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A UNIQUENESS THEOREM FOR SECOND ORDER QUASILINEAR HYPERBOLIC EQUATIONS

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A uniqueness theorem is proved for weak solutions of quasilinear second-order hyperbolic equations of the form

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, t, u, u_1 \cdots, u_n) = b(x, t, u)$$

in many space variables. The weak solutions are assumed to satisfy a time-wise upper Lipschitz bound

$$\frac{u_k(x, t_1) - u_k(x, t_2)}{t_1 - t_2} \leqq K(t)$$

for all $0 < t \leqq t_1, t_2$ where $K(t)$ is an L^1-function. Together with the obvious assumptions, the equation is supposed to satisfy a symmetry condition

$$\frac{\partial a^i}{\partial u_j} = \frac{\partial a^j}{\partial u_i}$$

along with convexity of the a^i in u and u_k. As a corollary, a uniqueness theorem for systems proved by Oleǐnik is generalized.

In recent years a number of authors have studied quasilinear hyperbolic equations and systems with the goal of obtaining general existence and uniqueness theorems for the initial problem. Regular or smooth solutions do not usually exist for these problems, and so one tries to establish the existence of weak or generalized solutions of various types. The uniqueness question for such solutions is then somewhat more tenuous than that for smooth solutions, and usually involves the assumption of some sort of one-sided Lipschitz estimate on the solution.

The first comprehensive attack on these problems began with Oleǐnik's paper [6] in which she established existence and uniqueness results for generalized solutions of first order equations of the form

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi}{\partial x}(x, t, u) + \psi(x, t, u) = 0 . \tag{1}$$

The function φ was subject to a convexity assumption

$$\frac{\partial^2 \varphi}{\partial u^2} \geqq 0 . \tag{2}$$

To prove uniqueness she used a variant of the method of Holmgren (see [3]), assuming that the generalized solutions u were bounded measurable, and satisfied a Lipschitz bound of the form

$$\frac{u(x_1, t) - u(x_2, t)}{x_1 - x_2} \leqq K(x_1, x_2, t) . \tag{3}$$

Since Oleĭnik's paper much effort has been directed to generalizing her results in two directions;

(a) to systems of first order equations (see e.g. [4]), and

(b) to equations in many space variables ([1], [2]).

However, little progress has been made on a corresponding general theory for higher order equations, with no existence theorems having yet been produced.

This paper is devoted to proving a uniqueness theorem for weak solutions of the initial value problem for second order symmetric (Assumption II) quasilinear hyperbolic equations in several space variables. A variant of the Holmgren method is again used, except that energy estimates are used in place of pointwise estimates. The same method has been applied to symmetric first order systems in [5]. We also require convexity-type assumptions on the equation (Assumption IV). But in interesting contrast with the case of first order equations, we are led to impose time-wise Lipschitz bounds on the solution (Assumption B) in place of the space-wise Lipschitz bounds (3).

In the last part of the paper our result is used to generalize a uniqueness theorem for a hyperbolic system of two first order quasilinear equations which was obtained by Oleĭnik [7]. The generalization essentially amounts to weakening the convexity condition, and replacing a constant Lipschitz bound by an L^1 function. It would seem that more substantial extensions of Oleinik's result are possible using the technique presented here.

2. The uniqueness theorem. In the region

$$D = \{(x, t): x = (x_1, \cdots, x_n) \in R_n; t \text{ real}, 0 < t < \infty\}$$

we consider the second order quasilinear hyperbolic equation

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, t, u, u_1, \cdots, u_n) - b(x, t, u) = 0 \tag{4}$$

for the function $u(x, t)$, where we have used the notation

$$\frac{\partial \psi}{\partial x_k} = \psi_k \quad \text{and} \quad \frac{\partial \psi}{\partial t} = \psi_t .$$

The equation will be subject to the following assumptions.

I. The functions $a^i(x, t, u, u_k)$ and $b(x, t, u)$ are defined for all

x, t, u, u_k satisfying $-\infty < x, u, u_k < \infty, 0 < t < \infty$, and are differentiable with respect to these variables, the derivatives being uniformly Lipschitz continuous on compact subsets of $\bar{D}$. Also $\partial a^i/\partial u_j$ is continuously differentiable with respect to t, u and u_k.

