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COMPOSITION SERIES IN CHEVALLEY ALGEBRAS

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This paper continues the study of how the ideal structure of a Chevalley algebra (a Lie algebra obtained by transferring the scalars of a finite dimensional simple Lie algebra over C to a commutative ring R with identity in which 2 and 3 are not zero divisors) depends on the ideal structure of R. Specifically, we find that composition series of ideals for the Chevalley algebras exist only in case R has composition series of ideals, and in the latter case give explicit descriptions of the composition series in the Chevalley algebras. We also give a necessary and sufficient condition for the composition series in the algebra to exactly parallel those in the ring.

In two earlier papers ([3] and [4]), we have used the fundamental procedures of Chevalley [1] to construct certain Lie algebras from the finite dimensional complex simple algebras through replacement of the scalars by elements of a commutative ring R with identity in which 2 and 3 are neither zero nor zero divisors. The main results of these papers concerned the question of to what extent simplicity of the original algebras reflects itself in the ideal structure of the new algebras, which we call Chevalley algebras. In the case when the ring R is a field of prime characteristic, what amounts to this same question was previously considered by Dieudonné [2], Ono [7], and, as a tool for studying automorphisms, Steinberg [8]. The major emphasis in [2] however was upon the nature of the composition series of ideals in the nonsimple Chevalley algebras, with explicit results being obtained for the exceptional algebras and implicit results noted in the still earlier work of Jacobson ([5] and [6]). In the present paper, we take up this topic in the setting of an arbitrary commutative ground ring with identity, with our methods once more requiring exclusion of the cases when 2 or 3 are zero or zero divisors. We obtain results which give the extent to which the nature of composition series of ideals in the Chevalley algebras is determined by the nature of the composition series of ideals in the ring R.

Let L be a simple Lie algebra of finite dimension over the complex field, H an n-dimensional Cartan subalgebra, Σ the (ordered) set of nonzero roots determined by H, and Π the set of simple roots. For r and s in Σ, we denote the Cartan integer $2(r, s)/(s, s)$ by $c(r, s)$. When referring to the length of a root r, we shall mean simply $\sqrt{(r, r)}$.

Let $B = \{e_r\} \cup \{h_i\}$ be a Chevalley basis of L. Let L_Z be the free additive abelian group generated by B. Since the structural constants of L relative to B are all integers, if we define L_R to be $R \otimes_Z L_Z$, then L_R can be viewed as a Lie algebra over R, where the multiplication table for B is used with all integers interpreted in R. Under the obvious identification, we may regard B as a basis of L_R.

Let $\{h'_1, \cdots, h'_n\}$ be a complex basis of H which is dual to the system $\Pi = \{r_1, r_2, \cdots, r_n\}$. Then $H_Z \subseteqq H'_Z$ where H_Z is the additive group generated by $\{h_1, \cdots, h_n\}$ and H'_Z is that generated by $\{h'_1, \cdots, h'_n\}$. In fact, $h_i = \sum_{j=1}^n c(r_j, r_i)h'_j$ and $H_R = R \otimes H_Z \subseteqq H'_R = R \otimes H'_Z$. There exist then basis $\{\bar{h}_1, \cdots, \bar{h}_n\}$ of H_R and $\{\bar{h}'_1, \cdots, \bar{h}'_n\}$ of H'_R such that $\bar{h}_i = d_i\bar{h}'_i$, with d_i the i^{th} elementary divisor of the Cartan matrix C of L. In the sequel we also use C to represent the linear transformation on H whose matrix relative to $\{h_1, \cdots, h_n\}$ is the Cartan matrix of L. Recalling that a simple algebra has at most two distinct root lengths, we use s and t as generic symbols for short and long roots respectively, and define E_R, E_S, and E_L to be the R-submodules of L_R generated by $\{e_r \mid r \in \Sigma\}$, $\{e_s \mid s \text{ a short root}\}$ and $\{e_t \mid t \text{ a long root}\}$. H_S and H_L are defined similarly.

The basic relationship between ideals in L_R and ideals in R tells us that only for a narrow class of our rings R will L_R possess a composition series. We remark first that the existence of a composition series of ideals in L_R is equivalent to the presence of the ascending and descending chain conditions on ideals in L_R, since the lattice of ideals of L_R is, as usual, modular. The following lemma, a consequence of this remark, now limits our ensuing discussion to rings having a composition series of ideals.

LEMMA. *If R is a ring with no composition series (in the sense of* [9]), *then L_R has no composition series of ideals.*

Proof. If R has no composition series, then there exists an infinite sequence of ideals J_i of R which is either strictly increasing or strictly decreasing. It is then easily seen that the corresponding J_iL_R are ideals in L_R and together form an infinite sequence of the same sort as the J_i form. Thus L_R has no composition series of ideals.

