of L with respect to the cap operation. The Rees quotient $S = L/I$ is a compact, connected one-dimensional metric semilattice with identity which admits no nontrivial semilattice homomorphisms into the unit interval.

Proof. It is easily verified that the set $I = \{(0, (x_i)) \in L\}$ is a closed ideal of L with respect to the cap operation. Hence $S = L/I$, the Rees quotient, is a compact topological semilattice.

Since S is topologically a subset of the cone over the Cantor set, S is metric and one-dimensional. If $(t, (x_i)) \in L$, then $\{(r, (y_i)): r \leqq t, y_i = x_i \text{ for all } i\}$ is a connected subset of L which meets I. Hence in S each element lies in the component of 0; thus S is connected.

Assume that there does exist a nontrivial continuous homomorphism h from S into $[0, 1]$. Then $h(1) > h(0)$. If f denotes the natural homomorphism from L onto S, then hf is a continuous homomorphism from L into the unit interval such that $hf(I) = h(0)$. Choose r such that $h(1) > r > h(0)$. Then $(hf)^{-1}[r, h(1)]$ is a neighborhood of 1 in L, a subsemilattice of L, and misses I. However, no subset of L has these properties. Hence no nontrivial h exists.

EXAMPLE 3. Let S' denote the set of all closed ideals of S, the semilattice of Example 2. Then S' is a compact connected metrizable distributive topological lattice. With respect to the cap operation, S' has no nontrivial finite-dimensional homomorphic images; hence, in particular, S' admits no nontrivial lattice homomorphisms into the unit interval.

Proof. By Theorem 1.2 S' is a compact, connected, metrizable, distributive topological lattice and the mapping G from S into S' sending s into $L(s)$ is a topological isomorphism. Since $G(0) = 0$ and $G(1) = 1$, S' admits no nontrivial cap homomorphisms into the unit interval, because any such composed with G would be a nontrivial homomorphism from S into the interval.

Suppose that h is a continuous cap homomorphism from S' onto T, a finite-dimensional topological semilattice. Since S' is a compact, connected topological lattice, it is locally connected [1]; hence T is locally connected. But then, by [11], if T is nontrivial, it possesses nontrivial homomorphisms into the interval. The composition would be a nontrivial homomorphism from S' into the interval, and we have just seen such does not exist. Hence T is trivial.

These examples shed some light on the subject of intrinsic topologies in lattices. Birkhoff [4] describes several ways a lattice may be topologized from its algebraic structure. It has been shown that in a compact topological lattice which is metrizable the topology of the lattice agrees with the order topology (see [12] or [10]). Hence Examples 1 and 3 both have the order topology.

The question has been asked whether the topology of a compact topological lattice agrees with the interval topology [10]. Strauss [12] showed that if this is true and if the lattice is distributive, then the lattice admits nontrivial continuous homomorphisms into the unit interval. Hence the lattice of Example 3 does not have the interval topology.

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