II. (*Symmetry*) If

$$\frac{\partial a^i}{\partial u_j}(x, t, u, u_k) = a^{ij}(x, t, u, u_k)$$

then

$$a^{ij}(x, t, u, u_k) = a^{ji}(x, t, u, u_k) \, . \tag{5}$$

III. Given positive constants M and T there are corresponding constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} c_1 \sum_{i=1}^{n} \xi_i^2 &\geqq \sum_{i,j=1}^{n} a^{ij}(x, t, u, u_k)\xi_i\xi_j \\ &\geqq c_2 \sum_{i=1}^{n} \xi_i^2 \end{aligned} \tag{6}$$

for all vectors $(\xi_1, \cdots, \xi_n)$ if $(x, t) \in R_n \times [0, T]$ and

$$|u| + \sum_{j=1}^{n} |u_j| \leqq M \, .$$

IV. (*Convexity*). For all vectors $(\xi_1, \cdots, \xi_n)$ we have

$$\sum_{i,j=1}^{n} \frac{\partial a^{ij}}{\partial u}(x, t, u, u_k)\xi_i\xi_j \leqq 0 \tag{7a}$$

and

$$\sum_{i,j=1}^{n} \frac{\partial a^{ij}}{\partial u_k}(x, t, u, u_k)\xi_i\xi_j \leqq 0 \qquad (k = 1, \cdots, n) \, . \tag{7b}$$

V. The functions

$$\frac{\partial a^{ij}}{\partial t}(x, t, u, u_k), \frac{\partial a^i}{\partial u}(x, t, u, u_k) \quad \text{and} \quad \frac{\partial b}{\partial u}(x, t, u)$$

are uniformly bounded on compact subsets of (x, t, u, u_k) space.

We will be concerned with weak solutions of (4).

DEFINITION. Let $f(x)$ and $g(x)$ be essentially bounded measurable functions on R_n. A weak solution $u(x, t)$ of (4) on D with initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x)$$

is an essentially bounded measurable function possessing essentially bounded measurable weak (i.e., distribution) derivatives u_k and u_t a.e. on D which satisfy the following conditions:

A. For every twice continuously differentiable test function $\varphi(x, t)$ on $\bar{D}$ which vanishes for large $|x| + t$, $(|x| = (x_1^2 + \cdots + x_n^2)^{1/2})$ we have

$$\begin{aligned}\iint_D\Big[u\,\varphi_{tt} + \sum_{i=1}^n a^i(x, t, u, u_k)\varphi_i - b(x, t, u,)\varphi\Big]dxdt \\ + \int_{R_n} f(x)\varphi_t(x, 0)dx - \int_{R_n} g(x)\varphi(x, 0)dx = 0\,. \end{aligned} \tag{8}$$

B. Given any compact subset R of R_n, and T, $0 \leqq T < \infty$ there is a function $K(t) \in L^1(0, T)$ such that

$$\frac{u_k(x, t_1) - u_k(x, t_2)}{t_1 - t_2} \leqq K(t) \qquad (k = 1, \cdots, n) \tag{9a}$$

holds a.e. for all $x \in R$, and $0 < t_1 < t_2 \leqq t < T$.[1]

Any twice continuously differentiable (smooth) solution of (4) is a weak solution as is easily seen by applying the divergence theorem.