In the sequel, the converse of this lemma is essentially obtained through consideration first of Lie algebras L of one root length, then nonsymplectic L with two root lengths, and finally symplectic L. We in fact obtain explicit characterizations of composition series in L_R in terms of a given composition series in R.

2. Statement of results. Let

$$L_R = M_0 \supset M_1 \supset \cdots \supset M_m \supset M_{m+1} = 0$$

be a composition series of ideals for L_R. We say that this composition series is determined by a composition series in R if there exists a composision series $R = J_0 \supset J_1 \supset \cdots \supset J_m \supset J_{m+1} = 0$ such that for each i, $M_i = J_i L_R$.

THEOREM 1. *A necesssary and sufficient condition for every composition series in L_R to be determined by one in R is that* $\det C$ *and* $(t, t)/(s, s)$ *both be invertible in R.*

THEOREM 2. *Let L be of type A_n, $n \geqq 2$, D_n, n even $\geqq 4$, E_6, or E_7. Then there exists a composition series $\{J_0, J_1, \cdots, J_k, J_{k+1}\}$ in R such that*

$$\begin{aligned} M_1 &= J_1 E_R + J_1 \bar{h}_1 + \cdots + J_1 \bar{h}_{n-1} + R\bar{h}_n , \\ M_2 &= J_1 L_R , \\ M_3 &= J_2 E_R + J_2 \bar{h}_1 + \cdots + J_2 \bar{h}_{n-1} + J_1 \bar{h}_n , \\ M_4 &= J_2 L_R, \cdots M_{m-1} = J_k L_R, \qquad M_m = J_k \bar{h}_n . \end{aligned}$$

If L is of type E_8, then every composition series for L_R is determined by one in R.

THEOREM 3. *Let L be of type D_n, $n \geqq 3$ odd. Then there exists a composition series $\{J_0, J_1, \cdots, J_k, J_{k+1}\}$ in R such that*

$$\begin{aligned} M_1 &= J_1 E_R + J_1 \bar{h}_1 + \cdots + J_1 \bar{h}_{n-2} + R\bar{h}_{n-1} + R\bar{h}_n , \\ M_2 &= J_1 E_R + J_1 \bar{h}_1 + \cdots + J_1 \bar{h}_{n-2} + J^{(1)} \bar{h}_{n-1} + J^{(2)} \bar{h}_n \quad \text{where one of} \\ &\quad J^{(1)}, J^{(2)} \text{ is } R \text{ and the other is } J_1 , \\ M_3 &= J_1 L_R , \\ M_4 &= J_2 E_R + J_2 \bar{h}_1 + \cdots + J_2 \bar{h}_{n-2} + J^{(3)} \bar{h}_{n-1} + J^{(4)} \bar{h}_n \quad \text{where one of} \\ &\quad J^{(3)}, J^{(4)} \text{ is } J_1 \text{ and the other is } J_2 , \\ &\vdots \\ M_{m-1} &= J_k \bar{h}_{n-1} + J_k \bar{h}_n , \\ M_m &= J_k \bar{h}_{n-1} \quad \text{or} \quad J_k \bar{h}_n . \end{aligned}$$

THEOREM 4. *Let L be of type B_n, $n \geqq 3$. Let $\{\bar{h}_i\}$ be the basis of Theorem 7.3 of* [3]. *Then there exists a composition series $\{J_0, J_1, \cdots, J_k, J_{k+1}\}$ in R such that*

$$\begin{aligned} M_1 &= J_1 E_L + RE_S + R\bar{h}_1 + J_1 \bar{h}_2 + R\bar{h}_3 + J_1 \bar{h}_4 + \cdots + J_1 \bar{h}_{n-1} + R\bar{h}_n , \\ M_2 &= J_1 E_L + RE_S + R\bar{h}_1 + J_1 \bar{h}_2 + J_1 \bar{h}_3 + J_1 \bar{h}_4 + \cdots + J_1 \bar{h}_{n-1} + R\bar{h}_n \\ &\quad \text{or } J_1 E_L + RE_S + R\bar{h}_1 + J_1 \bar{h}_2 + R\bar{h}_3 + J_1 \bar{h}_4 + \cdots + J_1 \bar{h}_{n-1} + J_1 \bar{h}_n , \\ M_3 &= J_1 E_L + RE_S + R\bar{h}_1 + J_1 \bar{h}_2 + J_1 \bar{h}_3 + J_1 \bar{h}_4 + \cdots + J_1 \bar{h}_{n-1} + J_1 \bar{h}_n , \\ M_4 &= J_1 E_R + J_1 \bar{h}_1 + \cdots + J_1 \bar{h}_{n-1} + R\bar{h}_n , \end{aligned}$$

$$
\begin{aligned}
M_5 &= J_1 L_R ,\\
&\vdots\\
M_{m-2} &= J_k L_R ,\\
M_{m-1} &= J_k E_S + J_k \bar{h}_1 + J_k \bar{h}_3 + J_k \bar{h}_n ,\\
M_m &= J_k \bar{h}_n .
\end{aligned}
$$