Before presenting the main result of the paper we establish a lemma concerning energy inequalities for twice continuously differentiable solutions of hyperbolic equations of the form

$$\begin{aligned} L\varphi = \varphi_{tt} - \sum_{i,j=1}^n (\alpha^{ij}(x, t)\varphi_i)_j \\ + \sum_{i=1}^n \alpha^i(x, t)\varphi_i - \beta(x, t)\varphi = F(x, t)\,. \end{aligned} \tag{10}$$

The energy of the solution $\varphi(x, t)$ at time t is defined to be

$$E(t) = \int_{R_n}\Big[\varphi_t^2(x, t) + \sum_{i=1}^n \phi_i^2(x, t)\Big]dx\,. \tag{11}$$

We will assume that our solution has uniformly bounded special support on any given finite time interval, in the sense that, given $T \geqq 0$ there is a rectangle $R \subset R_n$ such that the support of $\varphi(x, t)$ as a function of x lies in R for all t, $0 \leqq t \leqq T$. A uniform bound on $E(t)$ for all such t will be obtained under the following assumptions:

I′. The functions $\alpha^{ij}(x, t)$, $\alpha^i(x, t)$ and $\beta(x, t)$ are continuously differentiable functions of x and t.

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[1] Since u has essentially bounded weak derivatives it follows (c.f. Serrin [**9**]) that

$$\frac{u(x, t_1) - u(x, t_2)}{t_1 - t_2} \leqq K \tag{9b}$$

holds a.e. in $R \times [0, T]$, where K is some constant.

II′. *For all* $i, j = 1, \cdots, n$ we have $\alpha^{ij}(x, t) = \alpha^{ji}(x, t)$.

III′. There are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \sum_{i=1}^{n} \xi_i^2 \geqq \sum_{i,j=1}^{n} \alpha^{ij}(x, t)\xi_i\xi_j \geqq c_2 \sum_{i=1}^{n} \xi_i^2 \tag{12}$$

for all $(x, t) \in R \times [0, T]$ and all vectors $(\xi_1, \cdots, \xi_n)$.

IV′. There is a function $K_1(t) \in L^1(0, T)$ such that

$$\sum_{i,j=1}^{n} \alpha_t^{ij}(x, t)\xi_i\xi_j \leqq K_1(t) \sum_{i=1}^{n} \xi_i^2 \tag{13}$$

for all $(x, t) \in R \times [0, T]$ and all vectors $(\xi_1, \cdots, \xi_n)$.

V′. There are constants $A \geqq 0$ and $B \geqq 0$ such that

$$|\alpha^i(x, t)| \leqq A , \qquad |\beta(x, t)| \leqq B \tag{14}$$

for all $(x, t) \in R \times [0, T]$.

VI′. The function $F(x, t)$ is square integrable on $R \times [0, T]$. Then we have

LEMMA. *Under assumptions* I′ *through* VI′ *there is a constant* C *such that*

$$E(\tau) \leqq C$$

for all $\tau, 0 \leqq \tau \leqq T$.

Proof. We have

$$\begin{aligned} 2\varphi_t L\varphi = \Big[\varphi_t^2 + \sum_{i,j=1}^{n} \alpha^{ij}\varphi_i\varphi_j\Big]_t \\ - 2\sum_{i,j=1}^{n} (\alpha^{ij}\varphi_i\varphi_t)_j - Q(x, t) \end{aligned} \tag{15}$$

where

$$Q(x, t) = \sum_{i,j=1}^{n} \alpha_t^{ij}\varphi_i\varphi_j - 2\sum_{i=1}^{n} \alpha^i\varphi_i\varphi_t + 2\beta\varphi\varphi_t . \tag{16}$$

Integrating (15) over $R \times [0, \tau]$, $0 \leqq \tau \leqq T$, using the divergence theorem, and the fact that φ vanishes on the boundary of R for all t, $0 \leqq t \leqq \tau$, we obtain

$$\begin{aligned} \int_R \Big[\varphi_t^2 + \sum_{i,j=1}^{n} \alpha^{ij}\varphi_i\varphi_j\Big]_0^\tau dx = \int_0^\tau\int_R Q(x, t)dxdt \\ + 2\int_0^\tau\int_R \varphi_t F(x, t)dxdt . \end{aligned}$$

Denoting all constants generically by C, there results

$$E(\tau) \leqq C\Big[E(0) + \int_0^\tau\int_R Q(x, t)dxdt + 2\int_0^\tau\int_R \varphi_t F(x, t)dxdt\Big] .$$