THEOREM 5. *Let L be of type F_4 and $\{\bar{h}_i\}$ be the basis of Theorem* 7.5 *of* [3]. *Then there exists a composition series $\{J_0, J_1, \cdots, J_k, J_{k+1}\}$ of ideals in R such that*

$$
\begin{aligned}
M_1 &= J_1 E_L + R E_S + R\bar{h}_1 + R\bar{h}_2 + J_1\bar{h}_3 + J\bar{h}_4 ,\\
M_2 &= J_1 L_R ,\\
M_3 &= J_2 E_L + J_1 E_S + J_1\bar{h}_1 + J_1\bar{h}_2 + J_2\bar{h}_3 + J_2\bar{h}_4 ,\\
M_4 &= J_2 L_R ,\\
&\vdots\\
M_{m-1} &= J_k L_R ,\\
M_m &= J_k E_S + J_k\bar{h}_1 + J_k\bar{h}_2 .
\end{aligned}
$$

THEOREM 6. *Let L be of type G_2 and let $\{\bar{h}_i\}$ be the basis of Theorem* 7.7 *of* [3]. *Then there is a composition series*

$$\{J_0, J_1, \cdots, J_k, J_{k+1}\}$$

of R such that

$$
\begin{aligned}
M_1 &= J_1 e_1 + R e_2 + J_1\bar{h}_1 + R\bar{h}_2 ,\\
M_2 &= J_1 L_R ,\\
M_3 &= J_2 e_1 + J_1 e_2 + J_2\bar{h}_1 + J_1\bar{h}_2 ,\\
M_4 &= J_2 L_R ,\\
&\vdots\\
M_{m-1} &= J_k L_R ,\\
M_m &= J_k e_2 + J_k\bar{h}_2 .
\end{aligned}
$$

3. Proof. The proofs of our results depend of course on the nature of the ideals in L_R, a characterization of which is found in [3]. When appropriate, we shall refer to results in [3] by number without giving the explicit statements themselves.

3.1. *Proof of Theorem* 1. By Theorem 3.3 of [3], every ideal in L_R has the form JL_R for some ideal J in R if and only if the two integers $\det C$ and $(t, t)/(s, s)$ are invertible in R. If these integers are invertible and a composition series $\{M_0, M_1, M_2, \cdots, M_m, M_{m+1}\}$ is given in L_R, then we have $M_i = J_i L_R$ for some ideal J_i in R, $i = 1, 2, \cdots, m$. Since no ideals exist in L_R between M_i and M_{i+1}, neither do any exist in R between J_i and J_{i+1}. Hence $\{J_0, J_1, \cdots, J_m, J_{m+1}\}$ is a composition series in R which determines the given series in L_R. Conversely, if every composition series in L_R consists of terms of the form $J_i L_R$ where $\{J_i\}$ is some composition series in R, then no ideals can exist in L_R which are not of the form JL_R for some ideal J in R. Then $\det C$ and $(t, t)/(s, s)$ are invertible in R.

3.2. *Proof of Theorems* 2 *and* 3. Given the composition series in L_R, we know that M_1, being a maximal ideal, has the asserted form for J_1 a maximal ideal in R, by virtue of Theorem 6.3 of [3] in all cases except E_8. For E_8 however, the conclusion of Theorem 1 is available since $\det C = 1$ and there is only one root length. In view of Theorems 3.4 and 6.2 of [3], in order for no ideal of L_R to exist between M_1 and M_2, it must be that M_2 has the asserted form also, and similarly for M_3 in the case D_n, n odd. Again by the above quoted theorems, if no ideals in L_R exist between M_2 and M_3 (M_3 and M_4 in case D_n, n odd), then there must exist an ideal J_2 of R, with J_2 maximal among the ideals of R contained in J_1 and having the property that M_3 (M_4 in case D_n, n odd) has the asserted form. Repetition of this reasoning at each stage yields the desired composition series in R and completes the proof.