Now to bound the right-hand side we note that from (13)

$$\int_0^\tau\int_R\Big[\sum_{i,j=1}^n \alpha_t^{ij}\varphi_i\varphi_j\Big]dxdt \leqq \int_0^\tau K_1(t)E(t)dt$$

and from (14)

$$-\int_0^\tau\int_R 2\Big[\sum_{i=1}^n \alpha^i\varphi_i\varphi_t\Big]dxdt \leqq C\int_0^\tau E(t)dt .$$

Also from (14),

$$\begin{aligned}\int_0^\tau\int_R 2\,\beta\varphi\varphi_t dxdt &\leqq C\int_0^\tau\int_R [\varphi^2 + \varphi_t^2]dxdt \\ &\leqq C\int_0^\tau\int_{R_n} \varphi^2 dxdt + C\int_0^\tau E(t)dt ,\end{aligned}$$

so that

$$\begin{aligned}&\int_0^\tau\int_R Q(x, t)dxdt \\ &\leqq \int_0^t [K(t) + C]E(t)dt + C\int_0^\tau\int_{R_n} \varphi^2 dxdt .\end{aligned}$$

To estimate the last integral on the right we have

$$\varphi(x, t) = \int_0^t \varphi_s(x, s)ds + \varphi(x, 0) , \qquad x \in R_n .$$

By Schwarz' inequality

$$\varphi^2(x, t) \leqq 2\Big\{t\int_0^\tau \varphi_s^2(x, s)ds + \varphi^2(x, 0)\Big\}$$

and so

$$\int_0^\tau \varphi^2(x, t)dt \leqq \int_0^\tau (\tau^2 - t^2)\varphi_t^2 dt + \tau\varphi^2(x, 0) .$$

Thus

$$\begin{aligned}\int_0^\tau\int_R \varphi^2(x, t)dxdt \leqq \int_0^\tau &(\tau^2 - t^2)E(t) \\ &+ \tau\int_{R_n} \varphi^2(x, 0)dx\end{aligned}$$

yielding

$$\begin{aligned}\int_0^\tau\int_R Q(x, t)dxdt \leqq \int_0^\tau &[K(t) + \tau^2 - t^2 + C]E(t)dt \\ &+ \tau\int_{R_n} \varphi^2(x, 0)dx .\end{aligned}$$

Finally,

$$2\int_0^\tau\int_R \varphi_t F(x, t)dxdt \leqq \int_0^\tau E(t)dt + \int_0^\tau\int_{R_n} F^2(x, t)dxdt ,$$

and so in all

$$E(\tau) \leqq f(\tau) + \int_0^\tau \chi(t)E(t)dt ,$$

where

$$f(\tau) = C\left[E(0) + \tau\int_{R_n} \varphi^2(x, 0)dx + \int_0^\tau\int_{R_n} F^2(x, t)dxdt\right],$$

and

$$\chi(t) = C[K(t) + \tau^2 - t^2 + 1] .$$

From Gronwall's inequality it follows that

$$E(\tau) \leqq f(\tau) + \int_0^\tau \chi(t)f(t) \exp\left(\int_t^\tau \chi(s)ds\right)dt$$

and hence the uniform boundedness of $E(\tau)$ on $0 \leqq \tau \leqq T$.

In the proof of the theorem we will actually use the following immediate

COROLLARY. *Under the assumptions of the lemma we have*

$$\int_0^T\int_R\left[\varphi_t^2 + \sum_{i=1}^n \varphi_i^2\right]dxdt \leqq \text{constant} .$$

The fact of crucial importance in this lemma, as far as the application to quasilinear equations is concerned, is that the bounds on the solution follow only from upper and not two-sided bounds on α_t^{ij}.

We now come to the main result of this paper.

THEOREM. *Weak solutions of* (4) *are uniquely determined by their initial conditions.*

Proof. If $u^1(x, t)$ and $u^2(x, t)$ are two weak solutions of (4) with the same initial conditions we will show that if $\omega = u^1 - u^2$, then

$$\iint_D F(x, t)\omega(x, t)dxdt = 0 \tag{17}$$

for every twice continuously differentiable function $F(x, t)$ having compact support in D thus showing that ω is zero a.e. in D.

For any test function $\varphi(x, t)$ we have

$$\iint_D \Big\{\omega\varphi_{tt} + \sum_{i=1}^n [a^i(x, t, u^1, u_k^1) - a^i(x, t, u^2, u_k^2)]\varphi_i - [b(x, t, u^1) - b(x, t, u)]\varphi\Big\} dxdt = 0 . \tag{18}$$

Now

$$a^i(x, t, u^1, u_k^1) - a^i(x, t, u^2, u_k^2) = \alpha^i(x, t)\omega + \sum_{j=1}^n \alpha^{ij}(x, t)\omega_j$$

where

$$\alpha^i(x, t) = \int_0^1 \frac{\partial}{\partial u} a^i(x, t, \tau u^1 + (1-\tau)u^2, \tau u_k^1 + (1-\tau)u_k^2)d\tau \tag{19}$$

and

$$\alpha^{ij}(x, t) = \int_0^1 a^{ij}(x, t, \tau u^1 + (1-\tau)u^2, \tau u_k^1 + (1-\tau)u_k^2)d\tau . \tag{20}$$

Similarly

$$b(x, t, u^1) - b(x, t, u^2) = \beta(x, t)\omega$$

where

$$\beta(x, t) = \int_0^1 \frac{\partial}{\partial u} b(x, t, \tau u^1 + (1-\tau)u^2)d\tau . \tag{21}$$

Thus for any test function φ we have

$$\iint_D \Big\{\omega\varphi_{tt} + \sum_{i,j=1}^n \alpha^{ij}\omega_j\varphi_i + \sum_{i=1}^n \alpha^i\omega\varphi_i - \beta\omega\varphi\Big\} dxdt = 0 . \tag{22}$$

The identity (17) will be established by constructing an appropriate sequence of test functions $\varphi^m (m = 1, 2, \cdots)$, using (22), and taking limits.

Let $\omega_m(x, t)$ be the Gaussian averaging kernel on $R_n \times (-\infty < t < \infty)$ with support contained in the sphere

$$|x|^2 + t^2 \leqq \frac{1}{m^2} ;$$

thus

$$\iint_D \omega_m(x, t)dxdt = 1$$

for all m. If the function $\psi(x, t)$ is in $L^2_{\text{loc}}(\bar{D})$ we extend it to $R_n \times (-\infty < t < \infty)$ by putting $\psi(x, t) = 0$ for $t \leqq 0$ and then define

$$\psi_m(x, t) = \psi * \omega_m$$

where the $*$ indicates convolution. It is well known that the functions ψ_m are smooth in $R_n \times (-\infty < t < \infty)$ and converge to ψ in mean square on compact subsets of $\bar{D}$. If in addition the function ψ is uniformly bounded on a compact subset of $\bar{D}$ then the functions ψ_m possess the same bound on that subset.

The functions $\alpha^{ij}_m(x, t)$, $\alpha^i_m(x, t)$ and $\beta_m(x, t)$ are now defined by the formulas for $\alpha^{ij}(x, t)$, etc., except that u^i_m replaces u^i $(i = 1, 2)$. Using our assumptions it is easy to see that

$$\begin{aligned} &| \alpha^{ij}_m(x, t) - \alpha^{ij}(x, t) | \\ &\qquad \leqq \text{const} \left[\sum_{i=1}^{2} | u^i - u^i_m | + \sum_{i=1}^{2} \sum_{k=1}^{n} | u^i_k - u^i_{k,m} | \right], \end{aligned}$$

uniformly on compact subsets of $\bar{D}$, from which it follows that the sequence α^{ij}_m converges to α^{ij} in mean square on compact subsets of $\bar{D}$. Similarly, α^i_m and β_m converge to α^i and β in mean square on compact subsets of $\bar{D}$.