3.3. *Proof of Theorems* 4, 5, *and* 6. We reason as in 3.2, this time calling upon the relevant theorems in [3] for the nonsymplectic algebras of two root lengths. The maximal ideal M_1 has the form asserted for some maximal ideal J_1 in R by appeal to Theorems 7.4, 7.6, and 7.8 of [3] in the respective cases B_n, F_4 and G_2. Since no ideals in L_R exist between M_1 and M_2, we use Theorems 3.5, 7.3, 7.5, and 7.7 of [3] to determine the nature of M_2. We know in each case that $M_2 \cap E_R = J_1 E_L + RE_S$ and that

$$J_1 H_L + RH_S \subseteqq M_2 \cap H_R \subseteqq C^{-1}(RH_S + J_1 H_L) .$$

Thus to preclude ideals between M_1 and M_2 we need only make $M_2 \cap H_R$ a maximal R-submodule of $M_1 \cap H_R$, all in view of 3.5 of [3]. The subsequently listed results merely prescribe that M_2 then has the form asserted in Theorems 4, 5, and 6 in the respective cases B_n, F_4, and G_2. The same combination of references is effective in producing the ideals of R needed to complete the composition series below J_1 and with it the proof.

4. **The symplectic algebras.** If L is of type C_n, $n \geqq 2$, the ideal structure of L_R is far less tidy than in the other cases, so much

so that the concrete representations of the ideals (and so of the composition series of ideals) in L_R given above in terms of simply chosen bases just no longer exist. Using Theorem 3.6 of [3] however, we can at least describe a composition series in L_R module the nature of composition series of R-submodules in H_R. Since M_1 is a maximal ideal of L_R, we know that $M_1 \cap E_S = J_1E_S$ for some maximal ideal J of R. Moreover, $M_1 \cap (E_L + H_R) \subseteqq J_1E_L + C^{-1}(J_1H_R)$. By maximality then M_1 must be $J_1E_R + C^{-1}(J_1H_R)$. We have that $C^{-1}(J_1H_R) = J_1H'_R$, and $\bar{h}_i = \bar{h}'_i$, $i = 1, \cdots, n-1$, with $\bar{h}_n = 2\bar{h}'_n$. If J_1 contains 2, then writing $h = \sum n_i\bar{h}_i$ in $C^{-1}(J_1H_R)$, we have $n_n \in (1/2)J_1 = R$. The same is true if J_1 fails to contain 2, except that $(1/2)J_1 = J_1$. In the latter case, $M_1 = J_1L_R$; in the former $M_1 = J_1E_R + J_1\bar{h}_1 + \cdots + J_1\bar{h}_{n-1} + R\bar{h}_n$. Now for any R-module $N' = J'E_L + \tilde{H}$ where $2J_1 \subseteqq J' \subseteqq J_1$ and $J'H_L + J_1H_S \subseteqq \tilde{H} \subseteqq C^{-1}(J'H_L + J_1H_S)$, $N' + J_1E_S$ will be an ideal in L_R. The first step in constructing M_2 then is to find a J_2 in R maximal among the R-ideals contained in J_1 which also contain $2J_1$. Then one constructs M_2 and the next few M_i by determining which $\tilde{H}$ can be fitted into a composition series through $C^{-1}(J_2H_L + J_1H_S)$ so as to contain $J_2H_L + J_1H_S$. Then the whole process breaks into two possibilities. One either constructs an M with $M \cap E_S = J_2E_s$, and repeats the above steps with J_2 in place of J_1, or else finds a J_3 maximal in J_2 which contains $2J_1$ and looks for additional $\tilde{H}$. As can be seen, numerous alternative paths exist for finishing the composition series in L_R through construction of one in R.

References

1. C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2) **7** (1955), 14-66.
2. J. Dieudonné, *Les algèbres de Lie simples associées aux groupes simples algébriques sur un corps de charactéristique* $p > 0$, Rend. Circ. Mat. Palermo (2) **6** (1957), 198-204.
3. J. Hurley, *Ideals in Chevalley algebras*, Trans. Amer. Math. Soc. **137** (1969), 245-258.
4. ———, *Extensions of Chevalley algebras* (to appear)
5. N. Jacobson, *Classes of restricted Lie algebras of characteristic p I*, Amer. J. Math. **63** (1941), 481-515.
6. ———, *Restricted Lie algebras of characteristic p*, Trans. Amer. Math. Soc. **50** (1941), 15-25.
7. T. Ono, *Sur les groupes de Chevalley*, J. Math. Soc. Japan **10** (1958), 307-313.
8. R. Steinberg, *Automorphisms of classical Lie algebras*, Pacific J. Math. **11** (1961), 1119-1129.
9. O. Zariski and P. Samuel, *Commutative algebra*, Volume 1, Princeton, Van Nostrand, 1958.

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