The test functions φ^m are now chosen to satisfy the equation

$$\varphi^m_{tt} - \sum_{i,j=1}^{n} (\alpha^{ij}_m \varphi^m_i)_j + \sum_{i=1}^{n} \alpha^i_m \varphi^m_i - \beta_m \varphi^m = F(x, t) \tag{23}$$

and the conditions $\varphi^m(x, T) = \varphi^m_t(x, T) = 0$, where it is assumed that the support of $F(x, t)$ is contained in $D \cap \{0 < t < T\}$. Such functions are obtained by solving the backward initial-value problem with zero initial conditions at $t = T$. More precisely, we find solutions $\psi^m(x, t)$ of

$$\psi^m_{tt} - \sum_{i,j=1}^{n} (\tilde{\alpha}^{ij}_m \psi^m_i)_j + \sum_{i=1}^{n} \tilde{\alpha}^i_m \psi^m_i - \tilde{\beta}_m \psi^m = F(x, t) \tag{24}$$

on $0 \leqq t \leqq T$, subject to the initial conditions $\psi^m(x, 0) = \psi^m_t(x, 0) = 0$, where $\tilde{\alpha}^{ij}_m(x, t) = \alpha^{ij}_m(x, T - t)$, etc., and then put $\varphi^m(x, t) = \psi^m(x, T - t)$. The standard existence theory [3] guarantees that we can find smooth solutions of this initial-value problem which are then admissible test functions.

From (22) and (24) we obtain, after some integration by parts, the identity

$$\begin{aligned} \iint_D \omega F dx dt = \iint_D \Big\{ &\sum_{i,j=1}^{n} (\alpha^{ij}_m - \alpha^{ij}) \omega_j \varphi^m_i \\ &+ \sum_{i=1}^{n} (\alpha^i_m - \alpha^i) \omega \varphi^m_i + (\beta_m - \beta) \omega \varphi^m \Big\} dx dt \, . \end{aligned} \tag{25}$$

Our result will be established by showing that the right-hand side of (25) approaches zero as $m \to \infty$. Now as is easily seen,

$$c_1 \sum_{i=1}^{n} \xi_i^2 \geqq \sum_{i,j=1}^{n} \tilde{\alpha}_m^{ij}(x, t)\xi_i\xi_j \geqq c_2 \sum_{i=1}^{n} \xi_i^2$$

for all vectors $(\xi_1, \cdots, \xi_n)$, where c_1 and c_2 are constants independent of m, F has compact support and the existence theorems then show that the supports of the functions ψ^m are uniformly contained in some rectangular region $R \times [0, T]$ where R is a fixed rectangle in R_n; the integration on the right-hand side of (25) need only be extended over this region. Since the functions ω and ω_i are uniformly bounded, and the functions α_m^{ij}, etc., converge to α^{ij}, etc., in mean square on $R \times [0, T]$ we see, using Schwarz' inequality, that it suffices to show that φ_i^m and φ^m are uniformly bounded in mean square over this region. This will be achieved by applying the Corollary of Lemma 1 to obtain a similar bound for the functions ψ_i^m and ψ^m.

To apply Lemma 1 we need to show that assumptions I′ – VI′ are satisfied by the coefficients of equation (23), with bounds independent of m. The only assumption that is not immediately evident is IV′. To establish it we note that

$$\frac{\partial}{\partial t}\tilde{\alpha}_m^{ij}(x, t) = -\frac{\partial}{\partial t}\alpha_m^{ij}(x, t)$$

and so it suffices to demonstrate that

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial t}\alpha_m^{ij}(x, t)\xi_i\xi_j \geqq -K_1(t) \sum_{i=1}^{n} \xi_i^2 .$$

Now

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial t}\alpha_m^{ij}(x, t)\xi_i\xi_j$$
$$= \sum_{i,j=1}^{n} [A^{ij}(x, t) + B^{ij}(x, t) + C^{ij}(x, t)]\xi_i\xi_j$$

where

$$A_m^{ij}(x, t)$$
$$= \int_0^1 \frac{\partial a^{ij}}{\partial t}\Big(x, t, \tau u_m^1 + (1-\tau)u_m^2, \tau\frac{\partial u_m^1}{\partial x_k} + (1-\tau)\frac{\partial u_m^2}{\partial x_k}\Big)d\tau \tag{25a}$$

$$B_m^{ij}(x, t)$$
$$= \int_0^1 \frac{\partial a^{ij}}{\partial u}(x, t, \cdots)\Big[\tau\frac{\partial u_m^1}{\partial t} + (1-\tau)\frac{\partial u_m^2}{\partial t}\Big]d\tau \tag{25b}$$

(26c)
$$C_m^{ij}(x, t) = \sum_{k=1}^{n} \int_0^1 \frac{\partial a^{ij}}{\partial u_k} \left[\tau \frac{\partial u_m^1}{\partial t \partial x_k} + (1 - \tau) \frac{\partial u_m^2}{\partial t \partial x_k} \right] d\tau$$

and we need only show the lower boundedness of the three separate quadratic forms. Since the A^{ij} are uniformly bounded, the associated form is lower bounded. To show the lower boundedness of the two other forms we use assumptions IV and B. Using the properties of the averaging kernel it can be shown (see [6]) that the inequalities (9) imply that

$$\frac{\partial u_m^i}{\partial t} \leqq \tilde{K}(t)$$

and

$$\frac{\partial^2 u_m^i}{\partial t \partial x_k} \leqq \tilde{K}(t)$$

where the function $\tilde{K}(t) \in L^1(0, T)$. The same upper bound then holds for the convex combinations of these derivatives which occur in (26b) and (26c). Using IV we now see that

$$\sum_{i,j=1}^{n} (B^{ij} + C^{ij}) \xi_i \xi_j \geqq -2\tilde{K}(t) \sum_{i=1}^{n} \xi_i^2 ,$$

completing the proof.

It is clear from the proof that the uniqueness theorem will still be valid if the inequalities in (7) are reversed and the inequalities (9) are replaced by lower bounds

(9a)′
$$\frac{u(x, t_1) - u(x, t_2)}{t_1 - t_2} \geqq -K(t)$$

(9b)′
$$\frac{u_k(x, t_1) - u_k(x, t_2)}{t_1 - t_2} \geqq -K(t) .$$

Our assumptions were chosen to be consistent with those Oleĭnik [7]. In that paper she considered the system

(27a)
$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(x, t, v)}{\partial x} = 0$$

(27b)
$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0 .$$

For this system the proved uniqueness, under the assumption $\partial \varphi / \partial v < 0$ and the convexity assumption $\partial^2 \varphi / \partial v^2 > 0$, of pairs of weak solutions (u, v) (defined in the obvious way), where v satisfied a.e. a bound of the form

$$\frac{v(x, t_1) - v(x, t_2)}{t_1 - t_2} < K \tag{28}$$

where K is a constant. If we were dealing with smooth (i.e., twice continuously differentiable) solution pairs, then equations (27) could be replaced by the single equation

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x} a(x, t, v, v_x) = 0 \tag{29}$$

where

$$-a(x, t, v, v_x) = \frac{\partial}{\partial x} \varphi(x, t, v) = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial v}(x, t, v) v_x .$$

Inequality (7a) is then

$$\frac{\partial^2 \varphi}{\partial v^2} \geqq 0 , \tag{30}$$

which is even weaker than the strict convexity assumed by Oleĭnik, and inequality (7b) is vacuously satisfied.

But this reduction can also be made for weak solutions. Using a result of Schauder [8] we see that the weak form of (27b) implies the existence of a (locally) Lipschitz continuous potential function $J(x, t)$ which a.e. satisfies $J_x = v$ and $J_t = u$. (This function can be normalized so that $J(0, 0) = 0$.) It is then easy to see that J is a weak solution of

$$J_{tt} + \frac{\partial}{\partial x} \varphi(x, t, J_x) = 0 \tag{31a}$$

with the initial conditions

$$J(x, 0) = \int_0^x v(x, 0) dx$$

and

$$J_t(x, 0) = u(x, 0) . \tag{31b}$$

For (31a) the inequality (7b) is equivalent to (30) and Oleĭnik's other assumptions are sufficient for the application of our theorem. The uniqueness theorem applied to (31) then generalizes Oleĭnik's result. I am indebted to E. D. Conway for pointing out the possibility of this reduction.

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